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Abstract

In a recent paper in *Decision Sciences*, Barton proposes a method for finding a unique solution for the nucleolus. We argue that Barton's analysis is flawed because (1) the nucleolus is unique and (2) his proposed solution is inconsistent and ad hoc. Using Barton's own example, we reveal Barton's proposal to be a modified proportional method applied to the least core. Comparing the nucleolus with Barton's solution, we argue that his proposed method has little to recommend it, either conceptually or computationally.

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1 Introduction

In a recent issue of this Journal, Barton (1992) purports to identify "a problem in applying the nucleolus for allocating joint costs not envisioned in previous research" in which the nucleolus solution resulted in multiple joint costs. He proposes a solution to this problem, which involves minimizing the total propensity to disrupt (MTPD) over allocations in the least core. We believe that Barton's analysis is flawed, because

- The problem is non-existent — the nucleolus is always unique.
- His proposed solution is inconsistent and ad hoc.

Cooperative game theory has proved a fruitful vehicle for the exploration of the problem of allocating joint costs when multiple entities share a common resource (Young, Okada and Hashimoto 1982, Young 1985). The participating entities are modelled as players in a game in which the payoffs are the potential cost savings from cooperative action. Total cost savings are allocated according to the best alternatives open to the entities, either acting individually or in conjunction with other players (coalitions). The nucleolus is a game-theoretic solution concept which has many desirable properties as a cost allocation method.

The problem of cost allocation existed long before the emergence of game theory, and the engineering literature contains a host of practical but essentially ad hoc proposals and methods. Prominent amongst these is the so-called **separable costs - remaining benefits** (SCRB) method, which is used for the cost allocation of multipurpose reservoir projects in the United States and other countries (Young, Okada and Hashimoto 1982:463). The SCRB method allocates cost savings in proportion to each entity's marginal contribution to total cost savings.

Compared to the traditional approaches (such as SCRB), the advantages of a method founded in game theory are two-fold:

- allocations are based on all the strategic (alternative) opportunities available to the participants.
- game theoretic methods tend to have known properties which are regarded as desirable, such as consistency, monotonicity, and stability.

In comparison, allocations under the SCRB method are based solely on each entity's contribution to the grand coalition, ignoring all other possibilities of cooperation (coalitions). Furthermore, SCRB fails to exhibit **any** of the properties widely regarded as desirable in a cost allocation method (Young, Okada and Hashimoto 1982:472).

Barton's minimal total propensity to disrupt (MTPD) is an ad hoc modification of the SCRB method, and his proposed cost allocation method is a strange marriage of the best of the game theoretic literature (the nucleolus) and the most popular of the ad hoc approaches (SCRB) which fails to inherit the desirable characteristics of either parent.

2 Barton's model

Barton considers a cost allocation game comprising 4 producers or divisions (A, B, C, D) which utilize a common resource, which can be obtained at a total cost of

$$\text{Total cost} = 7.96218Q^{0.7}$$

where Q represents the quantity required. The requirements of the entities are

$$\begin{aligned} q_A &= 100 \\ q_B &= 300 \\ q_C &= 500 \\ q_D &= 700 \end{aligned}$$

respectively. Owing to declining marginal cost, the producers can minimize total costs by acting jointly. For example, the cost of providing sufficient resource to meet the requirements of A and B together is \$527.81 compared to the aggregate cost of \$631.54 if they purchase separately.

The total costs of various combinations (**coalitions**) of the entities are listed in Table 1. Clearly, total costs are minimized if all the producers purchase jointly. This raises the question of how the total cost (\$1392.92) should be shared amongst the producers. One approach to this problem is to regard the cost savings as payoffs in a game with the producers as players, and allocate the cost savings from cooperation in accordance with a suitable solution concept. This is the approach adopted by Barton, who seeks to apply the nucleolus. To model as a game, we define the characteristic function as the cost savings to each coalition of cooperating rather than each acting individually. That is

$$v(S) = \sum_{i \in S} c_i - c(S)$$

where

$$c_i = \text{Cost}(q_i)$$

is the cost to entity i of purchasing individually and $c(S)$ is aggregate cost to coalition S of purchasing jointly, as listed in the third column of the following table. The cost savings $(v(S))$ are listed in the last column.

Table 1: Basic Information for Coalitions

Coalition	Quantity	Total Cost $c(S)$	Average Cost	Coalition Savings $v(S)$
A	100	200.00	2.00	
B	300	431.54	1.44	
C	500	617.04	1.23	
D	700	780.91	1.12	
AB	400	527.81	1.32	103.73
AC	600	701.03	1.17	116.01
AD	800	857.42	1.07	123.49
BC	800	857.42	1.07	191.15
BD	1000	1002.38	1.00	210.07
CD	1200	1138.83	.95	259.12
ABC	900	931.11	1.03	317.46
ABD	1100	1071.54	.97	340.91
ACD	1300	1204.46	.93	393.49
BCD	1500	1331.36	.89	498.12
ABCD	1600	1392.89	.87	636.60

A solution to the cost allocation game is an allocation x of the total cost savings $v(\{A, B, C, D\}) = 636.60$ to the participants (players) in the game, that is a vector (x_A, x_B, x_C, x_D) such that $x_A + x_B + x_C + x_D = \$636.60$. Any allocation x of the cost savings implies a distribution of the total costs amongst the four participants according to the formula

$$\text{Cost share of } i = c_i - x_i$$

Clearly, not every allocation x of the cost savings (or cost shares) will be equally acceptable to all the participants. In particular, no individual or subgroup is likely to accept an allocation of the total cost savings which is less than they could achieve by unilateral action independent of the rest of the participants. That is, a minimal requirement for a cost allocation would appear to be that everyone gains from cooperation, that is

$$\sum_{i \in S} x_i \geq v(S) \qquad \forall S \tag{1}$$

The set of allocations which satisfy this requirement is called the **core** of the game. Any allocation in the core implies that costs borne by any group are less than they would incur if they purchased independently. Formally, an allocation is in the core if

$$\text{Cost share of group } \leq c(S)$$

Every group has an incentive to participate in a core allocation.

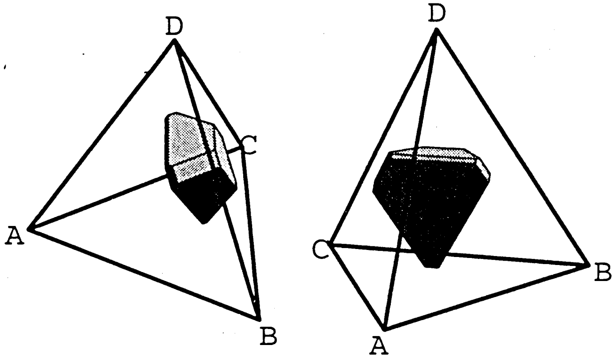

Figure 1: The core of the cost allocation problem: Two views

Typically, there are many allocations in the core.[1] A cost allocation method is required to select a particular allocation from the core to serve as **the solution** of the game. The nucleolus is one such solution; Barton's method is another. Indeed, a plethora of alternative allocation procedures have been advocated. One advantage of a game-theoretic approach to the problem is that it enables the properties of different proposals to be compared.

The nature of the cost allocation problem is illustrated in Figure 1. The set of all possible allocations in a four player game is a tetrahedron in 3–space. The shaded area is the core, which is a convex subset of the tetrahedron. A cost allocation method is required to select a unique point from this set.

3 The Nucleolus

The nucleolus as a solution concept for cooperative games was introduced by David Schmeidler (1969) specifically to overcome the multiplicity of outcomes characteristic of its antecedent concepts, the bargaining set and the kernel.

The **dissatisfaction** of a coalition with an allocation x can be measured by its **excess**, which is difference between the cost savings it enjoys at the allocation x and the cost savings it could obtain by acting alone

$$e(x, S) = v(S) - x(S)$$

which can be expressed as

$$e(x, S) = \sum_{i \in S}(c_i - x_i) - c(S)$$

[1]There may be no such allocations, that is the core may be empty. However cost allocation games (including Barton's example) are typically convex, which implies that core allocations always exist. For an introduction to cooperative game theory in general and cost allocation games in particular, see for example Carter (1993).

The first term is the total cost share of the coalition S under the proposed allocation. The second term is the total costs they would bear if they acted unilaterally. The smaller the excess, the better off is the coalition at the allocation x. An allocation belongs to the core if and only if $e(x, S) \leq 0$ for all coalitions S. That is, in the core, joint action is better than unilateral action for every coalition.

Within the core, the excess provides a measure of the degree to which the coalition is sharing the benefits of cooperation. An allocation y is more favorable to coalition S than an allocation x whenever

$$e(y, S) < e(x, S)$$

Following Osborne and Rubinstein (forthcoming), we say that a coalition S has an **objection to** x if there exists another allocation y which is more favorable to S, that is $e(y, S) < e(x, S)$. Their objection has a **counterobjection** if there exists another coalition T who is worse off at y and furthermore whose dissatisfaction (excess) with y is greater than coalition S's dissatisfaction with x. That is, T has a counterobjection to y if $e(y, T) > e(x, T)$ **and** $e(y, T) \geq e(x, S)$. The nucleolus of the allocation game is the set of all allocations x with the property that for every objection (y, S) there is a counterobjection.

For example, in Barton's game, consider the allocation which shares the cost-savings in proportion to usage, that is

$$\begin{aligned} x^P &= \frac{q}{\sum_{i \in N} q_i} v(N) \\ &= (39.79, 119.36, 198.94, 278.51) \end{aligned}$$

to which the coalition $\{A, B, C\}$ can object by proposing the more favorable allocation

$$y = (69.24, 146.25, 198.83, 222.28)$$

since

$$e(y, \{A, B, C\}) = -96.8582$$

is less than

$$e(x^P, \{A, B, C\}) = -40.6231$$

Furthermore, since no coalition has a bigger excess at y than S has at x, there is no counter-objection. Therefore, coalition S's objection to the proportional division can be sustained. On the other hand, from y, every attempt to make some coalition better off makes some other coalition worse off. Every objection to the allocation y can be met with a corresponding counterobjection. Thus, the allocation y belongs to the nucleolus. The nucleolus is the set of allocations to which no group can validly object. Somewhat surprisingly, in any cost allocation game, there is always **exactly one** allocation to which there is no valid objection. That is, the nucleolus is non-empty and comprises a single point. Indeed, as Schmeidler (1969: p1164) himself remarked in his introduction, uniqueness is "one of the most appealing properties of the nucleolus as a solution concept" .

Other desirable properties of the nucleolus are **consistency** with respect to reduced games and the fact that the nucleolus always belongs to the core (provided the core is non-empty). This latter property is especially valuable in cost allocation games, since core allocations provide incentives for cooperation.

4 Computing the Nucleolus

In the previous section, we claimed that the allocation

$$y = (69.24, 146.25, 198.83, 222.28)$$

sustains no valid objection and is the nucleolus of Barton's cost game. However, it is not at all obvious from the definition how to compute the nucleolus of this or any other game. It is necessary to provide an overview of the computation of the nucleolus in order to understand Barton's proposal. Typical algorithms for calculating the nucleolus proceed by solving a sequence of minimization problems.

Let X be the set of feasible allocations (called imputations).

$$X = \{(x_A, x_B, x_C, x_D) \in \Re^4 \mid x_i \geq 0 \text{ and } x_A + x_B + x_C + x_D = v(N)\}$$

Consider the following linear program.

$$\min_{x \in X} r$$

$$\text{such that } e(x, S) \leq r \quad \forall S$$

The solution to this linear program comprises a minimum value r^1 and a set of allocations $x \in X$ at which this minimum value is obtained.[2] Let X^1 denote the set of optimal solutions to the linear program. X^1, which is known as the **least core**, is the set of allocations which minimize the excess (potential objections) over all coalitions. The greater the excess, the greater the incentive for a coalition to object. In this sense, the least core X^1 identifies those allocations which are least objectionable. The following well-known proposition highlights two standard properties of the least core.

Proposition 1 *The least core X^1*

1. *is a compact convex subset of X defined by the binding constraints to the linear program,*

2. *which contains the nucleolus.*

Proof. [3] The first property is a standard result of the theory of linear programming. To see the second, let x be any allocation not in X^1. We show that there

[2] In general this linear program has multiple optimal solutions.

[3] Although this proposition is standard, we provide a proof of the second part using the Osborne and Rubinstein characterization of the nucleolus, since it is crucial to understanding the algorithm.

exists a valid objection to x. There exists an allocation in the least core X^1 that can be used to object to x. That is, there exists a coalition S and allocation $y \in X^1$ such that

$$e(y, S) \leq r^1 < e(x, S)$$

since y is a solution of the linear program while x is not. Furthermore, there is no counterobjection to y. Every coalition that is worse off at y ($e(y,T) > e(x,T)$) is however better off at y than S was at x, since

$$e(y, T) \leq r^1 < e(x, S)$$

We have shown that for every $x \notin X^1$ there exists a $y \in X^1$ that provides an unassailable objection to y. The least core X^1 is the set of least objectionable allocations. □

With respect to the least core, proper coalitions divide into two classes. The first group $\mathcal{A}$ comprises coalitions for which the excess is constant and equal to r^1 throughout the least core. For the remaining coalitions $\mathcal{B}$, there are points in the least core that have an excess less than r^1. Formally let

$$\mathcal{A} = \{S \subset N \mid e(x, S) = r^1 \quad \forall x \in X^1\}$$
$$\mathcal{B} = \{S \subset N \mid e(x, S) < r^1 \text{ for some } x \in X^1\}$$

We make the following observations, which follow easily from the definitions or from the previous proposition.

1. Every coalition belongs to either $\mathcal{A}$ or $\mathcal{B}$
2. $\mathcal{A} \neq \emptyset$
3. Coalitions in $\mathcal{A}$ cannot object to any point in X^1
4. For every coalition in $\mathcal{B}$, there exist points in X^1 to which that coalition can object with another point in X^1.
5. No coalition in $\mathcal{A}$ can counterobject to any such objection.
6. If $\mathcal{B} = \emptyset$ (there are no objections), then the least core X^1 contains a single point, which is the nucleolus.

Coalitions in $\mathcal{A}$ can neither object to any allocation in X^1 nor counterobject to any objection in X^1. By restricting attention to allocations in X^1 (the least-objectionable allocations), their bargaining power has been exhausted and they are effectively neutralized. If all coalitions belong to $\mathcal{A}$ ($\mathcal{B} = \emptyset$), then the least core X^1 contains a single point to which there are no valid objections. We have located the nucleolus.

If not, we simply repeat the preceding minimization with respect to those coalitions that still have the potential to object (that is $\mathcal{B}$). Formally, we solve the following linear program

$$\min_{x \in X^1} r$$

$$\text{such that } e(x, S) \leq r \quad \forall S \in \mathcal{B}$$

where X^1 is defined as

$$X^1 = \{x \in X \mid \text{ such that } e(x,S) = r^1 \quad \forall S \in \mathcal{A}, \\ e(x,S) \leq r^1 \quad \forall S \in \mathcal{B}\}$$

This identifies a set X^2 of still less objectionable allocations that again satisfies the properties in Propositions 1. We continue in this fashion until we exhaust the set of coalitions with valid objections. At each iteration, we reduce the set of coalitions with the potential to object, leading eventually to a single point, which is the nucleolus.

This procedure provides a algorithm for the computation of the nucleolus by solving a sequence of linear programs.[4] The algorithm also reveals an alternative characterization of the nucleolus, namely as the **lexicographic center** of the core of the game. The nucleolus is that point in the core which is as far away as is possible from all its bounding hyperplanes. In this sense, it is the centre of the polyhedron in Figure 1. This is another justification for its claim to be a "fair" share of the gains from cooperation.

In Barton's game, the least core is given by the following system of equalities and inequalities

$$\begin{aligned} x_A &= 69.24 \\ 103.73 \leq x_B &\leq 173.87 \\ 116.01 \leq x_C &\leq 226.45 \\ 123.49 \leq x_D &\leq 249.90 \\ x_B + x_C + x_D &= 567.36 \end{aligned} \tag{2}$$

which is a plane in the three dimensional simplex,[5] which is illustrated in Figure 2.

In the least core, the payoff to player A is fixed at 69.24. In other words, player A cannot validly lay claim to a larger share of the cost savings. Any such claim could be met by a justifiable objection from the other players. By the same token, player A would have a justifiable objection against any allocation which awarded him a smaller share. Any unobjectional allocation must award player A exactly 69.24. After satisfying A, there remains some flexibility in the shares awarded to the other players. While their total payoff to the other players is constrained to

$$x_B + x_C + x_D = 567.36$$

their individual shares may vary within the bounds given by the inequalities in (2). Even within this range, there are allocations to which the some

[4]The only difficulty at each stage is in identifying all the optimal solutions (the sets $X^1, X^2, \ldots$), since most linear programming packages are content with providing a single optimal solution. A *Mathematica* implementation of this algorithm is available in Carter (1993).

[5]Barton lists these inequalities on p.369, omitting the equation

$$x_B + x_C + x_D = 567.36$$

and therefore implying that they are independent inequalities. This is potentially very misleading.

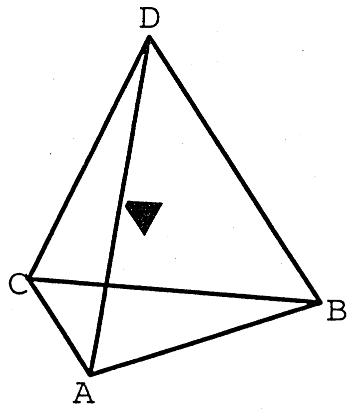

Figure 2: The least core in Barton's game

of the players may justifiably object. For example, consider the allocation $(69.24, 173.87, 192.99, 200.50)$, that allocates the maximum possible of the remaining gains to player B (173.87) and shares the rest (over and above their minima) equally between C and D. The coalition $\{C, D\}$ can justifiably object with the amended allocation $(69.24, 146.25, 198.83, 222.28)$ to which B has no counterobjection.

Attempting to find the least objectional allocations in the least core as defined by (2), we solve a second linear program

$$\min_{x \in X^1} r \tag{3}$$

$$\text{such that } e(x, S) \leq r \quad \forall S \in \mathcal{B}$$

where X^1 is defined as

$$X^1 = \{x \in X \mid \text{such that } e(x, S) = r^1 \quad \forall S \in \mathcal{A}\}.$$

This has a unique solution

$$x^N = (69.24, 146.25, 198.83, 222.28)$$

which therefore is the nucleolus of the game. It the only allocation to which no coalition can justifiably object. Every objection to x^N can be met with a counterobjection.

The least core is illustrated in Figure 3 which offers a planar view of the least core in Figure 2. The outer triangle is the two dimensional simplex representing all allocations consistent with a fixed share of 69.24 to player A.[6]. The shaded area is the least core which is a subset of the this simplex.

[6] It is a cross-section of the three dimensional simplex (Figure 2) parallel to the A axis at the level 69.24)

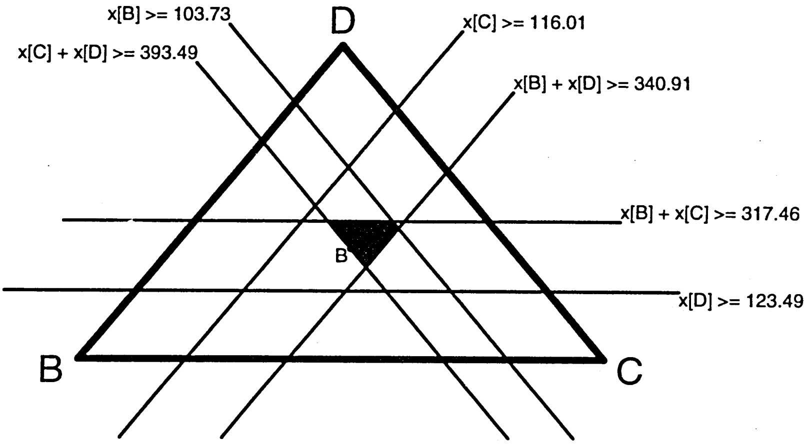

Figure 3: A planar view of the least core

The nucleolus (marked **N**) is the central point of the least core. Barton's solution, which is labelled **B**, lies on the boundary of the least core. It is the allocation $(69.24, 173.87, 192.99, 200.50)$ discussed above, that allocates the maximum possible share of the gains within the least core to player B.

5 Barton's method

The preceding algorithm lies at the heart of Barton's confusion. He successfully computes the least core (X^1) which he confuses with the nucleolus. Confronted with a multiplicity of acceptable solutions in this first round, he abandons the previous objective of minimizing objections, substituting the alternative objective of minimizing the non-linear function $a/x^A + b/x^B + c/x^C + d/x^D$, where a, b, c and d are constants related to the contributions of the coalitions to total savings. In short, Barton substitutes the non-linear program

$$\min_{x \in X^1} \sum_{i \in N} \frac{a_i}{x_i} \tag{4}$$

for the linear program (3). As a convex function on a convex domain, this problem is guaranteed to have a unique solution. This is Barton's proposed solution to the alleged multiplicity of the nucleolus[7].

[7] Barton reports that he "discovered" this problem in an earlier experimental study (Barton 1988), where he there applied "the somewhat arbitrary choice" (Barton 1992:369) of selecting the midpoint rather than solving the second stage optimization problem. It is ironic that in that game, which involved only three players, his somewhat arbitrary choice correctly selected the nucleolus while his "solution" would have selected a different and less satisfactory point!

Leaving aside the fact that Barton's proposal is designed to tackle a non-existent problem, let us examine it as a solution in its own right. We raise three fundamental objections to Barton's proposal:

- it is inconsistent to apply one objective (minimizing objections) in the first round and an entirely different objective in the second round.
- the second objective function is ad hoc yielding a solution without desirable properties or rationalization.
- it makes only selective use of the available information in the second round.

In order to understand how Barton arrives at his particular solution and our objections to it, it is necessary to refer back to some earlier work. Barton's starting point is a proposal by Gately (1973), who proposed equalizing the propensities to disrupt as a practical solution for a very real cost allocation problem[8]. Gately argued that a coalition S would have a propensity to disrupt an allocation x if its share of the gains from cooperation were small relative to those of remaining players. Accordingly, he defined the propensity of a coalition S to disrupt an allocation x as the ratio of the excesses of the coalition and its opponents at x, that is

$$d(x,S) = \frac{e(x,S^c)}{e(x,S)}$$

where S^c is the complement of S. For individual players, this can be equivalently expressed as

$$d(x,i) = \frac{\varphi_i}{x_i} - 1$$

where

$$\begin{aligned} \varphi_i &= v(N) - v(N-i) \\ &= c_i - (c(N) - c(N-i)) \end{aligned} \tag{5}$$

is player i's (marginal) contribution to total cost savings. Gately suggested that an allocation at which the individual propensities to disrupt were equalized would be an acceptable solution to the cost-sharing problem. It is easily shown that such a point exists and is achieved where the maximum propensity to disrupt is minimized. At this point, total cost savings are shared in proportion to marginal contributions, that is

$$x_i^{\mathrm{PD}} = \frac{\varphi_i}{\sum_{i\in N}\varphi_i} v(N).$$

This is precisely the allocation obtained under the **separable costs – remaining benefit** (SCRB) method (Young 1985, p. 12). Although it does

[8] Gately's problem involved the sharing the costs of joint hydoelectric development amongst states in southern India. His work was roughly contemporaneous with Schmeidler's introduction of the nucleolus, and it is tempting to speculate that he may have been unaware of this concept.

not exhibit the same desirable properties as competing methods, such as the nucleolus and the Shapley value (Young, Okada and Hashimoto, 1982), Gately's method does have an impressive practical pedigree, being well established in the engineering literature.

Having cited Gately's work with approval, Barton makes a significant amendment. Instead of selecting an allocation which minimizes the maximum individual propensity to disrupt (which has an obvious affinity with nucleolus), Barton chooses to minimize the aggregate propensity to disrupt. In effect, he solves the nonlinear program

$$\min_{x \in X^1} \sum_{i \in N} \frac{\varphi_i}{x_i}.$$

A strictly convex function is bound to attain a unique minimum on a compact convex set, so a unique solution is guaranteed. But it is a solution with little to recommend it.

Applied to the original game, as opposed to the reduced game delineating the least core, both Barton's and Gately's methods belong to the class of **proportional methods**[9] in which the total cost savings $v(N)$ are divided amongst the participants in proportion to some set of weights $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Each players allocation is determined according to

$$x_i = \frac{\alpha_i}{\sum_{i \in N} \alpha_i} v(N) \tag{6}$$

In Gately's method, the weights are marginal contributions to total cost savings (5), that is

$$\begin{aligned} \alpha_i &= \varphi_i \\ &= c_i - (c(N) - c(N - i)) \end{aligned} \tag{7}$$

Barton's modification amounts to using different weights[10], namely

$$\alpha_i = \sqrt{\varphi_i} \tag{9}$$

[9]Most of the other methods referred to in Barton's earlier experimental study (1988) are also proportional methods.

[10]The first order condition for minimizing

$$\sum_{i \in N} \frac{\varphi_i}{x_i}$$

subject to the constraint

$$\sum_{i \in N} x_i = v(N)$$

is

$$\frac{\varphi_i}{x_i^2} = -\lambda \qquad \forall i \in N \tag{8}$$

where λ is the Lagrange multiplier. Summing over i gives

$$-\lambda = \left(\frac{\sum \sqrt{\varphi_i}}{v(N)} \right)^2$$

which can be substituted into the first order condition (8) to give (9).

Barton's modification of the weights seems to have little justification. He offers two reasons for favouring the total propensity to disrupt over the individual propensity to disrupt, namely because the total propensity to disrupt

- offers an overall measure of discordance
- can be expressed as a function of x_i having a unique minimum value.

When both are identified as proportional methods with different weights, $\sqrt{\varphi_i}$ versus φ_i, these two reason for preferring MTPD over PD are revealed to be vacuous, and his modification seems extremely ad hoc.

Moreover, debating the merits of alternative weights in a proportional method seems rather trivial compared to the fundamental shortcoming of all proportional methods. The fundamental deficiency of any proportional method is that it can only take account of a subset of information available in the problem. Allocation is entirely determined by a set of n numbers $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. In the case of both Barton and Gately, allocation depends solely on marginal contributions to the grand coalition, φ_i. The advantage of an allocation method based on a game theoretic solutions (such as the nucleolus or the Shapley value) is that it incorporates in some fashion information regarding all the alternatives available to the participants. Allocation is based on all the 2^n numbers which constitute the characteristic function of the game, rather than just a subset of n numbers as in Barton or Gately or some other set of n numbers unrelated to individual contributions. The only advantage attributable to proportional methods is their simplicity, which undoubtedly accounts for the popularity in practice.

Of course, Barton does not apply his method of minimizing aggregate propensity to the original game, but to its least core, for the reason given in the preceding paragraph. In his own words

> *Actually, the nucleolus is a much better stand-alone method than the MTPD. The MTPD has a very serious deficiency as a stand-alone method: It looks at the problem from the viewpoint of the set of last entitities added to coalitions that contain $(n-1)$ members.* (p. 370)

To illustrate, he shows that total propensity to disrupt is minimized over all allocations at $x^{\text{MTPD}} = (120.02, 159.02, 175.37, 182.19)$. "This is probably unsatisfactory", Barton argues, because the share awarded to the three largest entities is too small relative to that awarded to A. The coalition comprising B, C and D is liable to object to this allocation, since they receive only 13% of the gains from A's participation, compared to A's 87%. In Gately's terms, the coalition comprising the three largest entities $\{B, C, D\}$ has a high propensity to disrupt this allocation (6.5), since their excess at x^{MTPD} of 18.46 is very small relative to A's excess of 120.02.

Barton goes on to claim

> *While this characteristic is undesirable for a stand-alone method, it it is not troublesome in an extension to the nucleolus because*

the nucleolus generates a range of feasible values ... [for the least core] prior to the extension's being invoked. In other words, the extension is being applied at the second-order level. (p. 370-371)

As we interpret this passage, Barton is claiming justification for his method because it constrained to select an allocation from the least core. Hence, his method is guaranteed to select a core allocation provided one exists.[11] But selecting a particular allocation in the core is precisely the problem of cost allocation. Barton's particular selection has little if anything to recommend it as a cost allocation method.

This is well illustrated by Barton's own game. In this game, Barton's method selects the allocation

$$x^B = (69.24, 173.87, 192.99, 200.50)$$

which is point B on the boundary of the least core in Figure 3. As we showed in the previous section, this allocation seems unduly favourable to B, awarding B all the maximum possible gains subject to satisfying A. What possible justification can be offered to players C and D to persuade them that this allocation is fair or reasonable. A random selection of a point in the least core would have more to recommend it.

Though we have not investigated this issue formally, it seems to us intuitive that boundary solutions are likely to occur in Barton's method, since the domain of the objective function is the whole two dimensional simplex in Figure 3 while the feasible region is the shaded area (the least core). To attain a justifiable **proportional method** of selecting a point within the least core, we suggest that the allocation constants in α_i (equation 6) and the total to be allocated ($v(N)$) should be defined relative to the feasible region (the least core) rather than the overall game.

Furthermore, as the number of players increases, the dimension of the least core can grow proportionately. As the dimension of the least core increases, the number of possible coalitions and strength of possible objections to extreme allocations within the least core will also increase. Barton's objections to applying his method to the overall game will apply with increasing strength to his own proposal as the number of players increases. To underscore the point, we could construct a five player game in which the least core will be identical to Barton's four player game. In this game, his comments made in the above quotations apply *mutatis mutandis* to his own method!

For this reason, his method seems designed with four player games in mind. It is unnecessary in three player games, since the midpoint of the least core yields the nucleolus. Its application to games with more than four players seems destined, for the reasons just discussed, to incur the very problems it is designed to overcome.

The reader might reasonably infer from Barton's paper that the first iteration (the least core) always determines the allocation of at least one player,

[11] Recall that cost allocation games are typically convex and therefore have a non-empty core.

thereby reducing the dimensionality of the optimization problem.[12] This is not so, as the four player counter-example in Maschler, Peleg and Shapley (1979, p. 355) demonstrates. While we can see no bar to applying Barton's method to such a game, it does somewhat undermine the justification.

Does Barton's method offer any computational advantage? We think not. The computational burden in finding the nucleolus involves identifying the bounds of the least core. Barton must do this also. In the second stage, he substitutes a constrained non-linear program (4) for the linear program (3). In general, this optimization problem could be rather difficult to solve, and Barton's own demonstration of the solution to his model (p. 373) is anything but elegant. By comparison, the linear program (3) is straightforward and easy to solve. On the other hand, Barton's solution terminates at this point whereas the nucleolus might require further iterations. Computationally, the trade-off is a more complicated optimization problem at the second round against possibly fewer iterations. Given the efficiency of linear programming algorithms, we think it unlikely that Barton's method will have a computational advantage. For similar reasons, it cannot lay claim to virtue on grounds of simplicity.

6 Conclusion

In this paper, we have analysed the solution proposed by Barton to the problem of finding a unique solution for the nucleolus in cost allocation. We have argued that it is a poor solution to a non-existent problem. The problem is non-existent because the nucleolus as defined by Schmeidler (1969) is unique and widely accepted as a cost allocation method (Young, Okada and Hashimoto 1982, Young 1985). Moreoever, his proposed solution seems arbitrary and inconsistent, and does not exhibit the properties desirable in a cost allocation method.

In conclusion, we would like to offer some more reflective comments on Barton's method as a cost allocation procedure. Young (1985, p. vii) describes the cost allocation problem as a normative problem: "How should the common costs of an enterprise be shared 'fairly' among its beneficiaries?" Underlying any cost allocation method is some notion of fairness.

Barton proposes his solution as an extension of that of Gately. However, we believe they are founded on quite different notions of fairness. Gately notes that, in equalizing the propensity to disrupt, his solutions also equalizes the ratio of each player's share to their maximum possible share within the core. This ratio, he suggests, may be viewed as an index of the players satisfaction with their share. With this interpretation, his solution would make the players equally satisfied. For Gately, then, fairness means equality (according to an appropriate index). In contrast, at the MTPD, Barton is only concerned with the aggregate propensity to disrupt. There is no explicit concern for equality among the players.[13]

[12] He talks of seeking "unique values for the remaining x_is" (p. 370). In some games, all the x_is will be "non-unique".

[13] The individual propensities to disrupt in Barton's solution of vary from 0.4 to 1.

Further, Barton contends that the MTPD is similar in flavour to the nucleolus. This contention seems difficult to sustain, since the nucleolus is founded on a very different notion of fairness. Since it aims to minimize the maximum complaint (excess) of any coalition, the nucleolus has a distinctly Rawlsian notion of fairness, compared to the utilitarian flavour of Barton's proposal.

Young (1985) also gives the following empirical test of a **fairness principle**: "is it sufficiently compelling to cause parties with diverse interests to **voluntarily** agree to its application?" (p. viii). It seems to us unlikely that any informed participant would find Barton's fairness principle compelling. Rather a procedure which applies a single principle in a consistent and clear manner is more likely to be judged "fair" than a procedure which applies different methods at different stages in an apparently ad hoc and arbitrary fashion. In this regard, we note that Barton's MTPD procedure achieved least support in his own empirical test (Barton 1988, Table 2)!

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