Department of Economics
UNIVERSITY OF CANTERBURY
CHRISTCHURCH, NEW ZEALAND

THE LIMITING POWER OF POINT OPTIMAL AUTOCORRELATION TESTS

John P. Small

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Department of Economics, University of Canterbury
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Abstract

This paper considers the point optimal tests for AR(1) errors in the linear regression model. It is shown that these tests have the same limiting power characteristics as the Durbin-Watson test. The limiting power is zero or one when the regression has no intercept, but lies strictly between these values when an intercept is included.

Address for correspondence: J.P. Small, Department of Economics, University of Canterbury, Private Bag, Christchurch, New Zealand.

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1. Introduction

Consider the standard linear regression model, with possibly AR(1) errors:

$$y = X\beta + u$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad |\rho| < 1, \ \varepsilon_t \sim NID(0,\sigma_\varepsilon^2), \ t = 1,\ldots,T$$

where y and u are (T×1) vectors of observations on the dependent variable and random disturbances respectively, X is a (T×K) non-stochastic matrix with full column rank and β is a (K×1) vector of parameters. Let $E(uu') = V = \sigma_\varepsilon^2/(1-\rho^2)\Omega$ and denote the Cholesky decomposition of Ω by $L(\rho)$.

We are interested in those tests of H_0: $\rho = 0$ vs H_A: $\rho > 0$ for which the test statistics can be written as a ratio of quadratic forms in u,

$$d = u'Qu/u'Mu ,$$

where $M = I - X(X'X)X'$ is symmetric and idempotent and Q is some other (T×T) non-stochastic matrix. This specification encompasses two types of tests which have been shown to have high power in exact comparative studies (e.g. King (1985)).

The first type of test uses OLS residuals only, examples being the Durbin-Watson (DW) and the alternative Durbin-Watson (ADW) tests (see King (1981) for discussion of the relative strengths of these tests). For these tests Q = MAM where A is a first-differencing matrix of a slightly different form for each test.

Tests of the second type use GLS and OLS residuals and are most powerful invariant in particular regions of the parameter space. Examples are the BW test of Berenblut-Webb (1973) and King's (1985) $S(\rho_1)$ test, where $Q = B-BX(X'BX)X'B'$ and B is the inverse of the covariance matrix of u, for some value of ρ, ρ_1. The BW test is MPI as $\rho \to 1$ and the $S(\rho_1)$ tests are MPI when $\rho = \rho_1$.

It is known that the power of all of these tests can approach zero as ρ approaches unity. This has been demonstrated by Krämer and Zeisel (1990). For the DW test, analytic results prove that this limiting power is always zero or unity for regressions with no intercept (Krämer (1985)) but lies strictly between these two values when an intercept is included (Zeisel (1989)). Extension of these results to the ADW test is trivial as the precise form of the A matrix is not relevant to either proof. The following section will show that the same limiting power characteristics apply to the point optimal tests, $S(\rho_1)$ and BW.

2. Theoretical Discussion

The power of each test considered above can be expressed as

$$\Pr\left\{u'(Q-d^*M)u < 0\right\} = \Pr\left\{\sum_{j=1}^{T} \lambda_j Z_j^2 < 0\right\}$$

where d^* is the appropriate critical value, $Z_j^2 \sim \chi^2_{(1)}$ and independent and the λ_j's are the eigenvalues of

$$W(\rho) = L'(\rho)(Q-d^*M)L(\rho).$$

For the DW and ADW tests $Q = MAM$ and, since $MM = M$, we have

$$W(\rho) = L'(\rho)M(A-d^*I)ML(\rho).$$

The result of Krämer (1985) derives from observing that $L(1)$ contains a column of ones and using $MX = 0$ to conclude that if X does not contain an intercept then $W(1) \neq 0$ but is of unit rank. There is only one non-zero eigenvalue of $W(1)$, the sign of which determines whether zero or unity is the limiting power.

There are two ways of showing that this result also holds for the point optimal tests. The more direct method employs the following theorem, due to Evans and King (1985).

Theorem 1. If $Q = B - BX(X'BX)^{-1}X'B'$ and $M = I - X(X'X)^{-1}X'$, then $MQ = QM = Q$.

The proof follows directly from the definitions of M and Q. This result allows the point-optimal test statistic to be written as a Durbin-Watson type test, with a particular A matrix:

$$d = u'MQMu/u'Mu.$$

The power of a point optimal test can now be seen to depend on the eigenvalues of

$$W(\rho) = L'(\rho)M(Q-d^*I)ML(\rho)$$

and if the regression has no intercept then W(1) has exactly one non-zero eigenvalue and the result of Krämer (1985) holds.

When an intercept is present $W(1) = ML(1) = 0$ and the covariance matrix manipulations of Zeisel (1989) show that the limiting power depends only on the eigenvalues of $U'M(A-d^*I)MU$ for the DW (and hence ADW) test, where

$$U = \left[\begin{array}{c|ccccc} 0 & & & 0 & & \\ \hline & 1 & & & & \\ & 1 & 1 & & & \\ 0 & 1 & 1 & 1 & & \\ & & & & \ddots & \\ & 1 & \cdots & \cdots & \vdots & 1 \end{array}\right].$$

Defining $i = (1,1,\ldots,1)'$ and $F = [i|0] + U$, Zeisel notes that F is regular and $MF = MU$ when the model has an intercept. The congruent transformation $U'M(A-d^*I)MU = F'M(A-d^*I)MF$ does not change the number of positive and negative eigenvalues, by Sylvester's law of inertia. Now, since F is non-singular, $F'M(A-d^*I)MF$ is a congruent transformation of $M(A-d^*I)M$, the eigenvalues of which determine the size of the test. For a non-trivial test, some eigenvalues of $M(A-d^*I)M$ will be positive, some will be negative, and Sylvester's law of inertia ensures that this is also true of the eigenvalues of $F'M(A-d^*I)MF$. Thus the limiting power of the DW (and ADW) test lies strictly between zero and unity.

This result is readily extended to the point optimal tests by using Theorem 1 and noting that the particular form of the A matrix is not relevant to Zeisel's argument.

An alternative derivation of the power of point optimal tests, which highlights computational issues, is possible by using the following diagonalisation of M. There exists an orthogonal matrix P such that

$$PMP' = \begin{bmatrix} I_{T-K} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad PP' = P'P = I.$$

Partition P as

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

where P_1 is $(T-K) \times T$ and P_2 is $K \times T$ and observe the following consequences:

$$P_1MP_1' = I_{T-K}$$

$$P_2MP_2' = 0$$

$$P_1P_2' = 0$$

The rows of P are eigenvectors of M, with the rows of P_1 corresponding to the unit eigenvalues and the rows of P_2 corresponding to the zero eigenvalues. It follows that $P_1M = P_1$ and $P_2M = 0$, while $P'P = I$ implies that $P_1'P_1 = I - P_2'P_2$. Post-multiplying by M gives

$$P_1'P_1 = M. \qquad (1)$$

The matrices X and M together span R^n and the row space of M is the orthogonal complement of the column space of X (Searle, (1982), p.226). This implies that the rows of P_2 are linearly dependent on the columns of X while the rows of P_1 are orthogonal to the columns of X, and we can write

$$P_1X = 0 \qquad (2)$$

$$P_2' = XG, \tag{3}$$

where G is T × T and nonsingular.

Following King (1980) we use the following result from Rao (1973, p.77).

Lemma 1: If V is (nxn) and positive definite and U and T are (nxk) and $\left(n\times(n-k)\right)$ matrices respectively, such that if W = (U:T) then $W'W = WW' = I_n$, then

$$V^{-1} - V^{-1}U(U'V^{-1}U)^{-1}U'V^{-1} = T(T'VT)^{-1}T'$$

Lemma 2:

$$d = u'(V^{-1}-V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1})u/u'Mu = u'P_1'(P_1VP_1')^{-1}P_1u/u'P_1'P_1u$$

Proof:

Apply Lemma 1 with $T = P_1'$ and $u = P_2'$ and use (1) and (3):

$$d = u'\left(V^{-1}-V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}\right)u/u'Mu$$

$$= u'\left(V^{-1}-V^{-1}P_2'(P_2'V^{-1}P_2)^{-1}P_2'V^{-1}\right)u/u'P_1'P_1u$$

$$= u'P_1'(P_1VP_1')^{-1}P_1u/u'P_1'P_1u.$$

We can now use the usual manipulations to see that the power of a point optimal test depends on the eigenvalues of

$$W^*(\rho) = L'(\rho)P_1'\left\{(P_1VP_1')^{-1} - d^*I\right\}P_1L(\rho). \tag{4}$$

Using (2) it is clear that if the regression has no intercept then $P_1L(1) \neq 0$ but $r\left(W^*(1)\right) = 1$, so the power is uniquely determined by the only non-zero eigenvalue.

When an intercept is present, $P_1F = P_1U$ and the limiting power of a point optimal test must lie strictly between zero and unity.

A further advantage can be gained from (4). Under H_0 the rejection probability depends on the eigenvalues of

$$P_1'\left\{(P_1VP_1')^{-1} - d^*I\right\}P_1$$

However, since $P_1P_1' = I$ these are the same as the eigenvalues of

$$(P_1VP_1')^{-1} - d^*I$$

or of

$$(VM)^{-1} - d^*I$$

This allows a relatively simple method of finding a 100α% critical value by solving for d* in

$$pr\left\{\sum_{j=1}^{T-K} \lambda_j Z_j^2 < d^*\right\} = \alpha$$

where the λ_j are the reciprocals of the non-zero eigenvalues of VM and $Z_j^2 \sim \chi^2_{(1)}$.

Furthermore, once a point optimal test statistic has been calculated, an exact prob-value is easily obtained by this method. This could be included as an option in a computer package, as SHAZAM (White et al., 1990) does for the Durbin-Watson Test.

3. Conclusion

It has been shown that the well-known importance of including an intercept in the regression when using the Durbin-Watson test also applies to the point optimal tests for autocorrelation. In particular, the limiting power of the Berenblut-Webb test, which is LMPI as ρ approaches unity, can be zero if there is no intercept in the model.

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