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L_p-NORM CONSISTENCIES OF NONPARAMETRIC ESTIMATES OF REGRESSION, HETEROSKEDASTICITY AND VARIANCE OF REGRESSION ESTIMATE WHEN DISTRIBUTION OF REGRESSOR IS KNOWN

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WHEN DISTRIBUTION OF REGRESSOR IS KNOWN[1,2]

by

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ABSTRACT

When dealing with heteroskedastic models $Y = \mu(X) + \varepsilon$ in econometrics and other disciplines, situations often arise (especially with structural models) where the probability distribution of the ($\mathbb{R}^d$-valued) regressor vector X is known, but postulations about the functional form of the regression $\mu(x)$, the heteroskedasticity $\sigma^2(x) = \text{var}(\varepsilon | X=x)$ and the distribution of the disturbance term ε are made. These three postulations generally lead to misspecification of the models. This paper, based on a data set on (Y,X), considers nonparametric kernel estimators $\hat{\mu}(x)$, $\hat{\sigma}^2(x)$ and $\hat{V}(\hat{\mu}(x))$, respectively, of the regression $\mu(x)$, the heteroskedasticity $\sigma^2(x)$ and the asymptotic variances $V(\hat{\mu}(x))$ of the regression estimate $\hat{\mu}(x)$ for situations where only the probability distribution of X, say λ is known. For an arbitrary subset A in the interior of the support of λ and for $1 \le p < \infty$, we establish convergences to zero, as the data set gets large, of the L_p-norms $\|\hat{\mu}-\mu\|^p = \int_A E|\hat{\mu}(x)-\mu(x)|^p d\lambda(x)$ (with $A \equiv \mathbb{R}^d$ for p = 1), $\|\hat{\sigma}^2-\sigma^2\|^p$ (with $A \equiv \mathbb{R}^d$ for p = 1) and $\|\hat{V}(\hat{\mu})-V(\hat{\mu})\|^p$ under certain moment conditions on Y but with no assumptions on the joint distribution of (Y,X) or the continuity of $\mu(x)$, $\sigma^2(x)$ or the density of X.

1. INTRODUCTION

Let (Y,X) be a $\mathbb{R}^1 \times \mathbb{R}^d$ - valued random vector defined on a probability space $(\Omega,\mathbb{B},P)$. Then the general heteroskedastic regression model is given by $Y = \mu(X) + \varepsilon$. A measurement on response Y at a value x of the regressor X is the value of the unknown regression $\mu(x)$ at $X = x$ contaminated with an unobservable disturbance term ε, which is random on $(\Omega,\mathbb{B},P)$ with conditional mean $E(\varepsilon|X) = 0$ and unknown conditional variance $Var(\varepsilon|X) = \sigma^2(X)$, called the heteroskedasticity of the regression model. Estimations of the regression function $\mu(x) = E(Y|X=x)$ and the heteroskedasticity function $\sigma^2(x) = var(Y|X=x)$ are invariably handled in various sciences by postulating a certain fixed model (functional form) for the regression, by assuming a constant conditional variance, $\sigma^2(x) \equiv \sigma^2$ (the homoskedasticity), for the response variable Y at any value x of the regressor X and by assuming a knowledge of the functional form of the distribution of the disturbance term ε. However any postulations regarding the functional form of the regression $\mu(x)$, the heteroskedasticity $\sigma^2(x)$ or the distributions of disturbances ε are all questionable, and often lead to misspecifications of the models, which thereby lead to serious impact on decisions and plannings.

This problem of misspecification of the models, however, can be avoided by assuming no specific parametric form for the regression or heteroskedasticity function; and by estimating them completely nonparametrically.

Whereas a vast literature on nonparametric estimation of regression μ is at hand, (for example, Watson (1964), Rosenblatt (1969), Schuster (1972), Schuster and Yakowitz (1979), Noda (1976), Greblicki and Krzyzak (1980), Härdle (1984), among others, discussed pointwise consistency; and

Nadarya (1964, 1970), Deheuvels (1974), Hall (1981), Mack and Silverman (1987), Härdle and Luckhans (1984) Singh and Ahmad (1987), among others discussed uniform consistency of nonparametric kernal estimators of regression), nonparametric estimation of heteroskedasticity function $\sigma^2(x)$ has drawn little attention. Hildreth and Houck (1968), Fraehlich (1973), Box and Hill (1974), Jobson and Fuller (1980) and Judge et al (1988) adopted a sort of parametric approach in which they assumed that $\sigma^2(x)$ has a known functional form involving a finite number of unknown parameters. Fuller and Rao (1978), White (1980), Carroll (1982) and MacKinnon and White (1985) took a somewhat semi-non-parametric approach in which they assumed that the regression function $\mu(x)$ has a known functional form involving a finite number of unknown parameters, estimated these parameters and then used the residuals and the estimated variance-covariance of the parameter estimates to estimate $\sigma^2(x)$ nonparametrically. In the latter works, the assumed functional form of $\mu(x)$ is a linear regression. Müller and Stadtmüller (1987) considered estimation of $\sigma^2(x)$ with nonstochastic ordered regressors assuming, among others, that the density of Y and $\sigma^2(x)$ satisfy certain order of Lipschitz condition. Singh and Ullah (1985) and Singh et al. (1987) discussed estimation of $\sigma^2(x)$ under the continuity of $\sigma^2(x)$, the density of X and the joint density of (Y,X) at x.

In most of the literature on the nonparametric estimation of regression or heteroscedasticity (e.g. the references cited in the preceding paragraph), it is invariably assumed, particularly when discussing consistencies like weak, strong, mean square or asymptotic normality that the regression $\mu(x)$ and heteroskedasticity $\sigma^2(x)$ (in case of its estimation) are continuous, and the joint density of (Y,X), the density of Y and the density of X (and hence the conditional density of Y given X = x) not only exist but are also continuous. Recently Singh (1989) considered kernel estimations of μ and σ^2 under no condition on the distribution of (Y,X) other than that the distribution function of X is absolutely continuous w.r.t. the Lebsegue measure on $\mathbb{R}^d$ so that the probability density function (pdf) of X exists. He established the weak and strong consistencies as well as the asymptotic normality of the estimators $\mu(x)$ and $\sigma^2(x)$ only under some appropriate moment conditions on Y. Thus for his results the joint density of (Y,X), the density of Y or the conditional density of Y given X need not exist and $\mu(x)$, $\sigma^2(x)$ or the density of X need not be continuous at x. Under similar weaker conditions he has also established weak and strong consistencies of the estimates of the variances of his regression and heteroskedasticity estimators by establishing similar properties for the estimators of a general function, namely $\mu_r(x) = E(Y^r|X=x)$ for $r \geq 0$.

Greblicki and Krzyzak (1980), Johnston (1982) and Härdle (1986), among others considered Watson-Nadaraya type kernel estimators of regression when the pdf of the regressor X, say ϕ, is known. Under the assumption that $\phi(x)$ and $\mu(x)$ are continuous and $EY^2 < \infty$, pointwise weak consistency of such estimators $\hat{\mu}(x)$ is established by Greblicki and Kryzyzak, while Johnston and Härdle obtained results, respectively, on the asymptotic distribution of the maximal deviation of $\hat{\mu}(x)$ from $\mu(x)$ and an $L_2(0,1)$-norm $\int_0^1 E|\hat{\mu}(x)-\mu(x)|d\lambda(x)$, where λ is the probability measure of X. Considering stituations where the distributions of the regressor vectors are known is not quite unrealistic. For example, in structural regression models in econometrics quite often the regression vector is consisted of endogenous variables generated from some known functions of exogenous variables plus independent multivariate normal errors with mean zero and known variances.

In this note we further look at the situation considered in the preceding paragraph, i.e. when the pdf of X is known. We estimate the regression as well as the heteroskedasticity and the variance of the regression estimator, and examine the L_p-norm ($p \geq 1$) consistencies of these estimators. Weak and strong pointwise consistencies of these estimators follow from those established in Singh (1989) under weaker conditions stated in preceding paragraph. However, note that weak or strong consistency of an estimator does not imply L_p-norm consistency. Without any continuity assumption on $\mu(\cdot)$, $\sigma^2(\cdot)$ or the p.d.f. $\phi(\cdot)$ of X, and without making any assumption on the distribution of (Y,X), the distribution of Y or the conditional distribution of Y given X, we establish the convergence to zero, as the sample size gets large, of the L_p-norm distances, $1 \leq p < \infty$, of the estimators from theirrespective true values.

In Section 2, we introduce some notations, give a proper definition of "L_p-norm convergence" considered here and define estimators $\hat{\mu}_r(x)$ of the regression function $\mu_r(x) = (EY^r|X=x)$ of Y^r on X for $r \geq 0$, estimators $\hat{\sigma}^2(x)$ of the heteroskedasticity $\sigma^2(x)$ and estimators $\hat{V}(\hat{\mu}(x))$ of the asymptotic variance of the regression estimate $\hat{\mu}(x) = \hat{\mu}_1(x)$. In Section 3 we establish L_p-norm consistencies of the estimator $\hat{\mu}_r$ (Theorem 3.1), $\hat{\sigma}^2$ (Theorem 3.2) and $\hat{V}(\hat{\mu})$ (Theorem 3.3). The paper is concluded with a few remarks in Section 4.

2. NOTATIONS, DEFINITIONS AND ESTIMATORS

Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space with the usual norm $\|z\| = (z'z)^{1/2} = (\Sigma_1^d z_i^2)^{1/2}$ for $z = (z_1,\ldots,z_d)' \in \mathbb{R}^d$. Let (Y,X), (Y_1,X_1), ..., (Y_n,X_n) be independent identically distributed $\mathbb{R}^1 \times \mathbb{R}^d$ - valued random vectors defined on a common probability space $(\Omega,\mathbb{B},P)$. We assume that the regression $\mu(x) = E(Y|X)$ and the heteroskedasticity $\sigma^2(x) = var(Y|X)$ are properly defined and exist a.e.(λ), where λ is the probability measure of X. We assume nothing about the joint distribution of (Y,X), the distribution of Y or the conditional distribution of Y given X. We do however assume that the probability measure λ of X is known and is dominated by the Lebesgue measure on $\mathbb{R}^d$ so that the probability density, say ϕ, of X exists and is known. Thus the joint density of (Y,X), the density of Y or the conditional density of Y given X need not exist and the functions $\mu(x)$, $\sigma^2(x)$ or $\phi(x)$ need not be continuous. When ϕ is unknown but the response variable Y is bounded w.p.1, estimators of the regression and heteroskedasticity functions, which are some retracted versions of those considered in Singh (1989), have been recently studied by Singh and Giles (1991) and L_1-norm consistencies (which also imply L_p-norm consistencies for $p \geq 1$ in their case) have been established.

All real-valued functions on $\mathbb{R}^d$ in this paper carrying an argument x (explicitly or implicitly) are only defined a.e. λ, i.e. on the support of the pdf ϕ. Also all convergences of sequences (in n) of functions carrying an argument x (explicitly or implicitly) are only a.e. λ.

For $r \geq 0$, define

$$\mu_r(x) = E(Y^r|X=x)$$

(so that $\mu_1(x) = \mu(x)$). In the heteroscedastic model $Y = \mu(X) + \varepsilon$, with

$E(\varepsilon|X) = 0$, *the regression function is* $\mu(x) = E(Y|X=x) = (\mu_1(x)$, according to our notation) and the heteroskedasticity function is

$$\sigma^2(x) = \mathrm{var}(\varepsilon|X=x) = E(\varepsilon^2|X=x),$$

which can be expressed as $\sigma^2(x) = \mu_2(x) - \mu_1^2(x)$.

Define $\psi_r = \mu_r\phi$ so that

$$\mu_r(x) = \frac{\psi_r(x)}{\phi(x)}. \tag{2.1}$$

Notice that $\psi_0 = \phi$.

Let K be an arbitrary Borel-measurable function on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} K(u)du = 1$. Let $\{h_n\}$ be a sequence of positive real numbers such that $h_n \to 0$ as $n \to \infty$. Define

$$\hat{\psi}_r(x) = (nh_n^d)^{-1}\Sigma_{j=1}^n Y_j^r K\left(\frac{X_j - x}{h_n}\right), \tag{2.2}$$

and

$$\hat{\phi}(x) = \hat{\psi}_0(x).$$

Throughout this paper, the notation $\{a\}_B$ will stand for $-B$, a or B depending on whether $a < -B$, $|a| \le B$ or $a > B$.

In view of our knowledge of ϕ, nonparametric kernel estimate of $\mu_r(x)$ is

$$\hat{\mu}_r(x) = \frac{\hat{\psi}_r(x)}{\phi(x)} \tag{2.3}$$

For estimation of σ^2, we assume that $|E(Y|X)| \le B$ w.p.1 for some $B < \infty$. We estimate $\sigma^2(x)$ by

$$\hat{\sigma}^2(x) = \hat{\mu}_2(x) - (\tilde{\mu}(x))^2, \text{ where } \tilde{\mu} = \{\hat{\mu}_1\}_B . \tag{2.4}$$

Estimators $\tilde{\theta}_r(x) = (\hat{\psi}_r(x)/\hat{\phi}(x))$ *for* $\mu_r(x)$ and $\tilde{\delta}^2(x) = \{\tilde{\theta}_2(x)-(\tilde{\theta}_1(x))^2\}$ for $\sigma^2(x)$ are considered in Singh (1989). Only under some moment conditions on Y and *the existence of* ϕ, *it has been established there that* the asymptotic variance of the regression estimate $\tilde{\theta}_1(x) = \tilde{\theta}(x)$, up to the order $o\left((nh_n^d)^{-1}\right)$, is given by

$$V(\tilde{\theta}(x)) = (nh_n^d)^{-1} \frac{\sigma^2(x)\int K^2}{\phi(x)} , \tag{2.5}$$

$$\text{where } \int K^2 = \int_{\mathbb{R}^d} K^2(x)dx ,$$

(see also Singh and Ullah (1985), Singh et al (1987). It follows from the same arguments that *the asymptotic variances of the regression estimates* $\tilde{\mu}(x)$ and $\hat{\mu}(x) = \hat{\mu}_1(x)$ are the same as given in (2.5). Thus we estimate $V(\hat{\mu}(x))$ by

$$\hat{V}(\hat{\mu}(x)) = (nh_n^d)^{-1} \frac{\hat{\sigma}^2(x)\int K^2}{\phi(x)} \tag{2.6}$$

where $\hat{\sigma}^2(x)$ is as given in (2.4).

Notice that $\tilde{\theta}_1(x) = \Sigma_{j=1}^n Y_j W_{nj}(x)$ and $\tilde{\theta}_2(x) = \Sigma_{j=1}^n Y_j^2 W_{nj}(x)$, where $W_{nj}(x) = \{K((X_j-x)/h_n)\}[\Sigma_{j=1}^n K((X_j-x)/h_n)]^{-1}$, are well known Nadaraya-Watson type kernel estimators of the regressions of Y and Y^2 on X, respectively. Therefore $\tilde{e}_j(x) = (Y_j - \tilde{\theta}_1(x))$ are the nonparametric kernel estimators of the residuals $e_j(x)$ and $\tilde{\delta}^2(x) = (\tilde{\theta}_2(x)-(\tilde{\theta}_1 x))^2 = \Sigma_{j=1}^n (Y_j-\tilde{\theta}_1(x))^2 W_{nj}(x)$ is the Nadaraya-Watson type kernel estimator of the regression of the square of the residuals $(Y_j-\tilde{\theta}_1(x))$ on X, which is a natural way to estimate $\sigma^2(x) = E(\varepsilon^2|X=x) = var(\varepsilon|X=x)$.

Throughout this paper, let

$$A \subseteq \mathbb{R}^d \text{ such that ess-inf(w.r.t.}\lambda)_{x\in A}\phi(x) > 0. \tag{2.7}$$

Definition:

For a real valued statistic $\theta^*(x) = \theta^*((Y_1,X_1),\dots,(Y_n,X_n);x)$ defined for $x \in \mathbb{R}^d$ and for $p \geq 1$, we say θ^* is L_p-norm consistent estimator of a real valued function $\theta(x)$ on $\mathbb{R}^d$ if

$$(2.8) \qquad \|\theta^*-\theta\|^p = \int_A E|\theta^*(x)-\theta(x)|^p d\lambda(x) \to 0 \text{ as } n \to \infty.$$

We remark that L_p-norm convergence of the type (2.8) over a subset A of $\mathbb{R}^d$ satisfying condition similar to (2.7), has been used as a standard by several authors for examining a global consistency of a regression estimate, see Section 4, for example.

For a real valued function g on $\mathbb{R}^d$ we say $g \in L_p$ if $\int_{\mathbb{R}^d} |g(u)|^p du < \infty$, and for a measure ν on $\mathbb{R}^d$, we say $g \in L_p(\nu))$ if $\int_{\mathbb{R}^d} |g(\cdot)|^p d\nu(\cdot) < \infty$. For the sake of simplicity in writing, we denote, whenever convenient, $\int_{\mathbb{R}^d} g(u)du$ by $\int g$ and $\int g(\cdot)d\nu(\cdot)$ by $\int g d\nu$. Unless stated otherwise, all integrals throughout this paper are taken over the space $\mathbb{R}^d$. For $t \in \mathbb{R}^1$, let $K^*(t) = \text{ess-sup}_{\|u\|>t}|K(u)|$, where the ess-sup is taken with respect to the Lebergue measure on $\mathbb{R}^d$. (Note that $K^* \in L_p$ whenever $|K(\cdot)|$ is nonincreasing in $\|\cdot\|$ and belongs to L_p). Throughout this paper it is assumed that K and K^* belong to $L_1 \cap L_2$, and no further indication of it will be made in the paper. Unless stated otherwise, all convergences are w.r.t. $n \to \infty$.

3. THE L_p-NORM CONVERGENCES OF ESTIMATORS

In this section we establish three theorems. Theorem 3.1 establishes the L_p-norm convergence of $\hat{\mu}_r$ under certain moment conditions on Y. The statistics $\hat{\mu}_r$ are the kernel estimators of μ_r, the regression of Y^r on X for $r \geq 0$, though only the particular cases of our interest here are with r = 1 and 2. The L_p-norm convergences of $\hat{\sigma}^2$ and $\hat{V}(\hat{\mu})$ are established in Theorems 3.2 and 3.3 respectively.

Theorem 3.1. *Let* $h_n \to 0$, $r \geq 0$ *and* $1 \leq p < \infty$. *Then*

(3.1) $$\|\hat{\mu}_r - \mu_r\|^p = o(1) \quad \text{with A, in (2.6), } \equiv \mathbb{R}^d \text{ for } p = 1,$$

provided for $1 \leq p \leq 2$, $E|Y|^{2r} < \infty$ *and* $nh_n^d \to \infty$; *and for* $p > 2$, *there exists an* $0 < \eta < 2$ *such that with* $w = 2(p-\eta)/(2-\eta)$, $E|Y|^{rw} < \infty$, $K \in L_w$ *and* $n^{\eta/2}h_n^{d(p-1)} \to \infty$.

Theorem 3.2. *Let* $1 \leq p < \infty$, *and the conditions of Theorem* 3.1 *hold for* r = 1 *and* 2. *Further let there exist a finite constant* B *such that* $|E(Y|X)| \leq B$ w.p.1, *and* $\hat{\sigma}^2$ *be defined as in* (2.4). Then

(3.2) $$\|\hat{\sigma}^2 - \sigma^2\|^p = o(1) \text{ with A, in (2.6), } \equiv \mathbb{R}^d \text{ for } p = 1.$$

Theorem 3.3. *Let* $1 \leq p < \infty$, *and the conditions for Theorem* 3.2 *hold. Let* $\hat{V}(\hat{\mu}(x))$ *be defined as in* (2.6) *and* $V(\hat{\mu}(x))$ *stand for the asymptotic variance of* $\hat{\mu}(x)$. *Then*

(3.3) $$(nh_n^d)^p \|\hat{V}(\hat{\mu}) - V(\hat{\mu})\|^p = o(1).$$

To prove Theorems 3.1, 3.2 and 3.3 we establish first the following lemmas.

Lemma 3.1 *Let* $h_n \to 0$, $1 \le p < \infty$ *and* $r \ge 0$. *If* $E|Y|^{rp} < \infty$, *then*

(3.4) $$\int \phi^{-p}(x)|E\hat{\psi}_r(x)-\psi_r(x)|^p d\lambda(x) = o(1)$$

Proof.

Our proof involves the following steps. We first prove that $|E\hat{\psi}_r(x)-\psi_r(x)| = o(1)$ a.e. $\lambda(x)$. Then we show that the sequence (in n) of functions $\phi^{-p+1}(x)|E\hat{\psi}_r(x)-\psi_r(x)|^p \le g_n(x)$ a.e. $\lambda(x)$ for some sequence of functions $g_n(x)$ which converges to $g(x)$ as $n \to \infty$ a.e. $\lambda(x)$. We then show that $g \in L_1$ and then we complete the proof by using generalized denominated convergence theorem.

Since $\{(Y_1,X_1),\ldots,(Y_n,X_n)\}$ is a random sample on (Y,X),

(3.5) $$E\hat{\psi}_r(x) = h_n^{-d} \int K\left(\frac{u-x}{h_n}\right) \psi_r(u)du$$

Next note that $\int|\psi_r(u)|du = E|Y|^r < \infty$. Therefore, $\psi_r \in L_1$ and hence almost all points $u \in \mathbb{R}^d$ are in the Lebsegne set of ψ_r (see, Natansan (1955), pp255-266 or Wheeden and Zygmund (1977), pp.100-109). Hence, since $K^* \in L_1$, by Theorem 1.25 of Stein and Weiss (1975), p.13,

(3.6) $$\lim_{n\to\infty} h_n^{-d} \int K\left(\frac{u-x}{h_n}\right) \psi_r(u)du = \psi_r(x)\int K \quad \text{a.e. in } x \in \mathbb{R}^d$$

Hence since $\int K = 1$, we conclude from (3.5) and (3.6) that $|E\psi_r(x)-\psi_r(x)| \to$ o a.e. $\lambda(x)$.

Next notice that

$$\phi^{-p+1}(x)|E\hat{\psi}_r(x)-\psi_r(x)|^p \le 2^{p-1}\phi^{-p+1}(x)\left\{|E\hat{\psi}_r(x)|^p+|\psi_r(x)|^p\right\}$$

$$\le 2^{p-1}\phi^{-p+1}(x)\left\{|h_n^{-d}\int K\left(\frac{u-x}{h_n}\right)\psi_r(u)du|^p\right.$$

$$\left.+ |\psi_r(x)|^p\right\} = g_n(x), \text{ say.}$$

Notice that from (3.6), $g_n(x) \to 2^p\ \phi^{-p+1}(x)|\psi_r(x)|^p = g(x)$, say, a.e. in $x \in \mathbb{R}^d$. But $g \in L_1$ since $\int\phi^{-p+1}(x)|\psi_r(x)|^p dx \le E|Y|^{rp} < \infty$, which completes the proof of the Lemma. □

Lemma 3.2. *Let* $r \ge 0$ *and* $1 \le p \le \infty$. *If for* $1 \le p \le 2$, $E|Y|^{2r} < \infty$ and $nh_n^d \to \infty$, *then*

(3.7) $$\int_A \phi^{-p}(x)E|\hat{\psi}_r(x)-E\hat{\psi}_r(x)|^p d\lambda(x) = o(1) \quad \textit{with } A \equiv \mathbb{R}^d \textit{ for } p = 1.$$

Further (3.7) *holds for* $p > 2$ *if there exists an* $0 < \eta < 2$ *such that, with* $w = 2(p-\eta)/(2-\eta)$, $E|Y|^{rw} < \infty$, $K \in L_w$ and $n^{\eta/2}h_n^{d(p-1)} \to \infty$.

Proof.

First consider the case $1 \le p \le 2$. By Hölder's inequality $E|\hat{\psi}_r(x)-E\hat{\psi}_r(x)|^p \le (\text{var}\hat{\psi}_r(x))^{p/2}$. But since (Y_1,X_1), ..., (Y_n,X_n) are i.i.d.,

$$\text{var } \hat{\psi}_r(x) \le \left(nh_n^{2d}\right)^{-1} E\left(Y_1^r K\left(\frac{X_1-x}{h_n}\right)\right)^2$$

Therefore

(3.8) $$\left(nh_n^d\right)^{p/2} E|\hat{\psi}_r(x) - E\hat{\psi}_r(x)|^p \le \left(h_n^{-d}\int K^2\left(\frac{u-x}{h_n}\right)\psi_{2r}(u)du\right)^{p/2}$$

The r.h.s. of (3.8) converges to $(\psi_{2r}(x)\int K^2)^{p/2}$ a.e. λ, since $\psi_{2r} \in L_1$ by virtue of the fact that $E|Y|^{2r} < \infty$ and the arguments used to prove the convergence of the rhs (3.5) to $\psi_r(x)\int K$ can be applied. Hence, we conclude that if $nh_n^d \to \infty$, then

$$\text{(3.9)} \qquad \phi^{-p+1}(x)E|\hat{\psi}_r(x) - E\hat{\psi}_r(x)|^p \to o$$

Now for $p = 1$, the l.h.s. of (3.9) is bounded by $2(nh_n^d)^{-1}\Sigma_{j=1}^n E|Y_j^r K((X_j-x)/h_n)| = 2h_n^{-d} \int |K(u-x)/h_n|\,|\Psi_r(u)|du = g_n(x)$, say, where $\Psi_r(x) = E(|Y|^r|X=x)\phi(x)$. Again, by the arguments given earlier, $g_n(x) \to g(x) = 2\Psi_r(x)\int|k|$ which belongs to L_1 since $E|Y|^r < \infty$. Hence (3.9) followed by the generalized dominated convergence theorem gives (3.7) for $p = 1$.

For $1 < p \le 2$, the l.h.s. of (3.7) from (3.8) is bounded by

$$(nh_n^d)^{-p/2} \int_A \phi^{-p}(x)\left(\int K^2(u)\psi_{2r}(x+h_n u)du\right)^{p/2} d\lambda(x)$$

$$\le \left(nh_n^d\right)^{-p/2} \left(\alpha(A)\right)^{-p/2} \left(\iint K^2(u)\psi_{2r}(x+h_n u)dudx\right)^{p/2}$$

$$= \left(nh_n^d\right)^{-p/2} \left(\alpha(A)\right)^{-p/2}\left(\left(\int K^2\right) E|Y|^{2r}\right)^{p/2}$$

by Hölder's inequality. The proof of (3.7) for $1 < p \le 2$ is now complete since $nh_n^d \to \infty$.

Now we prove (3.11) for $p > 2$. Let $0 < \eta < 2$ and $w = 2(p-\eta)/(2-\eta)$. By Hölder's inequality,

$$\text{(3.10)} \qquad E|\hat{\psi}_r(x) - E\hat{\psi}_r(x)|^p \le \left(E|\hat{\psi}_r(x)-E\hat{\psi}_r(x)|^w\right)^{(2-\eta)/2}\left(\text{var}(\hat{\psi}_r(x)\right)^{\eta/2}$$

Now, since $w > 1$, $E|\hat{\psi}_r(x) - E\hat{\psi}_r(x)|^w$ is bounded by $2^w h_r^{-dw} E|Y_1^r K((X_1-x)/h_n)|^w$ $= 2^w h_n^{-d(w-1)}$ times $\int |K(u)|^w \Psi_{rw}(x+h_n u)du$ and $\mathrm{var}(\hat{\psi}_r(x))$ is bounded by $(nh_n^d)^{-1} E|Y_1^r K(X_1-x)/h_n|^2 = (nh_n^d)^{-1} \int K^2(u)\Psi_{2r}(x+h_n u)du$. Hence, from (3.10) and by Hölder inequality, the l.h.s. of (3.7) for $p > 2$ is bounded by $2^w h_n^{-d(w-1)(2-\eta)/2}(nh_n^d)^{-\eta/2} = 2^w n^{-\eta/2} h_n^{-d(p-1)}$ times

$$\int_A \phi^{-p}(x)\left\{\int |K(u)|^w \Psi_{rw}(x+h_n u)du\right\}^{(2-\eta)/2}\left\{\int |K(u)|^2 \Psi_{2r}(x+h_n u)du\right\}^{\eta/2} d\lambda(x),$$

which, again by Hölder Inequality, is bounded by $c_n . c_n'$, where

$$c_n = \left\{ \int_A \phi^{-p+1}(x)\int |K(u)|^w \Psi_{rw}(x+h_n u)dudx\right\}^{(2-\eta)/2}$$

$$\le \left((\alpha(A))^{-(p-1)} E|Y|^{rw}\int |K|^w\right)^{(2-\eta)/2}$$

and

$$c_n' = \left\{ \int_A \phi^{-p+1}(x)\int |K(u)|^2 \Psi_{2r}(x+h_n u)dudx\right\}^{\eta/2}$$

$$\le \left((\alpha(A))^{-(p-1)} E|Y|^{2r}\int |K|^2\right)^{\eta/2}.$$

Hence we conclude that the l.h.s. of (3.7) for $p > 2$ is $O(n^{-\eta/2} h_n^{-d(p-1)})$ under the moment condition in Y stated there. Now the proof of the Lemma is complete. □

Proof of Theorem 3.1

It can be easily seen that

$$|\hat{\mu}_r(x) - \mu_r(x)|^p \le 2^{p-1}\{|\hat{\mu}_r(x)-\mu_r(x)|^p + |\hat{\mu}_r(x)-E\hat{\mu}_r(x)|^p\}$$

$$= 2^{p-1}\phi^{-p}(x)\{|E\hat{\psi}_r(x)-\psi_r(x)|^p + |\hat{\psi}_r(x)-E\hat{\psi}_r(x)|^p\}$$

from (2.1) and (2.3). Therefore

$$\|\hat{\mu}_r-\mu_r\|^p \le 2^{p-1}\left\{ \int_A \phi^{-p}(x)|E\hat{\psi}_r(x)-\psi_r(x)|^p d\lambda(x) \right.$$

$$\left. + \int_A \phi^{-p}(x)E|\hat{\psi}_r(x) - E\hat{\psi}_r(x)|^p d\lambda(x)\right\}.$$

The proof is complete from Lemmas 3.1 and 3.2. □

Proof of Theorem 3.2

Note that

$$|\hat{\sigma}^2(x) - \sigma^2(x)|^p \le |(\hat{\mu}_2(x)-\mu_2(x)) - (\tilde{\mu}^2(x)-\mu^2(x))|^p$$

$$\le 2^{p-1}\left\{|\hat{\mu}_2(x)-\mu_2(x)|^p + (2B)^p\ (|\hat{\mu}(x)-\mu(x)|\wedge B)^p\right\}$$

since $|E(Y|X)| = |\mu(X)| \le B$ w.p.1 and by definition, $\tilde{\mu}$ is the retraction of $\hat{\mu}_1$ to the interval $[-B,B]$. Thus proof for (3.2) follows from (3.1) applied with $r = 1$ and 2. □

Proof of Theorem 3.3

To prove (3.3), recall from Section 2 that the asymptotic variance of $\hat{\mu}(x) = \hat{\mu}_1(x)$, up to under $o(nh_n^d)^{-1}$ is

$$V(\hat{\mu}(x)) = (nh_n^d)^{-1}\frac{\sigma^2(x)\int K^2}{\phi(x)}.$$

Therefore, from (2.5)

$$(nh_n^d/\textstyle\int K^2)|\hat{V}(\hat{\mu}(x)) - V(\hat{\mu}(x))| \le \phi^{-1}(x)|\hat{\sigma}^2(x)-\sigma^2(x)|$$

$$\le (\alpha(A))^{-1}|\hat{\sigma}^2(x)-\sigma^2(x)|$$

Hence proof for (3.3) follows from (3.2). □

4. CONCLUDING REMARKS

Note that for p = 1 the integrals in (3.1) and (3.2) are over the whole space $\mathbb{R}^d$, and for p > 1, the set A in the definition (2.8) could be any arbitrary subset of the support of ϕ satisfying the restriction (2.7). We remark that Stone (1982), in consideration of "in probability convergence" of $\int_A |\theta^*(x) - \theta(x)|^p dx$ for a class of regression estimators θ^* has used the same restriction (2.7) on A, and Härdle (1984) in consideration of some results on mean integrated squared error (i.e. the case of p=2) of regression estimators $\hat{\mu} = \hat{\mu}_1$ and $\mu^* = (\hat{\psi}_1/\hat{\phi})$ have taken A = $(0,1)^d$ with the same restriction (2.7) on A. In consideration of uniform (over A) weak and strong convergences of $\mu = (\hat{\psi}_1/\hat{\phi})$, A is invariably taken as a compact subset of the support of ϕ with the restriction (2.7), (e.g. see Nadaraya (1964, 1965, 1970), Schuster and Yakowitz (1979), Mack and SIlverman (1981), among others). The condition (2.7) on A is also imposed in Singh and Ahmad (1987) in consideration of uniform (over A) mean square consistency of their regression estimator $\mu_* = (\hat{\psi}_1/\hat{\phi})_b$, where b is an w.p.1 bound for $|Y|$.

If ϕ is known, it makes sense to consider Nadaraya-Watson type estimator $\hat{\mu} = \hat{\mu}_1$, utilizing the knowledge of ϕ. Greblicki and Krzyzak (1980), Johnston (1982) and Härdle (1986) have considered such estimators as well. Under the assumptions that ϕ and μ are continous at x and $E(Y^2) < \infty$, Greblicki and Kryyzak established weak consistency of $\hat{\mu}$ at x and Johnston examined the asymptotic distribution of the maximal deviation of $\hat{\mu}(x)$ from $\mu(x)$ while Härdle established L_2-norm consistency of $\hat{\mu}$ with A = (0,1).

As indicated earlier, estimations of heteroskedasticity or of asymptotic variance of the regression estimates under weaker conditions have drawn little attention in the literature, and certainly none of the works mentioned in the preceding two paragraphs have discussed such estimations. Recently Singh (1989) considered estimators $\tilde{\theta}_r(x) = (\hat{\psi}_r(x)/\hat{\phi}(x))$ for $r \geq 0$ and $\tilde{\delta}^2(x) = \{\tilde{\theta}_2^2(x)-(\tilde{\theta}_1(x))^2\}$ respectively for the rth order regression $\mu_r(x) = E(Y^r|X=x)$ and heteroskedasticity $\sigma^2(x)$, and established weak and strong consistencies, as well as the asymptotic normality of $\tilde{\theta}_r$ and $\tilde{\delta}^2$ only under certain moment conditions on Y. Weak and strong consistencies of the estimator $\tilde{V}(\tilde{\theta}_1,(x)) = (nh_n^d)^{-1}(\tilde{\delta}^2(x)/\hat{\phi}(x))\int K^2$ of the asymptotic variance $V(\tilde{\theta}_1(x)) = (nh_n^d)^{-1}(\sigma^2(x)/\phi(x))\int K^2$ of the regression estimate $\tilde{\theta}_1(x)$ follow inplicitly from his similar results on $\tilde{\theta}_r$ and $\tilde{\delta}^2$. We wish to point out that the weak and strong consistencies of $\hat{\mu}_r$, $\hat{\sigma}^2$ and $\hat{V}(\hat{\mu})$ also follow by the same arguments. However, we note that weak and strong consistencies do not imply L_p-norm consistency for any $p \geq 1$. In this paper we have established, for $1 \leq p < \infty$, the L_p-norm consistencies of $\hat{\mu}_r$, $\hat{\sigma}^2$ and $\hat{V}(\hat{\mu})$, again only under certain moment conditions on Y.

Singh and Giles (1991) considered regression estimator $\mu_* = \{\hat{\psi}_1/\hat{\phi}\}_b$ and heteroskedasticity estimator $\sigma_*^2 = (\{\hat{\psi}_2/\hat{\phi}\}_{b^2} - \mu_*^2)$ under the condition that $|Y|$ is bounded w.p. 1 by b, and established the L_1-norm consistencies (and hence L_p-norm consistencies for any $p \geq 1$, since $(\mu_*-\mu| \leq b$ and $|\sigma_*^2-\sigma^2| \leq b^2$). In our estimations of the regression, heteroskedasticity and the asymptotic variance of the regression estimate though ϕ is assumed to be known but Y need not be bounded.

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