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AN EXPOSITORY NOTE ON THE COMPOSITE COMMODITY THEOREM

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This note offers an alternative derivation of the composite commodity theorem using only elementary economic and mathematical tools. It offers some insight as to why constancy of relative prices induces separability in the consumers optimisation problem. It should be more readily accessible to student in upper level microeconomics courses and prove useful in the classroom in presenting this central theorem of economic analysis.

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An Expository Note on the Composite Commodity Theorem

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The composite commodity theorem is one of the cornerstones of economic analysis, justifying the frequent recourse to two-dimensional diagrams in analysing consumer demand. Conventional proofs of the theorem (eg. Hicks (1946), Samuelson (1947), Green (1976)) rely on aggregation properties of the substitution matrix together with an appeal to integrability of demand functions. This approach is technically difficult and offers little insight to the student.

Some texts (e.g. Cowell (1986), Deaton & Muelbauer (1980)) offer a more elegant proof by utilising the duality between expenditure and utility functions, showing that two utility maximisation problems give rise to the same expenditure function and hence must represent the same underlying preference ordering. This provides a good demonstration of the economy of duality theory but offers the student little insight into the nature of the composite commodity.

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[1] The author gratefully acknowledges the useful comments made by John Fountain, Leslie Young and Peyton Young.

In this note, we provide an intuitive demonstration of the composite commodity theorem by first showing how the result seems compelling in a special case in which utility is separable. We then show that the assumption of separability was innocent, and that the generalisation is trivial. Not only is this presentation more insightful for students, it depicts the composite commodity theorem as its true role, namely as a separability theorem.

Consider a consumer with preferences defined over three goods. Assume initially that the consumer's preferences are additively separable, i.e. that her preferences can be represented by the following utility function

$$U(x_1,x_2,x_3) = u(x_1) + w(x_2,x_3)$$

where u and w are strictly concave[2].

In these circumstances, it seems intuitive that the consumer's maximisation problem

$\mathbb{P}0$: $$\max_{x} U(x_1,x_2,x_3) \text{ s.t. } px = M$$

can be decomposed into two sub-problems

$\mathbb{P}1$: $$\max_{x_1,\bar{x}} u(x_1) + V(\bar{x}) \text{ s.t. } p_1x_1 + \bar{x} = M$$

and

$\mathbb{P}2$: $$\max_{x_2,x_3} w(x_2,x_3) \text{ s.t. } p_2x_2 + p_3x_3 = \bar{x}$$

where $V(\bar{x})$ is the maximum value function for $\mathbb{P}2$.

[2] The assumption of concavity will also be relaxed in the sequel.

To demonstrate this formally, let $x^* = (x_1^*, x_2^*, x_3^*)$ be the optimal solution to P1 and $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ be the optimal solution to P1 and P2 simultaneously. We show that $x^* = \hat{x}$. First note that the x^* must satisfy the first order conditions

$$\frac{\partial u(x^*)}{\partial x_1} = \lambda p_1$$

$$\frac{\partial w(x^*)}{\partial x_2} = \lambda p_2 \qquad (1)$$

$$\frac{\partial w(x^*)}{\partial x_3} = \lambda p_3$$

$$px = M$$

$\hat{x}$ must satisfy the first order conditions for P1

$$\frac{\partial u(\hat{x})}{\partial x_1} = \mu p_1$$

$$\frac{\partial V(\hat{x})}{\partial \bar{x}} = \mu \qquad (2)$$

$$p_1 \hat{x}_1 + \hat{\bar{x}} = M$$

and P2

$$\frac{\partial w(\hat{x})}{\partial x_2} = \delta p_2$$

$$\frac{\partial w(\hat{x})}{\partial x_3} = \delta p_3 \qquad (3)$$

$$p_2 \hat{x}_2 + p_3 \hat{x}_3 = \hat{\bar{x}}$$

simultaneously. But we note that since V is the maximum value function for P2,

$$\delta = \frac{\partial V(\hat{\overline{x}})}{\partial \overline{x}} = \mu \qquad \text{(equation (2))}$$

Combining the budget constraints

$$p_1\hat{x}_1 + p_2\hat{x}_2 + p_3\hat{x}_3 = M.$$

equations (2) and (3) become

$$\frac{\partial u(\hat{x}_1)}{\partial x_1} = \delta p_1$$

$$\frac{\partial w(\hat{x})}{\partial x_2} = \delta p_2$$

$$\frac{\partial w(\hat{x})}{\partial x_3} = \delta p_3$$

$$px = M$$

which have the same solution as equations (1).[3] We have verified that

$$\hat{x} = x^{*}.$$

Furthermore, the consumers preferences over x_1 and the composite commodity $\overline{x}$ can be represented by convex indifference curves. This follows trivially from the concavity of the maximum value function and additivity of concave functions.

These results can be illustrated in the following diagram, which shows how the consumer's optimisation problem can be decomposed into two sub-problems. The first quadrant depicts the allocation of expenditure between x_1 and the remaining commodities. The second quadrant shows how the

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[3] Strict concavity guarantees uniqueness of the solution to equations (1).

remaining expenditure is allocated between goods two and three.[4] The preceding analysis established (i) that the indifference curves in quadrant 1 are convex to the origin and (ii) that the diagram is consistent in the sense that the overall optimal consumption x* represents a point of tangency in both quadrants. In analysing the demand for good 1, we can focus our attention on quadrant 1 leaving quadrant 2 to look after itself. Ignoring the second quadrant will not lead us astray.

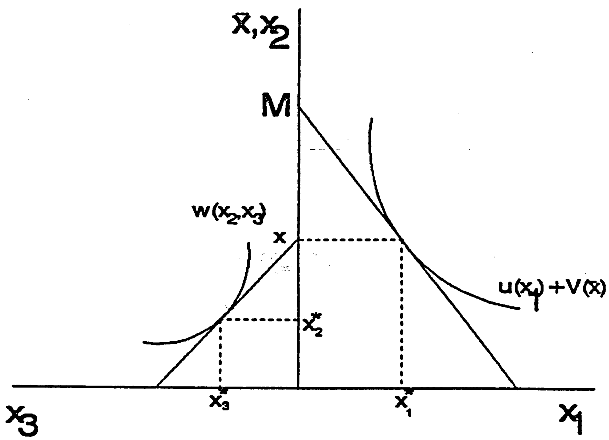

Figure 1

The essence of the composite commodity theorem is that preceding derivation and observations depend not on the separability of preferences but on the constancy the price of p_2 relative to p_3. Put differently, what makes Figure 1 commute is not the separability of the utility function but the constant slope of the budget line in quadrant 2. Without separability,

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[4]The diagram is drawn assuming that x_2 is numeraire.

the indifference curves in quadrant 1 will depend upon the quantities of x_2 and x_3 and the indifference curves in quadrant 2 will depend upon the quantity of x_1. The only interdependence which cannot be incorporated into the indifference curves in quadrant 1 is a change in the relative price p_2/p_3. As long as p_2/p_3 remains constant, the slope of the budget line in quadrant 2 remains unchanged, and hence the tangencies between indifference curves and budget lines in both spaces remain necessary and sufficient for an overall optimum. We can still focus our attention on quadrant 1 and leave quadrant 2 to look after itself.

To establish this result, we repeat the preceding analysis dispensing with the assumption of separability. The consumer's problem is:

$\mathbb{P}0'$: $$\max_{x} U(x_1,x_2,x_3) \quad \text{s.t. } px = M$$

with V strictly quasi-concave.

Let us decompose this into two subproblems

$\mathbb{P}1'$: $$\max_{x_1,\overline{x}} = V(x_1,\overline{x}) \qquad \text{s.t. } px_1 + \overline{x} = M$$

where $V(x_1,\overline{x})$ is the maximum value function of

$\mathbb{P}2'$: $$\max_{x_2,x_3} U(x_1,x_2,x_3) \quad \text{s.t. } p_2x_2 + p_3x_3 = \overline{x}$$

We claim that (i) $V(x_1,\overline{x})$ is quasi-concave and (ii) $\mathbb{P}0'$ has the same solution as $\mathbb{P}1'$ and $\mathbb{P}2'$.

To establish (i), we follow Diewert (1978, Theorem 2.4). Let $(x_1',\overline{x}')$ and $(x_1'',\overline{x}'')$ be two solutions to $\mathbb{P}1'$ and let (x_2',x_3'), (x_2'',x_3'') be the corresponding solutions to $\mathbb{P}2'$. That is, (x_2',x_3') maximises $U(x_1',x_2,x_3)$ subject to $p_2x_2+ p_3x_3 = \overline{x}'$. For any $\alpha \in (0,1)$, define

$$x^o = \alpha x' + (1-\alpha)x''$$

$$\overline{x}^o = \alpha\overline{x}' + (1-\alpha)\overline{x}''$$

Then
$$V(x^0,\bar{x}^0) = \max_{x_2,x_3} \{ U(x^0,x\ ,\ x\ ) : p_2x_2 + p_3x_3 = \bar{x}^0\}$$
$$\geq U(x_1^0,x_2^0,x_3^0)$$
$$> \min \{U(x'),U(x'')\} \qquad \text{(quasiconcavity of U)}$$
$$= \min \{V(x',\bar{x}'),V(x'',\bar{x}'')\}$$

Thus V is quasi-concave.

To establish (ii), we note that a commodity bundle x^* solves the consumer's optimization problem $\mathbb{P}0'$ if it satisfies the first order conditions

$$\frac{\partial U(x^*)}{\partial x_i} = \lambda p_i \qquad i = 1,2,3 \tag{4}$$
$$px = M$$

A commodity bundle $\hat{x}$ solves $\mathbb{P}1'$ and $\mathbb{P}2'$ jointly if it satisfies the first order conditions for $\mathbb{P}1'$

$$\frac{\partial V(\hat{x}_1,\hat{\bar{x}})}{\partial x_1} = \mu p_1$$

$$\frac{\partial V(\hat{x}_1,\hat{\bar{x}})}{\partial \bar{x}} = \mu \tag{5}$$

$$px_1 + \bar{x} = M$$

and $\mathbb{P}2'$

$$\frac{\partial U(\hat{x})}{\partial x_i} = \delta p_i \qquad i = 2,3$$
$$p_2x_2 + p_3x_3 = \bar{x} \tag{6}$$

simultaneously. Since $V(\bar{x})$ is the maximum value function for $\mathbb{P}2'$, the Lagrange multiplier δ is equal to its slope, i.e.

$$\delta = \frac{\partial V(\hat{\bar{x}})}{\partial \bar{x}}$$

Using (5), this implies that

$$\delta = \mu$$

The shadow price of the expenditure constraint in P2′ is equal to the marginal utility of income. Further we note that

$$\frac{\partial V(\hat{x}_1, \hat{\bar{x}})}{\partial x_1} = \frac{\partial U(\hat{x})}{\partial x_1}$$

so that first order conditions (5) and (6) can be amalgamated into

$$\frac{\partial U(\hat{x})}{\partial x_i} = \delta p_i \qquad i = 1,2,3$$

$$px + M$$

which have the same solution as (4). This establishes that $\hat{x} = x^*$. Problem P0′ has the same solution as P1′ and P2′.

This derivation also suggests another way of depicting the economic intuition underlying the composite commodity theorem. The first order conditions for the overall consumer problem (4) can be rearranged as follows:

$$\frac{\dfrac{\partial U(x^*)}{\partial x_i}}{p_i} = \lambda = \text{marginal utility on income.}$$

Optimality requires that expenditure be allocated across commodities so that a small increment in income yield the same utility no matter how it is spent. This applies <u>à fortiori</u> to the division of expenditure between commodity 1 and all other commodities,

i.e. $$\frac{\dfrac{\partial U(x^*)}{\partial x_1}}{p_1} = \frac{\partial V(\hat{x}_1, \hat{\bar{x}})}{\partial \bar{x}}$$

which is another way of expressing the tangency in quadrant 1. In this sense, the consumer's problem is <u>separable</u> into expenditure on commodity 1 and expenditure on the other commodities, assuming that expenditure on other commodities is allocated optimality.

In this note we have offered an alternative derivation of the composite commodity theorem which reveals it as a separability result arising from the structure of the utility maximisation process. The demonstration involves no more than simple manipulation of familiar first order conditions plus an elementary proof of quasi-concavity of the maximum value function. It should be readily accessible to intermediate and advanced level microeconomics classes. In conjuction with Figure 1, this should prove useful in the classroom in presenting this central theorem of economic analysis. Finally we note that nothing in the preceding discussion hinged on there being only two commodities in the fixed price group.

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