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Discussion Paper

No. 9001

Department of Economics University of Canterbury
Christchurch, New Zealand

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February 1990

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ABSTRACT

This paper shows a simple method of approximating the exact distribution of the Durbin-Watson Test Statistic for first-order autocorrelation in a nonlinear model. The proposed Approximate Nonlinear Durbin-Watson (A.N.D.) test has good size and power when compared to alternatives.

* This paper is circulated for discussion and comments. It should not be quoted without the prior approval of the author.

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* This paper was written while the author was a visitor at the University of Canterbury in Christchurch, New Zealand. I am grateful to David Giles for helpful suggestions and the University of Canterbury Computer Centre for computer support.

I. INTRODUCTION

In the linear regression model:

$$Y = X\beta + \varepsilon$$

Y is an (n x 1) vector of observations on the dependent variable, X is an (n x k) matrix of independent variables, β is a (k x 1) parameter vector, and ε is an (n x 1) normally distributed disturbance vector. It is common to use the Durbin-Watson [1950, 1951, 1971] test statistic:

$$d = \frac{e'Ae}{e'e} = \frac{\sum_{t=2}^{n}(e_t - e_{t-1})^2}{\sum_{t-1}^{n} e_t{}^2}$$

where e is the (n x 1) vector of least squares residuals and

$$\underset{n \times n}{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & 0 & & & \\ 0 & -1 & 2 & -1 & & & \\ 0 & 0 & -1 & 2 & & & \\ \cdot & & & & \cdot & & \\ \cdot & & & & & \cdot & \\ \cdot & & & & & & \cdot \\ & & & & & 2 & -1 \\ & & & & & -1 & 1 \end{bmatrix}$$

to test the hypotheses $H_0 : \rho = 0$ against $H_1 : \rho > 0$ in the first-order autoregressive error process:

$$\varepsilon_t = \rho \ \varepsilon_{t-1} + v_t$$

when v_t is a vector of independent normal errors.

Durbin and Watson found that evaluation of the exact cumulative distribution function F(d) required the computation of the n-k eigenvalues $\lambda_1, ..., \lambda_{n-k}$ of the matrix MA where $M = I - X(X'X)^{-1} X'$. These eigenvalues can be used in the Imhof [1961] algorithm to compute F(d) for any value of d. For example: if $F(d) < .05$ the null hypothesis H_0 could be rejected at the 5% level of significance. Some econometric computer programs (for example SHAZAM) automatically compute the exact F(d) for any linear regression so the Durbin-Watson bounds table is no longer required.

II. An Approximate Nonlinear Durbin-Watson Test

Unfortunately, Durbin-Watson distribution theory assumes a linear model so the exact F(d) test can not be used with a nonlinear model. However, the following Approximate Nonlinear Durbin-Watson (A.N.D.) test yields an approximation to F(d) which is suitable for nonlinear models.

Following Judge et al [1988, Chapter 12] a general nonlinear model can be written as:

$$Y = f(X, \beta) + \varepsilon$$

when X is an (n x k) matrix of independent variables and β is a ρ x 1 parameter vector. Next, define the (n x ρ) matrix of derivatives:

$$\underset{n \times \rho}{Z(\beta)} = \frac{\partial f(X, \beta)}{\partial \beta'} .$$

A first-order Taylor series expansion of f(X, β) around the converged nonlinear least squares estimates $\hat{\beta}$ yields:

$$f(X, \beta) \simeq f\left(X, \hat{\beta}\right) + Z\left(\hat{\beta}\right)\left(\beta - \hat{\beta}\right) .$$

Hence:

$$Y \approx f\left(X, \hat{\beta}\right) + Z\left(\hat{\beta}\right)\left(\beta - \hat{\beta}\right) + \varepsilon$$

which yields the so-called "linear pseudomodel":

$$\begin{aligned} \overline{Y}\left(\hat{\beta}\right) &= Y - f\left(X, \hat{\beta}\right) + Z\left(\hat{\beta}\right)\hat{\beta} \\ &= Z\left(\hat{\beta}\right)\beta + \varepsilon \ . \end{aligned}$$

So, an ordinary least squares regression of $\overline{Y}\left(\hat{\beta}\right)$ on $Z\left(\hat{\beta}\right)$ will reproduce the parameter estimate. $\hat{\beta}$ and the nonlinear residual vector e. Note that $f\left(X, \hat{\beta}\right)$ are simply the predicted values of the dependent variable from the nonlinear estimation and the covariance matrix is usually consistently estimated by:

$$\hat{V}\left(\hat{\beta}\right) = \hat{\sigma}^2 \left[Z\left(\hat{\beta}\right)' Z\left(\hat{\beta}\right) \right]^{-1}$$

where:

$$\hat{\sigma}^2 = \frac{e'e}{n-\rho} .$$

Comparison of $Z(\hat{\beta})$ in the linear pseudomodel with X in the linear regression model indicates that they play similar roles in the calculation of various statistics and it appears reasonable to compute an approximation to F(d) in the nonlinear model by computing the $n - \rho$ eigenvalues of $\tilde{M}A$ where

$$\tilde{M} = I - Z(\hat{\beta})\left[Z(\hat{\beta})' Z(\hat{\beta})\right]^{-1} Z(\hat{\beta})' .$$

The A.N.D. test then uses these eigenvalues with the usual d statistic to compute $\tilde{F}(d)$ which approximates the exact distribution function.

It is useful to compare the properties of the A.N.D. test to those of the asymptotic test based on the estimate of ρ where

$$\hat{\rho} = \frac{\sum_{t=2}^{n} e_t \ e_{t-1}}{\sum_{t=2}^{n} e_{t-1}^2}$$

and the approximate normal test statistic is:

$$W = \frac{\hat{\rho} - \rho}{\sqrt{(1-\hat{\rho})/n}}$$

as detailed in Judge et al [1988, p. 394].

Researchers have commonly used either

$$W_1 = \frac{\hat{\rho}}{\sqrt{\left(1-\hat{\rho}^2\right)/n}}$$

or

$$W_2 = \frac{\hat{\rho}}{\sqrt{1/n}}$$

to test the hypothesis $\rho = 0$ and rejection based on a normally distributed one-sided 5% test occurs if the test statistic exceeds 1.645.

As the small sample properties of $\tilde{F}(d)$, W_1, and W_2 are generally unknown in nonlinear models a Monte Carlo experiment was performed.

III. Experiment 1 - A Nonlinear Restriction

Judge [1982, p. 647] presents 20 observations on the nonlinear model:

$$Y = \beta_1 + \beta_2 X_2 + \beta_2^2 X_3 + \varepsilon$$

This model is interesting because it is a special case of the linear model

$$Y = \beta_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon$$

with the nonlinear restriction:

$$\beta_3 = \beta_2^2 \ .$$

The experiment uses a true parameter vector of $\beta' = (.8, 1.2)$ and 1000 sets of independent standard normal errors v_t were generated by the algorithm in Brent [1974] with values of ρ ranging from 0.0 to 0.9 to generate samples of Y.

For each sample, β was estimated by nonlinear least squares and the test statistics were computed from the results of the corresponding linear pseudomodel. Table 1 reports the power of the test statistics based on $\tilde{F}(d)$, W_1, and W_2 . In each case the power is computed as the percentage of times that the null hypothesis is rejected.

The table also reports the power of two other tests labelled $F_c(d)$ and $F_c(\hat{\rho})$. These are size corrected tests based on 5% critical values ($d_c = 1.2711$, $\rho_c = .322147$) obtained from the 1000 Monte Carlo values of d and $\hat{\rho}$ respectively. Naturally, the $F_c(d)$ and $F_c(\hat{\rho})$ critical values are not available in applied work so these results are less useful than those of the $\tilde{F}(d)$, W_1, and W_2 tests.

The results in Table 1 indicate that the A.N.D. $\tilde{F}(d)$ test is superior to either W_1 or W_2 which tended to under-reject and had lower power. If exact 5% critical values are available the $F_c(\hat{\rho})$ test was more powerful than the $F_c(d)$ test. With 1000 samples the sampling standard error of the numbers reported in Table 1 can be computed using the binomial formula. For example: when $\rho = .4$ the estimated power of the $\hat{F}(d)$ test is .461 with sampling standard error equal to $\sqrt{.461(1-.461)/1000} = .01576$.

TABLE 1

POWER OF TESTS FOR AUTOCORRELATION

$$Y = \beta_1 + \beta_2 X_2 + \beta_2^2 X_2 + \varepsilon_t$$

$$n = 20$$

	SIZE NOT CORRECTED			SIZE CORRECTED	
ρ	$\tilde{F}(d)$	W_1	W_2	$F_c(d)$	$F_c(\hat{\rho})$
0	.050	.043	.037	.050	.050
.1	.107	.092	.073	.103	.110
.2	.202	.174	.144	.197	.198
.3	.315	.284	.245	.305	.323
.4	.461	.427	.379	.450	.462
.5	.597	.575	.537	.591	.616
.6	.722	.699	.671	.718	.728
.7	.832	.803	.780	.829	.835
.8	.906	.893	.871	.904	.910
.9	.922	.905	.903	.919	.918

IV. Experiment 2 - C.E.S. Production Function

The second example uses the 30 observations from the very nonlinear CES production function described in Judge et al [1988, p. 512].

1000 samples of the model

$$\log Q = \beta_1 \log\left[\beta_2 L^{\beta_3} + (1-\beta_2) K^{\beta_3}\right] + \varepsilon$$

were generated where the independent errors v_t were drawn from a normal distribution with mean of zero and variance of 0.05. A true parameter vector of $\beta' = (-.5, .3, -2.0)$ was used. Table 2 reports the results of the experiment.

The results from the very nonlinear CES experiment support the conclusion that the A.N.D. test should be strongly considered for nonlinear models. The power of the A.N.D. test was superior to either the W_1 or W_2 tests and only had a slight tendency to over reject. The size corrected $F_c(d)$ test using the 5% Monte Carlo critical value of d_c=1.3679 have greater power than the size corrected $F_c(\hat{\rho})$ test which used a Monte Carlo critical value of $\rho_c = .2970$ Similar results were found on other examples which are not reported here.

TABLE 2

POWER OF TESTS FOR AUTOCORRELATION

$$\log Q = \beta_1 \log \left[\beta_2 L^{\beta_3} + (1-\beta_2) K^{\beta_3}\right] + \varepsilon$$

$$n = 30$$

	SIZE NOT CORRECTED			SIZE CORRECTED	
ρ	$\tilde{F}(d)$	W_1	W_2	$F_c(d)$	$F_c(\hat{\rho})$
0	.056	.059	.047	.050	.050
.1	.119	.119	.106	.112	.111
.2	.220	.221	.207	.210	.210
.3	.381	.384	.355	.371	.365
.4	.588	.586	.567	.575	.570
.5	.756	.746	.734	.743	.739
.6	.863	.855	.836	.853	.841
.7	.928	.922	.914	.928	.915
.8	.963	.949	.946	.955	.946
.9	.977	.972	.969	.974	.969

V. Conclusion

This paper has demonstrated that the exact distribution of the Durbin-Watson statistic can be easily approximated for nonlinear models and the test statistic has high power and may be superior to alternative asymptotic tests which have often been used in small samples.

It should be recognized that calculation of $\tilde{F}(d)$ is easy with currently available econometric software. The software package needs only to have the facility to save the results of the nonlinear estimation and the facility to compute the exact Durbin-Watson distribution in linear models. The following commands from Version 6.2 of the SHAZAM Econometric computer program were used to compute $\tilde{F}(d)$ in the CES example:

```
*FIRST RUN THE NONLINEAR MODEL
NL 1 / NCOEF=3  ZMATRIX=Z  COEF=BETA  PREDICT=YHAT
EQ LOGQ=B1*LOG(B2*L**B3+(1-B2)*K**B3)
END
* GENERATE THE LINEAR PSEUDOMODEL AND COMPUTE EXACT DURBIN-WATSON
MATRIX  YBAR=LOGQ-YHAT+Z*BETA
OLS YBAR Z / NOCONSTANT EXACTDW
STOP
```

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