18

Boston University

Center for Latin American Development Studies

THE DETERMINANTS OF CAPITAL UTILIZATION IN LABOR-MANAGED ENTERPRISES

Roger Betancourt

and

Christopher Clague

Discussion Paper Series

Number 18

March 1976

THE DETERMINANTS OF CAPITAL UTILIZATION IN LABOR-MANAGED ENTERPRISES

Roger Betancourt

and

Christopher Clague

Discussion Paper Series

Number 18   March 1976

THE DETERMINANTS OF CAPITAL UTILIZATION IN LABOR-MANAGED ENTERPRISES

R. R. Betancourt
Department of Economics
University of Maryland

and

C. K. Clague
Boston University and
University of Maryland

March 1976

I. Introduction

In the last few years, the theory of the labor-managed enterprise and the theory of the capital-utilization decision of the capitalist enterprise have been worked out in some detail.[1] This paper employs these two theories to develop a theory of the shift-work or capital-utilization decision of the labor-managed firm. It is not obvious from the analysis of the capital-utilization decision of the capitalist firm what the results will be for the labor-managed firm; thus, this topic is of scientific interest per se. In addition, it may also provide insights into the explanation of the remarkable growth performance of the Yugoslav economy. For, according to Vanek, one of the main explanations is the ability of a labor-managed economy to use efficiently the resources it withholds from present consumption.[2] But part of this 'efficiency' may simply be a much higher level of shift-work;[3] and indeed there is some empirical evidence that Yugoslav firms tend to work more shifts than enterprises in other countries.[4] Therefore,

---

[1] For examples of the former, see J. Vanek, The General Theory of Labor-Managed Market Economics (Ithaca: Cornell University Press, 1970), or B. Ward "The Firm in Illyria: Market Syndicalism," American Economic Review, XLVIII (September 1958), 566-589; for examples of the latter, see R. Betancourt and C. Clague "An Economic Analysis of Capital Utilization," Southern Economic Journal, XLI (July 1975), G. Winston, "Capital Utilization and Optimal Shift-Work", Bangladesh Economic Review, II (April 1974), 515-558, and for further references, G. Winston, "The Theory of Capital Utilization and Idleness, "Journal of Economic Literature, XII (December 1974), 1301-1320

[2] J. Vanek, The Participatory Economy (Ithaca: Cornell University Press, 1971), pp. 39-50.

[3] It has been shown in the context of a Harrod-Domar model that utilization has the same impact on the growth rate as efficiency or saving, see R. Marris The Economics of Capital Utilization (Cambridge: Cambridge University Press, 1964), Ch. 1.

[4] In a survey of industrial plants undertaken by UNIDO in various countries, about 90 percent of Yugoslav plants work two or more shifts whereas about 65 percent of plants in the nearest mixed economy country work two or more shifts, see UNIDO, Profiles of Manufacturing Establishments, Vol. I, II (ID/Ser.E/4 and 5).

the theoretical framework to be developed here can also be viewed as providing one of the components needed for an explanation of this growth performance. The actual explanation of this phenomenon, however, lies beyond the scope of this work. Instead, we concentrate on analyzing the determinants of shift-work in the labor-managed firm.

As is customary we shall assume the labor-managed firm attempts to maximize income per worker.[5] In the decision to work shifts, however, account must be taken of the workers' preferences for day-time or night-time work. We shall assume that the workers agree to the following procedure: the shift premium for night-time work will be set high enough to induce the required number of night workers to come forth voluntarily. Letting α be the premium for second-shift work, the marginal worker will be indifferent between receiving Y on the first shift and $Y(1+\alpha)$ on the second. In deciding whether to work shifts, the firm compares the incomes of the day workers under the single-shift and double-shift systems.[6]

---

[5]This theory has been criticized by A. Atkinson, "Worker Management and the Modern Industrial Enterprise," Quarterly Journal of Economics, LXXXVII (August 1973), 375-392, B. Horvat, "Prilog zasnivanju teorije jugoslovenskog produzéca" (A Contribution to the Theory of the Yugoslav Firm), Ekonomska Analiza, I-II(1967), 7-28 (as cited by Vanek and Miovic), Jan Vanek, The Economics of Workers' Management (London: George Allen and Irwin Ltd., 1972), and B. Horvat, "Critical Notes on the Theory of the Labor-Managed Firm and Some Macroeconomic Implications, Ekonomska Analiza, VI (1972), 288-293. For a partial defense see Jaroslav Vanek and P. Miovic, "Explorations into the 'Realistic' Behavior of the Yugoslav Firm," mimeo, Cornell University, April 1970.

[6]This rule tilts the balance slightly against shift-work because it ignores the consumers' surplus of the infra-marginal night-time workers.

We shall assume that the labor-managed firm makes a simultaneous decision regarding the size of the capital stock and its degree of utilization. Thus our analysis is long run in nature. We shall assume that capital is substitutable for labor ex ante, but once the machinery is installed, no substitution is possible. To our knowledge the only other paper on shift-work in the labor-managed enterprise is that of Abusada and Millan,[7] who allow for ex-post substitutability between capital and labor but take the capital stock as exogenous.

Following Vanek, we shall distinguish two kinds of technology. The technology of the first kind is that which gives rise to the familiar U-shaped long-run average cost curve for the capitalist firm; increasing returns to scale give way to decreasing returns to scale as output expands. The technology of the second kind is that of constant returns to scale. In both cases the CES production function will be utilized to examine the role of the capital intensity, the night-shift wage premium, and the elasticity of substitution in the decision to work shifts. In addition, the role of economies of scale and of the elasticity of demand will be brought out in our discussion of monopoly.

---

7 R. Abusada-Salah, and P. Millan, "Optima Utilización del Capital Instalado en Empresas con Participación de los Trabajadores en la Gestión," mimeo, Pontificia Universidad Católica, Lima, Perú, March 1974.

The analysis of the shift-work decision of the labor-managed firm will begin with the assumption of perfect competition (section II). Section III deals with monopoly. Comparisons with the capitalist firm are made in section IV and some concluding observations are contained in section V.

II. Perfect Competition

The following notation will be utilized:

P = price of output

X = daily rate of output

K = capital stock

L = employment

r = price of owning a unit of capital stock for a day

Y = income per worker

Using the superscripts 1 and 2 to refer to the single-shift and double-shift systems, we have

$$Y^1 = \frac{PX^1 - rK^1}{L^1}; \qquad Y^2 = \frac{PX^2 - rK^2}{L^2}$$

The subscript 1 or 2 identifies the first or second shift of the double-shift system. Since the ex post elasticity of substitution is zero, we have $L_1^2 = \frac{1}{2} L^2$. Letting α be the night-shift premium,

$$Y^2 = \frac{1}{2}(Y_1^2 + Y_2^2) = \frac{1}{2}[Y_1^2(2 + \alpha)]$$

Let us recall that the choice of system 1 or system 2 depends on the comparison of the income of the <u>day</u> workers under the two systems, or Y^1 and Y_1^2. We have

$$\frac{Y_1^2}{Y^1} = \frac{2}{2+\alpha} \frac{Y^2}{Y^1} \qquad (2.1)$$

The production function can be written

$$X^1 = F\,(S^1, L^1) \text{ and } X^2 = 2F(S_1^2, L_1^2) = 2X_1^2 \qquad (2.2)$$

where S refers to the capital services utilized from the capital stock. We assume the absence of wear-and-tear depreciation and the presence of perfect information and foresight. Under these conditions, there will be full capacity utilization within a shift; i.e., $S^1 = u^*K^1$ and $S_1^2 = u^*K^2$, where u^* is the maximum rate of utilization per shift.[8]

As Vanek shows, the first order conditions for maximizing Y under single-shift operation are: the value of marginal product of capital must equal the price of capital; and income per worker must equal the value of marginal product of labor.[9] More precisely,

$$p\,(\partial X^1/\partial K^1) = r, \text{ and } p\,(\partial X^1/\partial L^1) = Y^1 \qquad (2.3)$$

The corresponding conditions for the double shift system are:[1]

$$2P(\partial X_1^2/\partial K^2) = r\ ; \quad P\,(\partial X_1^2/\partial L_1^2) = Y^2 \qquad (2.4)$$

The second equalities in(2.3) and (2.4) imply

$$Y^2/Y^1 = (\partial X_1^2/\partial L_1^2)/(\partial X^1/\partial L^1) \qquad (2.5)$$

---

[8] See N. Georgescu-Roegen, "The Economics of Production, "American Economic Review, LX (May, 1970), 1-9.

[9] Vanek, The General Theory, op.cit., pp. 28-31

[1] $Y^2 = [P2X_1^2 - rK^2]/2L_1^2$, and

$\partial Y^2/\partial K^2 = 2P(\partial X_1^2/\partial K^2) - r = 0.$ Hence $2P(\partial X_1^2/\partial K^2) = r$

$\partial Y^2/\partial L_1^2 = \dfrac{2P(\partial X_1^2/\partial L_1^2) - 2Y^2}{2L_1^2} = 0.$ Hence $P(\partial X_1^2/\partial L_1^2) = Y^2$

These conditions hold whether the firm's technology is of the first or the second kind. The equality of the ratio of income per worker in the two systems to the ratio of the marginal products (2.5) is the basic equation for the analysis of perfect competition. Immediately below we develop an expression for the ratio of the marginal products under the technology of the second kind. In Section B we show that the same expression also holds for the technology of the first kind.

A. Technology of the Second Kind

With the technology of the second kind (constant returns to scale), the scale of the firm's operations is indeterminate, since income per worker plotted against the number of workers is a horizontal line. However, it is still possible to develop an expression for the ratio of the marginal products in the two systems. For this purpose, we assume that the production function is CES, i.e.,

$$X^1 = [\delta(u^*K^1)^{-\rho} + (1-\delta)(L^1)^{-\rho}]^{-\frac{1}{\rho}} \text{ and } X_1^2 = [\delta(u^*K^2)^{-\rho} + (1-\delta)(L_1^2)^{-\rho}]^{-\frac{1}{\rho}}$$

where $\rho = (1-\sigma)/\sigma$. The marginal products of labor are

$$f_1 = (1-\delta)(X^1/L^1)^{1+\rho} \quad \text{and} \quad f_1^2 = (1-\delta)(X_1^2/L_1^2)^{1+\rho}$$

where we write f_1 for $(\partial F/\partial L^1)$ and f_1^2 for $(\partial F/\partial L_1^2)$.

Hence $$\frac{f_1^2}{f_1} = \left[\frac{L^1}{L_1^2}\right]^{\frac{1}{\sigma}} \left[\frac{X_1^2}{X^1}\right]^{\frac{1}{\sigma}} \qquad (2.6)$$

Use of the marginal-product-of-capital equations in (2.3) and (2.4) with the CES production function leads to

$$K^1 = p^\sigma X^1 \delta^\sigma/(u^*)^{\rho\sigma} r^\sigma \text{ and } K^2 = 2^\sigma p^\sigma X_1^2/(u^*)^{\rho\sigma} r^\sigma.$$

Hence $$K^2/K^1 = 2^\sigma (X_1^2/X^1). \qquad (2.7)$$

Substituting (2.7) into (2.6) gives

$$\frac{f_1^2}{f^1} = \frac{1}{2} \left[\frac{L^1}{L_1^2}\right]^{\frac{1}{\sigma}} \left[\frac{K^2}{K^1}\right]^{\frac{1}{\sigma}}. \qquad (2.8)$$

From the production function[2]

$$\frac{K^2}{K^1} = 2^\sigma \frac{L_1^2}{L^1} \left[\frac{1-2^{\sigma-1}\psi}{1-\psi}\right]^{\frac{\sigma}{1-\sigma}}. \qquad (2.9)$$

---

[2] Using equation (2.7) and writing out X_1^2,

$$\frac{K^2}{K^1} = 2^\sigma \frac{X_1^2}{X^1} = 2^\sigma [\delta(u^* K^2)^{-\rho} + (1-\delta)(L_1^2)^{-\rho}]^{-1/\rho}/X^1$$

$$(X^1)^{-\rho} \left(\frac{K^2}{K^1}\right)^{-\rho} = 2^{-\rho\sigma}[\delta(u^* K^2)^{-\rho} + (1-\delta)(L_1^2)^{-\rho}]$$

$$\frac{K^2}{K^1} = \frac{2^\sigma (1-\delta)^{-\frac{1}{\rho}} L_1^2}{[(X^1)^{-\rho} - 2^{-\rho\sigma}(u^* K^1)^{-\rho}]^{-\frac{1}{\rho}}}$$

Cont. Footnote 2

Writing out X^1 gives

$$(1+\delta)^{-\frac{1}{\rho}} = \frac{1}{L^1}\left[(X^1)^{-\rho}-\delta(u^*K^1)^{-\rho}\right]^{-\frac{1}{\rho}}$$

and substituting into K^2/K^1

$$\frac{K^2}{K^1} = \frac{2^{\sigma}\frac{L_1^2}{L^1}\left[(X^1)^{-\rho}-\delta(u^*K^1)^{-\rho}\right]^{-\frac{1}{\rho}}}{\left[(X^1)^{-\rho}-2^{-\rho\sigma}(u^*K^1)^{-\rho}\right]^{-\frac{1}{\rho}}} \qquad \text{or}$$

$$\frac{K^2}{K^1} = 2^{\sigma}\frac{L_1^2}{L^1}\left[\frac{\left(\frac{pX^1}{rK^1}\right)^{-\rho} r^{-\rho}-\delta\, p^{-\rho}(u^*)^{-\rho}}{\left(\frac{pX^1}{rK^1}\right)^{-\rho\sigma} r^{-\rho}-2^{\sigma-1}\delta\, p^{-\rho}(u^*)^{-\rho}}\right]^{-\frac{1}{\rho}}$$

Noting that $\frac{rK^1}{pX^1} = \psi = (r/p)^{1-\sigma}\delta^{\sigma}(u^*)^{\sigma-1}$

$$K^2/K^1 = 2^{\sigma}\frac{L_1^2}{L^1}\left[\frac{\psi^{\rho}-\psi^{1/\sigma}}{\psi^{\rho}-2^{\sigma-1}\psi^{1/\sigma}}\right]^{-\frac{1}{\rho}} = 2^{\sigma}\frac{L_1^2}{L^1}\left[\frac{1-2^{\sigma-1}\psi}{1-\psi}\right]^{\frac{\sigma}{1-\sigma}}$$

where $\psi = rK^1/PX^1$ is the share of capital costs in value added under system 1. Substituting this into (2.8) yields

$$\frac{f_1^2}{f^1} = \left[\frac{1-2^{\sigma-1}\psi}{1-\psi}\right]^{\frac{1}{1-\sigma}} \qquad (2.10)$$

Hence, using (2.1) and (2.5)

$$\frac{Y_1^2}{Y^1} = \frac{2}{2+\alpha}\,\frac{Y^2}{Y^1} = \frac{2}{2+\alpha}\left[\frac{1-2^{\sigma-1}\psi}{1-\psi}\right]^{\frac{1}{1-\sigma}} \qquad (2.11)^3$$

Equation 2.11 can be analyzed to show the effects on the income ratio (Y_1^2/Y^1) of changes in α, ψ, and σ. Before doing that, however, we should note an anomalous case. As $1-2^{\sigma-1}\psi$ approaches zero from above, K^2/K^1 approaches infinity. If $1-2^{\sigma-1}\psi$ is less than zero, K^2/K^1 becomes negative. Clearly something in our assumptions has to give way; it seems reasonable to say that the elasticity of substitution would not remain constant as the capital-labor ratio becomes very high. In what follows we shall simply assume $1-2^{\sigma-1}\psi$ to be positive. For instance, if ψ has a value of .5, then a σ of less than 2 ensures that the expression will be positive.

It is apparent from (2.11) that shift-work will be more advantageous the weaker are workers' preferences against night-time work (the lower is α).

---

[3]The Cobb-Douglas case cannot be derived by direct substitution of $\sigma=1$ into (2.11) but replacement of the CES function by a Cobb-Douglas one leads to

$$Y_1^2/Y^1 = [2/(2+\alpha)][2^{\delta/(1-\delta)}] = [2/(2+\alpha)][2^{\psi/(1-\psi)}]$$

through a similar argument.

While not so apparent, it can be shown by differentiating the R.H.S. of (2.11) that shift-work will be more advantageous the more capital intensive the production process under single-shift operation (the higher is ψ). Formally, the impact of ψ on the income ratio is given by

$$\frac{\partial (Y_1^2/Y^1)}{\partial \psi} = \frac{1}{(1-\sigma)(1-\psi)} \; \frac{1-2^{\sigma-1}}{1-2^{\sigma-1}\psi} \; \frac{Y_1^2}{Y^1} > 0 \qquad (2.12)$$

Finally, taking the log of (2.11) and differentiating with respect to σ gives

$$\frac{\partial \ln R.H.S.}{\partial \sigma} = \frac{-2^{\sigma-1}(\ln 2)\psi}{(1-\sigma)\; B\; (1-\psi)} + \frac{\ln B}{(1-\sigma)^2} \qquad (2.13)$$

where $B = (1-2^{\sigma-1}\psi)/(1-\psi)$, and we assume $B > 0$. The sign of this expression is not apparent from inspection but numerical analysis shows it to be positive for all parameter combinations tried.[4] Why is shift-work more advantageous with a higher σ? The decision to work shifts lowers the price of capital services. In general, the firm benefits more from any factor price reduction when that factor can be more readily substituted for other factors. Application of this general principle to the factor capital leads to the conclusion that shift-work is more advantageous when the elasticity of substitution is higher.

B. Technology of the First Kind

Vanek shows that under perfect competition and with a technology of the first kind the firm will operate at the constant-returns-to-scale point

[4] This result is illustrated in Appendix A. Incidentally, the impact of changes in r or P on the income ratio is easily derived through their effect on ψ. That is, with a CES production function increases in (r/P) will increase, not affect, or decrease ψ depending on whether $\sigma<1$, $\sigma=1$, or $\sigma>1$.

in the production function because only at that point can the first-order conditions (2.3) hold simultaneously.[5] A similar argument leads to the conclusion for the double-shift system that each shift will be operated at the constant-returns-to-scale point in the production function. Otherwise, the two first-order conditions (2.4) cannot hold simultaneously. To see this, note that by definition

$$Y^2 = \frac{pX^2}{L^2} - \frac{rK^2}{L^2} = \frac{2pX_1^2}{L^2} - \frac{rK^2}{L^2} .$$

If the firm operates each shift at a point other than constant-returns-to-scale point, $X_1^2 \gtrless (\partial X_1^2/\partial K^2)K^2 + (\partial X_1^2/L_1^2)L_1^2$. Inserting this expression into the definition of Y^2 and using the first-order condition for the marginal product of capital from (2.4) leads to

$$Y^2 \gtrless 2r(\partial X_1^2/\partial L_1^2)\frac{L_1^2}{L^2} = p(\partial X_1^2/\partial L_1^2) .$$

If we now assume that at the constant-returns-to-scale point the production function is CES, the argument of the previous section leading from equation (2.6) to (2.11) goes through in exactly the same manner with the proviso that the two systems must be compared at the optimal level of output for each system. To conclude, the analysis of equation (2.11) in the previous section is equally applicable to a perfectly competitive labor-managed firm using a technology of the first kind.

III. Monopoly

The assumption under perfect competition that the firm would be able to market additional output with no loss in average revenue is frequently

---

[5] Vanek, op. cit., p. 30.

unrealistic. A reasonable assumption would be that, in order to sell additional output, the firm must either lower its price or incur higher selling costs per unit of output. Either of these possibilities can be represented by a downward-sloping demand curve.

Under monopoly we shall assume that the technology of the first kind is represented by a homothetic production function. Equation (2.2) will be written as $X = G[F(u^*K,L)]$ where F is a CES function with constant returns to scale. The first-order conditions for system 1 are:[6] the marginal-revenue product of capital must equal the price of capital and income per worker must equal the marginal-revenue product of labor, i.e.,

$$MR^1(\partial X^1/\partial K^1) = r, \text{ and } MR^1(\partial X^1/\partial L^1) = Y^1 \quad (3.1)$$

The corresponding conditions for the double shift system are:

$$2MR^2(\partial X_1^2/\partial K^2) = r, \text{ and } MR^2(\partial X_1^2/\partial L_1^2) = Y^2 \quad (3.2)$$

The homotheticity of the production function allows us to demonstrate that the optimal level of output will be chosen independently of the optimal capital-labor ratio in both systems of operation.[7] Consider the single-shift system first. From the definition of Y^1,

$$L^1 Y^1 + r K^1 = P^1 X^1$$

$$L^1 \cdot MR^1 G'(F^1)(\partial F/\partial L^1) + K^1 MR^1 G'(F^1)(\partial F/\partial K^1) = P^1 X^1$$

where we used (3.1). Since F is first-degree homogeneous,

$$G'(F^1) \cdot F^1/X^1 = P^1/MR^1 \quad (3.3)$$

---

[6]Vanek, op.cit., p. 101

[7]Note that the converse is not true, i.e., the choice of the optimal capital-labor ratio will depend on the optimal level of output.

Turning now to the double-shift system, the definition of Y^2, the first-order conditions (3.2), and the production function lead to

$$L_1^2 \cdot MR^2 \cdot G'(F_1^2)(\partial F/\partial L_1^2) + K^2 \cdot MR^2 \cdot G'(F_1^2) \cdot (\partial F/\partial K_1^2) = P^2 X_1^2, \text{ or}$$

$$G'(F_1^2) \cdot F_1^2/X_1^2 = P^2/MR^2 \qquad (3.4)$$

Condition (3.3) indicates that the optimal level of output under single-shift operation will be chosen to equate the degree of economies of scale, which is what the LHS of (3.3) measures, to the RHS which is related to the absolute value of the elasticity of demand at the level of output X^1 (that is, n_1) as follows $P^1/MR^1 = n_1/(n_1-1)$. Similarly condition (3.4) indicates that the optimal level of output, $X^2 = 2X_1^2$, for the double-shift system will be chosen by equating the degree of economies of scale <u>per shift</u> to the RHS which is related to the absolute value of the elasticity of demand at the level of output X^2.

These two conditions (3.3) and (3.4) can be used to explore the relationship between X^1 and X_1^2 and between X^1 and X^2. This is done in detail in Appendix B. At this point, however, we merely summarize the main results in order to preserve the continuity of the exposition. Our general conclusion is that in the "typical case", output per shift will be lower but total output will be higher under double-shift operation than under single-shift operation. That is, $X_1^2 < X^1 < X^2$. More precisely, if the production function exhibits a constant degree of homogeneity, β[8], then (3.3) and (3.4) imply that $X^1 = X^2$, hence $X_1^2 = (1/2)X^1$. If the demand function is of the constant-elasticity variety, and the production function is homothetic but not

[8] Note that in this special case the technology can't be of the first kind. Moreover, β must be greater than unity if the demand curve slopes downward; otherwise, equilibrium would be impossible as can be seen from either (3.3) or (3.4).

homogeneous,[9] then (3.3) and (3.4) imply that $X_1^2 = X^1$, hence $X^2 = 2X^1$. Finally, if the technology is of the first kind (but the production function is still homothetic, of course) and if the elasticity of demand declines in absolute value as output expands, then (3.3) and (3.4) imply that $X_1^2 < X^1 < X^2$.

As in the case of perfect competition we are interested in the behavior of the income ratio (Y_1^2/Y^1) and consequently in Y^2/Y^1. Using the first-order conditions (3.1) and (3.2) for the average product per worker in each system, and recalling that f^1 is defined as $(\partial F/\partial L^1)$,

$$\frac{Y^2}{Y^1} = \left[\frac{X_1^2/F_1^2}{X^1/F^1}\ \frac{P^2}{P^1}\right] \frac{f_1^2}{f^1} = H\,\frac{f_1^2}{f^1} \tag{3.5}$$

The term in brackets, denoted by H, reflects the influences of economies of scale and of demand conditions in ways that will be indicated below. Since H will also affect the ratio of the marginal products of labor in the two systems, we postpone any discussion of H until the derivation of an explicit expression for f_1^2/f^1 is completed.

Using the assumption that F is a CES production function, we have

$$\frac{f_1^2}{f^1} = \left[\frac{L^1}{L_1^2}\right]^{\frac{1}{\sigma}} \cdot \left[\frac{F_1^2}{F^1}\right]^{\frac{1}{\sigma}} \tag{3.6}$$

From the first-order conditions for the marginal product of capital for each system, i.e. (3.1) and (3.2),

$$K^1 = [MR^1 \cdot G'(F^1)]^{\sigma} F^1 \delta^{\sigma} (\mu^*)^{-\rho\sigma} r^{-\sigma} \tag{3.1)'$$

---

[9] With a constant degree of homogeneity and a constant elasticity of demand, equations (3.3) and (3.4) imply that the equilibrium of the labor managed firm is possible if and only if $\beta = n/(n-1)$.

$$K^2 = [2MR^2 \cdot G'(F_1^2)]^{\sigma} F_1^2 \delta^{\sigma} (u^*)^{-\rho\sigma} r^{-\sigma} \tag{3.2)'$$

$$\frac{K^2}{K^1} = 2^{\sigma} \left[\frac{MR^2}{MR^1} \quad \frac{G'(F_1^2)}{G'(F^1)}\right]^{\sigma} \quad \frac{F_1^2}{F^1}$$

Since (3.3) and (3.4) imply

$$\frac{MR^2}{MR^1} \quad \frac{G'(F_1^2)}{G'(F^1)} = \frac{X_1^2/F_1^2}{X^1/F^1} \quad \frac{P^2}{P^1} = H,$$

we have

$$K^2/K^1 = (2H)^{\sigma} \ . \ F_1^2/F^1. \tag{3.7}$$

substitute (3.7) into (3.6)

$$\frac{f_1^2}{f^1} = \frac{1}{2H} \left[\frac{L^1}{L_1^2}\right]^{\frac{1}{\sigma}} \left[\frac{K^2}{K^1}\right]^{\frac{1}{\sigma}}. \tag{3.8}$$

From the production function[1]

$$\frac{K^2}{K^1} = (2H)^{\sigma} \ \frac{L_1^2}{L^1} \left[\frac{1-(2H)^{\sigma-1}\psi}{1-\psi}\right]^{\frac{\sigma}{1-\sigma}}. \tag{3.9}$$

Substitute (3.9) into (3.8)

$$\frac{f_1^2}{f^1} = \left[\frac{1-(2H)^{\sigma-1}\psi}{1-\psi}\right]^{\frac{1}{1-\sigma}} \tag{3.10}$$

Note that if H=1, (3.10) collapses to (2.10), substitute (3.10) into (3.5) and use (2.1):

$$\frac{Y_1^2}{Y^1} = \frac{2}{2+\alpha} \ \frac{Y^2}{Y^1} = \frac{2}{2+\alpha} \ H \left[\frac{1-(2H)^{\sigma-1}\psi}{1-\psi}\right]^{\frac{1}{1-\sigma}} \tag{3.11}$$

Equation (3.11) can be analyzed to show the effects on the income ratio (Y_1^2/Y^1) of changes in α, ψ, σ and H. Similarly to the case of perfect

---

[1] See page 16.

---

[1] This derivation is similar to the one for the perfectly competitive case. Using equation (3.7) and writing out F_1^2,

$$\frac{K^2}{K^1} = (2H)^{\sigma}\frac{F_1^2}{F^1} = (2H)^{\sigma}[\delta(u^*K^2)^{-\rho}+(1-\delta)(L_1^2)^{-\rho}]^{-\frac{1}{\rho}}/F^1$$

$$(F^1)^{-\rho}\left(\frac{K^2}{K^1}\right)^{-\rho} = (2H)^{-\rho\sigma}[\delta(u^*K^2)^{-\rho}+(1-\delta)(L_1^2)^{-\rho}]$$

$$\frac{K^2}{K^1} = \frac{(2H)^{\sigma}(1-\delta)^{-1/\rho}L_1^2}{[(F^1)^{-\rho}-(2H)^{-\rho\sigma}(u^*K^1)^{-\rho}]^{-1/\rho}}$$

Writing out F^1 gives

$$(1-\delta)^{-1/\rho} = \frac{1}{L^1}[(F^1)^{-\rho}-\delta(u^*K^1)^{-\rho}]^{-\frac{1}{\rho}}, \text{ which leads to}$$

$$\frac{K^2}{K^1} = \frac{(2H)^{\sigma}(L_1^2/L^1)[(F^1)^{-\rho}-\delta(u^*K^1)^{-\rho}]^{-\frac{1}{\rho}}}{[(F^1)^{-\rho}-(2H)^{-\rho\sigma}(u^*K^1)^{-\rho}]^{-1/\rho}}, \text{ or}$$

$$K^2/K^1 = (2H)^{\sigma}\frac{L_1^2}{L^1}\left[\frac{\left(\frac{P^1F^1}{rK^1}\right)^{-\rho}r^{-\rho}-\delta(P^1)^{-\rho}(u^*)^{-\rho}}{\left(\frac{P^1F^1}{rK^1}\right)^{-\rho}r^{-\rho}-(2H)^{\sigma-1}\delta(P^1)^{-\rho}(u^*)^{-\rho}}\right]^{-\frac{1}{\rho}}.$$

Noting that the optimal level of output under single shift operation can be written as $X^1 = c(F^1)\cdot F^1$ where $c(F^1) > 1$, we have from equation (3.3) that $MR\ G'(F^1) = p\,c(F^1)$; therefore, using equation (3.1)

$$\psi = \frac{rK^1}{PX^1} = \frac{rK^1}{Pc(F^1)\cdot F^1} = (r/P)^{1-\sigma}\delta^{\sigma}(u^*)^{\sigma-1}c(F^1)^{\sigma-1}, \text{ and}$$

$$\frac{K^2}{K^1} = (2H)^{\sigma}\frac{L_1^2}{L^1}\left[\frac{c^{\rho}\psi^{\rho}-c^{\rho}\psi^{1/\sigma}}{c^{\rho}\psi^{\rho}-(2H)^{\sigma-1}c^{\rho}\psi^{1/\sigma}}\right]^{-1/\rho} = (2H)^{\sigma}\frac{L_1^2}{L^1}\left[\frac{1-(2H)^{\sigma-1}\psi}{1-\psi}\right]^{\frac{\sigma}{1-\sigma}}$$

competition, the expression $1-(2H)^{\sigma-1}\psi$ will be assumed to be positive in order to rule out nonsensical results.[2] As in the case of perfect competition, it is clear that shift-work will be more advantageous the weaker are workers' preferences against night-time work (the lower is α); and it can be easily shown that the more capital intensive the production process (the higher is ψ), the greater is the incentive to work shifts, as long as $1/2 < H < 1$, i.e.,

$$\frac{\partial (Y_1^2/Y^1)}{\partial \Psi} = \frac{1}{(1-\sigma)} \frac{1}{(1-\psi)} \left[\frac{1-(2H)^{\sigma-1}}{1-(2H)^{\sigma-1}\psi}\right] \frac{Y_1^2}{Y^1} > 0 \qquad (3.12)^*$$

Finally, taking the log of (3.11) and differentiating the RHS with respect to σ gives

$$\frac{\partial \ln RHS}{\partial \sigma} = -\frac{(2H)^{\sigma-1}(\ln 2H)\psi}{(1-\sigma)B(1-\psi)} + \frac{\ln B}{(1-\sigma)^2} \qquad (3.13)$$

where $B=(1-(2H)^{\sigma-1}\psi)/(1-\psi) > 0$ by assumption. Again the sign of this expression is not apparent from inspection but numerical analysis shows it to be positive for all parameter combinations, as long as $1/2 < H < 1$.[3]

Turning now to the distinctive characteristic of the analysis of monopoly, the derivative of (3.11) with respect to H, holding σ and ψ constant, is unambiguously positive, i.e.,

$$\partial (Y_1^2/Y^1)/\partial H = B^{1/(1-\sigma)}(1-\psi)/B > 0$$

---

[2] Note that if H is less than one, a wider range of combinations of ψ and σ will now satisfy this assumption.

[3] Appendix A illustrates this result for several values of H.

* Typographical note: the symbols ψ and Ψ are used interchangeably.

where B is defined as before.[4] Thus, anything which increases H will increase the income ratio. Below we shall indicate what sort of factors would cause H to increase and why it is reasonable to argue that $1/2 < H < 1$.

Two interesting special cases will reveal the role of H in (3.11). If the production function is homogeneous of degree β, we saw above that $X^2=X^1$. Clearly $P^2=P^1$. H may be written

$$H = \frac{X_1^2}{X^1}\,\frac{F^1}{F_1^2}\,\frac{P^2}{P^1} = \frac{1}{2}\cdot 2^{\frac{1}{\beta}}\cdot 1 = \frac{1}{2^{1-1/\beta}} \qquad (3.14)$$

H must be less than one since β is greater than 1 and the limiting (minimum) value which H approaches as β approaches infinity is 1/2. An increase in β clearly reduces H. Since a decline in H causes the income ratio to fall in (3.11), an increase in economies of scale, measured by β, makes shift-work less advantageous.

If the demand curve has a constant elasticity n while the production function is homothetic but not homogeneous, we saw earlier that $X_1^2=X^1$. Hence $X_1^2/F_1^2 = X^1/F^1$ and H may be written

---

[4] Differentiating the RHS of (3.11) with respect to H,

$$\frac{\partial RHS}{\partial H} = B^{\frac{1}{1-\sigma}} + \frac{H}{1-\sigma}B^{\frac{\sigma}{1-\sigma}}\left[\frac{-(\sigma-1)2^{\sigma-1}H^{\sigma-2}\psi}{1-\psi}\right]$$

$$= \left[B^{\frac{1}{1-\sigma}}\right]\left[1+\frac{(2H)^{\sigma-1}\psi}{1-(2H)^{\sigma-1}\psi}\right] = \left[\frac{1-(2H)^{\sigma-1}\psi}{1-\psi}\right]^{\frac{1}{1-\sigma}}\left[\frac{1}{1-(2H)^{\sigma-1}\psi}\right]$$

$$H = 1 \ . \ (P^2/P^1) = (X^1/X^2)^{1/n} = (1/2)^{1/n} \qquad (3.15)$$

H must be greater than one half since n must be greater than unity for the labor-managed monopoly facing a downward-sloping demand curve. Moreover H approaches 1 as its limiting (maximum) value as n approaches infinity, and an increase in n clearly increases H. Therefore, an increase in the elasticity of demand will increase the income ratio in (3.11) and will thereby favor shift-work.

More generally, if $X_1^2 < X^1 < X^2$, as we argued at the beginning of this section, then both $(X_1^2/F_1^2)/(X^1/F^1)$ and P^2/P^1 will be less than one and H must be less than one. In addition the two polar cases considered here buttress our confidence in the conjecture that H will also be greater than 1/2 when $X_1^2 < X^1 < X^2$. Therefore, any factor which increases the elasticity of demand or reduces the degree of economies of scale (while keeping the other constant) will increase H and, consequently, the income ratio.

IV. A Comparison with the Capitalist Firm

Comparing the behavior of the labor-managed firm with its capitalist counterpart yields further insights into the determinants of shift-work in both types of firms. In order to facilitate the exposition, we consider separately the cases of perfect competition and monopoly.

A. Perfect Competition

For the capitalist firm, the condition for shift-work to be profitable is that the rates of costs under system 1 to those under system 2 (the cost ratio CR) exceed unity.[5] This cost ratio can be written as

[5] The shift-work decision of the capitalist firm has been analyzed in Betancourt and Clague, op. cit., and all the comparisons in this section are made with respect to the results established there.

$$CR = \frac{2}{2+\alpha}\left[\theta(2+\alpha)^{\sigma-1}+(1-\theta)\right]^{\frac{1}{\sigma-1}} \qquad (4.1)$$

where θ is the share of capital costs in combined labor and capital costs under system 1, or $rK^1/(rK^1+W_1L^1)$.

When we speak of the labor-managed firm and its capitalist counterpart as being twins, we mean that they have the same production function, the same price of capital, the same α, and the same demand conditions. The wage rate for the capitalist firm, W_1, may or may not be the same as Y^1. If it is, then the capitalist firm would earn no profits under single-shift operation and Θ must equal Ψ. If W_1 is less than Y^1, then normally Θ will differ from Ψ.

Consider first the similarities between the labor-managed and the capitalist firm. The sign of the response of the pay-off function with respect to changes in the parameters σ, α, ψ(or θ), is the same. That is, increases in σ and ψ(or θ) increase the profitability of shift-work in the capitalist firm and the desirability of shift-work in the labor-managed firm. Similarly, decreases in α increase the profitability or the desirability of shift-work in both firms. Moreover, if $\Theta = \Psi$, the labor-managed firm and its capitalist twin will always reach the same shift-work decision. This proposition can be established by showing that the values of the parameters at which the two twins are indifferent between system 1 and system 2 are the same; that is, if Y_1^2/Y^1 equals one, then CR must also equal one, and vice-versa.

$$Y_1^2/Y^1 = \frac{2}{2+\alpha}\left[\frac{1-2^{\sigma-1}\psi}{1-\psi}\right]^{\frac{1}{1-\sigma}} = 1$$

$$2^{1-\sigma}-\psi = (1-\psi)(2+\alpha)^{1-\sigma}$$

$$(2+\alpha)^{\sigma} \quad 2^{1-\sigma} = \psi(2+\alpha)^{\sigma} + (2+\alpha)(1-\psi)$$

$$\left[\frac{2}{2+\alpha}\right]^{1-\sigma} = \psi(2+\alpha)^{\sigma-1} + (1-\psi)$$

$$1 = \frac{2+\alpha}{2}\left[\psi(2+\alpha)^{\sigma-1} + (1-\psi)\right]^{\frac{1}{1-\sigma}} = 1/CR \qquad (4.2)$$

Another interesting similarity between the two types of firms regarding their capital-utilization behavior is that the necessary (but not sufficient) condition for shift-work to be desirable and at the same time decrease capital productivity is the same: namely, the elasticity of substitution must be greater than unity. The capital productivity for the two systems in the labor-managed firm is easily derived by manipulating equation (2.7) to yield

$$(X^2/K^2)/(X^1/K^1) = 2^{1-\sigma} \qquad (4.3)$$

From this equation it can be seen that capital productivity will be lower under shift-work if σ is greater than one. This is a somewhat surprising conclusion since shift-work is usually thought of as a means of saving capital. However, the explanation of the paradox, as in the capitalist case, lies in the fact that shift-work involves a reduction in the price of capital services, which leads to the choice of a technique with a higher instantaneous capital-labor ratio.

While the similarities are surprising, it would be even more surprising if there were no differences in the behavior of the two types of firms in their capital utilization decision, but there are, of course, several differences. In the capitalist firm α plays a substantial role in determining the relative amounts of capital under the two systems through its impact on the cost ratio;[6] in the labor-managed firm, on the other hand, the relative amounts of capital that would be used under the two systems is independent of α, as (2.7) demonstrates. This difference in behavior stems from the difference in the objective functions of the two firms. In the capitalist firm an increase in α always reduces the profits of the double-shift system and the entrepreneur responds by adjusting the variables under his control; in a labor-managed firm, on the other hand, an increase in α merely changes the distribution of income between day and night workers and does not provide an incentive for the firm to change its behavior, as long as the change in α is too small to tip the balance in favor of single-shift operation.

Another difference in the role of α is that the percentage change in the income ratio (2.11) of the labor-managed firm due to a change in α depends solely on α; moreover, percentage changes in the income ratio with respect to Ψ and σ are independent of α. Neither of these results holds for the capitalist firm.[7]

[6] See Betancourt and Clague, op. cit., equation 7, p. 9

[7] This can be established directly from (4.1) by differentiation.

A final difference between the two types of firms arises when the capitalist twin is making profits under system 1, for then Y^1 will in general exceed W_1[8] and Ψ will normally differ from Θ. It can be shown that[9]

$$\frac{\Theta/(1-\Theta)}{\Psi/(1-\Psi)} = \left[\frac{Y^1}{W_1}\right]^{1-\sigma} \qquad (4.4)$$

Equation (4.4) shows that Θ will exceed Ψ, if $Y^1 > W_1$ and $\sigma<1$; however, if $Y^1>W_1$ and if $\sigma>1$, Ψ must exceed Θ. Since capitalist firms normally earn profits (most firms are infra-marginal) and σ is widely believed to be less than one, we may argue that $\Theta>\Psi$ is the normal case.

---

[8]Vanek, op. cit., p. 33

[9]The first-order conditions imply that the ratio of the marginal products of labor and capital equal (W/r) in the case of the capitalist firm, and (Y/r) in the case of a labor-managed firm. Writing out the marginal products for the CES and manipulating, we obtain

$$\left(\frac{K}{L}\right)_C = \left(\frac{W}{r}\ \frac{\delta}{1-\delta}\right)^{\sigma} \quad \text{and} \quad \left(\frac{K}{L}\right)_{LM} = \left(\frac{Y}{r}\ \frac{\delta}{1-\delta}\right)^{\sigma}$$

where the subscripts C and LM refer to capitalist and labor-managed firms. We have suppressed the superscript referring to system 1. Hence

$$\frac{(K/L)_{LM}}{(K/L)_C} = \left(\frac{Y}{W}\right)^{\sigma}.$$

Now

$$\frac{\theta}{1-\theta} = \frac{r}{W}\left(\frac{K}{L}\right)_C \quad \text{and} \quad \frac{\psi}{1-\psi} = \frac{r}{Y}\left(\frac{K}{L}\right)_{LM}.$$

Hence

$$\frac{\theta/(1-\theta)}{\psi/(1-\psi)} = \left(\frac{Y}{W}\right)^{1-\sigma}$$

B. Monopoly

A comparison of the two types of firms under monopoly is somewhat more difficult because of the increased complexity of the analysis in both cases. To simplify the analysis and to bring out the differences between the shift-work behavior of the two types of firms, we shall assume that for the capitalist firm the degree of homogeneity of the production function (β) is constant, and the elasticity of demand (n) is also constant. For the labor-managed firm, it is not very useful to assume that both β and n are constant, because in that case equilibrium is not possible unless $\beta = n/(n-1)$. (See note 9, p. 14 above). Although the labor-managed and capitalist firms will not face identical environments (and therefore we do not speak of them as twins), some interesting comparisons can be made between the two counterpart firms.

Under these assumptions the cost ratio for the capitalist firm becomes[3]

$$CR = \frac{2^{1/\beta}}{(2+\alpha)} \left[\theta(2+\alpha)^{\sigma-1} + (1-\theta)\right]^{1/\sigma-1} \tag{4.5}$$

and the ratio of profits under system 2 to those under system 1 becomes

$$\Pi^2/\Pi^1 = CR^{\beta(n-1)/(n+\beta-\beta n)} \tag{4.6}$$

The ratio of the optimal outputs under systems 2 and 1 is given by

$$X^2/X^1 = CR^{\beta n/(n+\beta-\beta n)} \tag{4.7}$$

---

[3] The analysis of the capitalist firm under these assumptions is contained in C. Clague, "The Theory of Capital Utilization: Some Extensions," mimeo, April 1975, Section II.

The second-order condition is that $\beta<n/(n-1)$, which ensures that the exponents in both (4.6) and (4.7) are positive.

As under perfect competition the sign of the response of the pay-off function with respect to changes in the parameters α, σ, and Ψ(or Θ) is the same for the capitalist and the labor-managed firms. But the determination of the relative output levels, X^2/X^1, is quite different for the two types of firms. Under capitalism, anything which increases the cost ratio (while holding β and n constant) will increase X^2/X^1, as (4.7) shows. Thus a decrease in α or in σ, or an increase in Θ (as a result of either a technological change or a change in the price of capital) will increase CR and thereby increase X^2/X^1. Under labor management, on the other hand, neither X^1 nor X^2 will be affected by a change in α, σ, or Ψ.

This difference in response to changes of parameters implies a corresponding difference in the level of X^2/X^1. While it is rather unusual for X^2 to be less than X^1 under labor management (it requires β to increase as output expands), there is nothing unusual about X^2 being less than X^1 for the capitalist firm: All it requires is that shift-work be unprofitable.

The elasticity of demand also plays a different role in the shift-work decisions of the two types of firms. The capitalist firm can determine the more profitable system of operations without knowing the elasticity of demand; shift-work is profitable if and only if the cost ratio in (4.5) exceeds unity.[4] But for the labor-managed firm, an increase in the elasticity of demand can easily change shift-work from being disadvantageous to advantageous.

---

[4] This statement does not hold if β is not constant.

V. Concluding Remarks

In this essay we have endeavored to develop the theory of the long-run capital-utilization decision of the labor-managed firm under both perfect competition and monopoly. In both cases we found that shift-work is more advantageous the more capital intensive the production process (measured by Ψ), the higher is the elasticity of substitution (σ), and the lower is the shift differential (α). In addition, in the case of monopoly, a high elasticity of demand (η) and a low degree of economies of scale (β) were shown to increase the desirability of shift-work.

The profitability of shift-work for the capitalist firm is also increased by a higher capital intensity (measured by Θ), a higher σ, and a lower α. Moreover, if $\Theta=\Psi$ under perfect competition, the labor-managed firm and its capitalist twin always reach the same shift-work decision. But there are a number of striking differences in the shift-work behavior of the two types of firms. An increase in α does not affect either the relative amounts of capital (K^2/K^1) or the relative outputs (X^2/X^1) under the two systems for the labor-managed firm; such an increase would affect both for the capitalist firm. Under monopoly the relative outputs (X^2/X^1) for the labor-managed firm would also be unaffected by a change in σ or a change in the price of capital. For the capitalist firm, on the other hand, such changes would normally affect X^2/X^1 through their effect on the cost ratio. While it is quite unusual for X^2 to be less than X^1 for the labor-managed firm under monopoly, it is not so unusual for the capitalist firm: All it requires is that shift-work be unprofitable.

To conclude we raise several questions which our results suggest should be analyzed empirically and which may help elucidate the issue discussed in the introduction as to the explanation of the remarkable growth performance of the Yugoslav economy. What explains the high level of shift-work in Yugoslavia? Is this explained by the large size of factories? If so, what role does the elasticity of demand play in explaining these large factories? Is shift-work explained by a lower effective α? If so, would this be explained by governmental subsidies or because the preferences of managers are not given excessive weight? Is shift-work in Yugoslavia explained by a very large capital share? If so, what accounts for this large capital share? The answers to the above questions will provide the information necessary to answer the question of whether or not Yugoslav growth can be explained by higher levels of incentive for shift-work in a labor managed system. While the above questions involve a substantial research program, this study may provide a framework in which to embed the answers.

APPENDIX A -- Percentage Changes in the Income Ratio Due to a Percentage Change in σ (eq. 3.13).

Perfect Competition
H= 1.0

ψ	σ = .2	σ = .8	σ = 1.4
.3	0.0804	0.1278	0.2152
.5	0.2051	0.3782	0.9531
.7	0.4954	1.1903	12.4327

Monopoly
H= .8

ψ			
.3	0.0445	0.6010	0.0869
.5	0.1187	0.1876	0.3400
.7	0.3144	0.6347	2.2735

H= .6

ψ			
.3	0.0085	0.0091	0.0115
.5	0.0254	0.0315	0.5308
.7	0.0807	0.1176	1.3897

APPENDIX B -- Relationships Among X^1, X_1^2, and X^2.

Conditions (3.3) and (3.4) can be understood more clearly with the aid of a graph (Fig. 1). Here we plot the ratio $G(F)/F$, or average output per unit of input, and $G'(F)$, or the marginal product of F. The ratio of these two, $G'(F)\cdot F/G(F)$, is denoted $S(F)$, and is our measure of economies of scale. We have drawn $S(F)$ such that it rises and then falls as F increases in order to be consistent with the assumption of a technology of the first kind.

The equilibrium condition is that $S(F)$ be equal to $n_1/(n_1-1)$ for system 1 and $n_2/(n_2-1)$ for system 2. Let us denote $n_1/(n_1-1)$ by N^1 and $n_2/(n_2-1)$ by N^2. Note that N^1 is a function of output X^1, or G^1. The shape of $N^1(G)$ depends on the shape of the demand curve. If n_1 falls as output rises (as would be true , for example, of a linear demand curve), $n_1/(n_1-1)$ rises, or $dN^1/dG^1>0$. On the other hand, if n_1 rises as output rises, $dN^1/dG^1<0$. Now there is a simple relationship between the curve $N^1(G^1)$ and the curve $N^2(G_1^2)$ (see Fig. 1, lower portion). If $G_1^2=G^1=G^*$, then the value of $N^2(G^*)$ is simply the value of $N^1(2G^*)$. (This is because if $G^1=G_1^2$, then X^2 must be twice as large as X^1.) Therefore, if $N^1(G^1)$ is monotonically rising, $N^2(G_1^2)$ must lie above it. If $N^1(G^1)$ is monotonically falling, $N^2(G_1^2)$ must be below it. The N^1 and N^2 curves can be translated to the F axis by using $N[G(F)]$.

It is clear from the graph that if $S(F)$ is declining and $N^2[G(F_1^2)]$ lies above $N^1[G(F^1)]$, then F_1^2 must be less than F^1. Hence $X_1^2<X^1$, i.e., output per shift is less under system 2 than under system 1. Other shapes of the $S(F)$ and $N[G(F)]$ curves will produce different results.

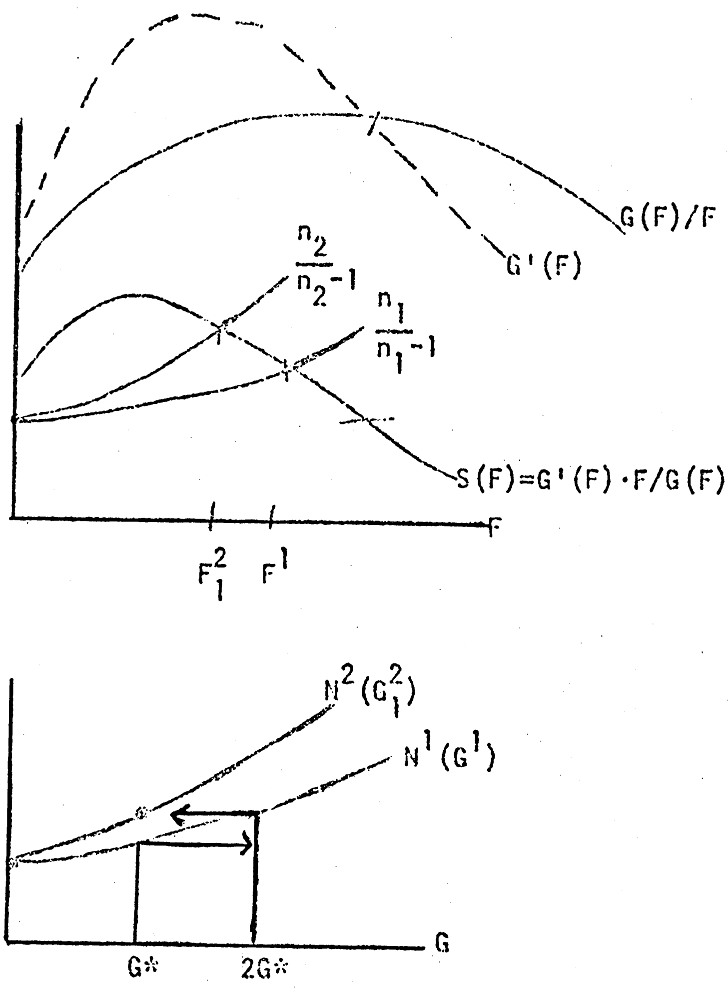

FIGURE 1

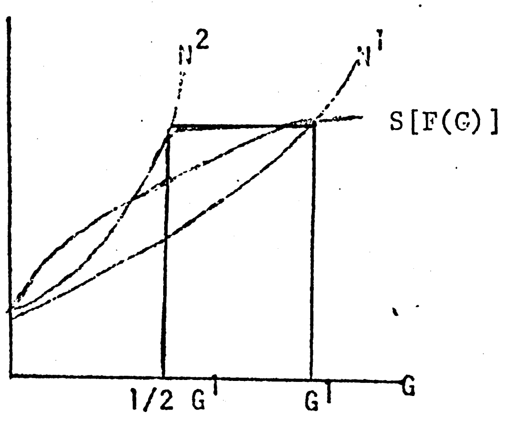

FIGURE 2

These other cases can be easily handled provided we first take note of the requirement imposed by the second-order condition, namely that at the point of intersection $dN/dF>S'(F)$.[5]

The second-order condition is clearly satisfied in the above case, when the $N[G(F)]$ curves are upward sloping and the $S(F)$ curve is downward sloping at the point of intersection. If the N curves are downward sloping (reflecting a demand curve for which the elasticity rises as output expands), they must be flatter than the S curve. In this case the N^2 curve would be below the N^1 curve and hence F_1^2 must be greater than F^1.

---

5. First we show that the maximization of Y implies the maximization of PX/F. By definition $Y=(PX-rK)/L$. Multiplying and dividing by F yields $Y=(PX/F - rK/F)F/L$. But the homogeneity of F implies that we may write $F=K.g(L/K)$. Hence

$$Y = \left[\frac{PX}{F} - \frac{r}{g(L/K)}\right]\frac{K}{L}\, g(L/K)$$

From this equation it can be seen that choosing X to maximize Y implies the maximization of PX/F.

Now let us treat PX/F as a function of G: $\phi(G)=P(G).G/F(G)$. The first-order condition for the maximization of ϕ is

$$\phi'(G) = \frac{F[P+G\frac{dp}{dG}]-P\cdot G\cdot F'(G)}{F^2} = \frac{P}{F}\left[\frac{n-1}{n} - \frac{G/F}{G'(F)}\right]$$

$$= \frac{P}{F}\left[\frac{1}{N(G)} - \frac{1}{S[F(G)]}\right] = 0$$

Hence $N(G)=S[F(G)]$. The second-order condition is

$$\phi''(G) = \frac{P}{F}\left[-\frac{N'(G)}{N^2} + \frac{dS/dG}{S^2}\right] + \left[\frac{1}{N(G)} - \frac{1}{S[F(G)]}\right]\frac{d(P/F)}{dG} < 0$$

Setting $N=S$ by virtue of the first-order conditions, this becomes

$$N'(G) > \frac{dS}{dG} = S'(F)\frac{dF}{dG}$$

Hence $\frac{dN}{dF} = N'(G)\cdot\frac{dG}{dF} > S'(F)$

since dG/dF must be positive.

If the S curve is upward sloping, the second-order condition requires that the N curves be upward sloping and be steeper than the S curve.[6] In this case, a graphical analysis can be used to show that $X^2<X^1$ (see Fig. 2). We translate the S(F) curve to the G axis by using S[F(G)]. In Fig. 2, we locate the intersection of $N^1(G^1)$ and S[F(G)] and thus find G^1. We can find a point on the $N^2(G_1^2)$ curve since we know $N^2(\frac{1}{2}G^1)=N^1(G^1)$. We draw the upward-sloping $N^2(G_1^2)$ curve through this point. It is clear that the intersection of $N^2(G_1^2)$ with S[F(G)] must lie where G_1^2 is less than $\frac{1}{2}G^1$. Hence $X^2<X^1$.

Turning to the two special cases considered in the text, if the production function has a constant degree of homogeneity β, then we may write $X^1=(F^1)^\beta$ and $X_1^2=(F_1^2)^\beta$ and the measure of economies of scale becomes

$$S(F^1) = \frac{G'(F^1)F^1}{G(F^1)} = \frac{\beta(F^1)^{\beta-1}F^1}{(F^1)^\beta} = \beta$$

That is, S is constant and becomes a horizontal line whether plotted against F or G. The second-order condition requires that the N curves slope upward at the point of intersection. Since S is constant, $N^1(G^1)$ must equal $N^2(G_1^2)$ and this implies that $G_1^2= \frac{1}{2}G^1$, or that $X^2=X^1$. Thus with a homogeneous production function, total output must be the same under the two systems.

The other special case arises when the demand curve has a constant elasticity. N^1 and N^2 are then constant and equal to each other. The

---

[6] Note that an upward-sloping S curve is not consistent with a technology of the first kind.

second-order condition requires that the S(F) curve slope downward at the point of intersection with the horizontal N[G(F)] curve. This intersection determines both F^1 and F_1^2, which must be equal to each other. A constant elasticity of demand therefore implies that output per shift will be equal under the two systems.

It is probably more common for elasticities of demand to decline as output expands than for the reverse to occur. And economies of scale within the firm probably tend to decline as output expands, in the region where most firms operate. If we accept these empirical assumptions, our model indicates that output per shift will be smaller but total output will be larger when the labor-managed monopolistic firm moves from one to two shifts, which is the case emphasized in the text.