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THE THEORY OF CAPITAL UTILIZATION AND THE PUTTY-RUBBER PRODUCTION FUNCTION

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The theory of capital utilization has been worked out on the assumption that the ex post elasticity of substitution is zero; that is, once the fixed capital is in place, the firm can no longer substitute labor for capital (Betancourt and Clague 1975), Winston (1974a). The theory has also been developed for the case where both the ex ante and the ex post elasticities of substitution are equal to one [Baily (1974), Millan (1974)]. The purpose of the present paper is to extend the theory of capital utilization to the presumably realistic case in which the ex post elasticity is greater than zero but is less than the ex ante elasticity.

In the theory of economic growth, economists have used the term "putty-putty" to describe models in which capital is equally malleable before and after installation, and "putty-clay" to depict the situation in which capital is malleable before installation but not afterwards. Extending this useful terminology, we shall use the term "putty-rubber" to characterize a production function in which the ex post elasticity of substitution (σ_2) is positive but is less than the ex ante elasticity (σ_1).[1] The image conveyed is that although the material out of which the machine is made is more malleable prior to construction, some malleability remains even after construction. Normally of course, ex post substitution between labor and capital does not take place by physically transforming the machine; ex post substitution involved assigning a larger or smaller number of workers to operate a given set of machines. Nevertheless, if the putty-clay image is accepted, putty-rubber seems a natural extension.

The literature on putty-rubber models is brief. The difference between ex ante and ex post substitution has been discussed in the context of technical change by Svennilson (1964) and in the context of short-run output fluctuations by Johansen (1972), and Hu (1972) has analyzed the behavior of a vintage-capital growth model with a putty-rubber production function. In another paper, Hu (1971) proposed a specific form for the ex post function: He employed a Cobb-Douglas ex ante function and a CES ex post function with σ_2 less than one. The present paper starts from Hu's basic conception of an ex post isoquant being tangent to an ex ante one, and we shall use a CES ex post function as one example. We shall point out, however, that the CES form is unsatisfactory for use as an ex post function, at least in the analysis of capital utilization, and we shall propose that a particular version of the variable elasticity of substitution (VES) production function be employed instead.

In the next section the necessary portions of the theory of capital utilization will be sketched. Section II will then compare the CES and the VES as ex post production functions in putty-rubber models. Section III shows how the model can be solved numerically and presents illustrative calculations of the effects of ex post substitutability on the profitability of shift-work. Some concluding comments are contained in Section IV.

I. A Sketch of the Theory of Capital Utilization

Prior to investment, the firm faces a decision as to the size of its fixed capital stock and its system of operations, i.e. whether to use a single-shift or a double-shift system. The theory is greatly simplified (without losing its essential points) if it is assumed that the firm must produce the same output under the two systems. (The theory is described in detail in Betancourt and Clague (1975). The output constraint is relaxed in the latter part of Betancourt and Clague (1975) and in Clague (1975)). The firm will maximize profits by choosing the system which minimizes costs for the given level of output. Shift-work (system 2) will be chosen if system 1 costs exceed system 2 costs, or

$$rK^1 + w_1L^1 > rK^2 + w_1L_1^2 + w_2L_2^2$$

where K is the stock of capital, L is the daily flow of labor services, r is the cost of owning the capital stock for one day, and w is the wage rate for a shift of labor. Throughout a superscript refers to the system of operations and a subscript refers to the shift. Under putty-clay assumptions, $L_1^2 = L_2^2$, but this will not be true if the production function is putty-rubber. Defining α as the night-shift wage premium ($\alpha = w_2/w_1 - 1$) and Θ as the share of capital costs in combined labor and capital costs under system 1 [$\Theta = rK^1/(rK^1 + w_1L^1)$], we divide the above inequality by its left-hand side and manipulate to obtain

$$1 > \left[\frac{rK^2}{w_1 L_1^2} + 1 + (1 + \alpha) \frac{L_2^2}{L_1^2} \right] \frac{L_1^2}{L^1} (1 - \theta)$$

The R.H.S. is the ratio of system 2 costs to system 1 costs and is called the cost ratio.

Shift-work is more profitable, the lower the cost ratio. With a CES putty-clay production function, it has been shown (Betancourt and Clague, 1975) that the cost ratio is reduced by an increase in Θ, and increase in σ, and a decrease in α. The cost ratio is greater than one (indicating that shift-work is unprofitable) if Θ is low and α is high. Baily (1974, chap. 2), on the other hand, in her putty-putty analysis of capital utilization, showed that with Cobb-Douglas ex ante and ex post production function (under constant returns to scale), double-shift shift operation is always profitable, no matter how high the night-shift premium nor how low the capital intensity of the production function. She developed her theorem for purposes of clarification and is fully aware of the lack of realism of her assumptions.

After discussing the CES and VES as ex post production functions in Section II, we shall develop expressions for K^2/L_1^2, L_2^2/L_1^2, and L_1^2/L^1, which can then be substituted into the cost ratio.

II. The CES and the VES Ex Post Production Functions

Let us suppose the ex ante production function is of CES form with constant returns to scale:

$$X = \gamma_1 [\delta_1 K^{-\rho_1} + (1 - \delta) L^{-\rho_1}]^{-\frac{1}{\rho_1}} \quad (1)$$

where $\sigma_1 = 1/(1+\rho_1)$ is the ex ante elasticity of substitution. The ex post production function yields the same output for some combination of capital and labor (say K_*, L_*); the isoquants of the two functions are tangent at this point, but the ex post isoquants exhibits a smaller elasticity of substitution (σ_2) at this point (see Fig. 1). An obvious candidate for the functional form of the ex post function is the CES:

$$X = \gamma_2 [\delta_2 K^{-\rho_2} + (1 - \delta_2) L^{-\rho_2}]^{-1/\rho_2} \quad (2)$$

where again for simplicity we assume constant returns to scale. If σ_2 (= $1/(1 + \rho_2)$) and the parameters of the ex ante function are given, the parameters δ_2 and γ_2 can be derived from the properties of the ex post function given above. The equality of the rates of technical substitution at the point K_*, L_* implies[2]

$$\frac{\delta_2}{1 - \delta_2} = \frac{\delta_1}{1 - \delta_1} \left(\frac{K_*}{L_*}\right)^{\rho_2-\rho_1} \quad (3)$$

from which δ_2 can be calculated. Then γ_2 can be calculated by setting the output in (1) equal to that in (2), for the inputs K_*, L_*.

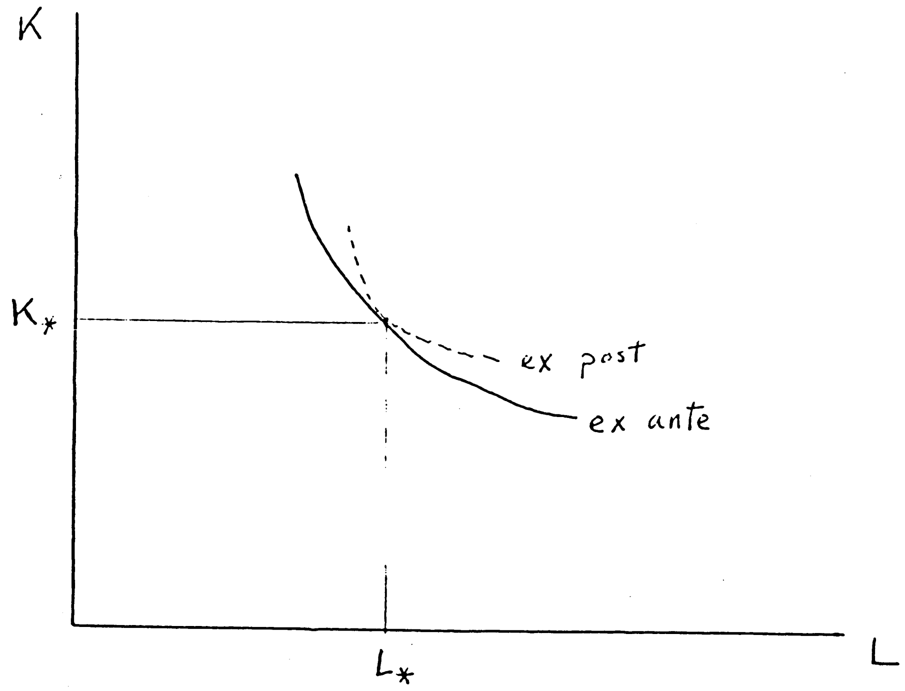

Fig. 1

$$\gamma_2 = \frac{\gamma_1 \, [\delta_1 \, (K_*)^{-\rho} + (1 - \delta_1)(L_*)^{-\rho_1}]^{-1/\rho_1}}{[\delta_2 \, (K_*)^{-\rho_2} + (1 - \delta_2)(L_*)^{-\rho_2}\,]^{-1/\rho_2}} \qquad (4)$$

A CES ex post function implies that the marginal product of labor can always be increased by reducing the number of workers employed to operate the capital stock K_*. Differentiating (2) yields

$$F_L(K_*,L) = \frac{1 - \delta_2}{\gamma_2^{\rho_2}}\left(\frac{X}{L}\right)^{1+\rho_2} = (1 - \delta_2)\gamma_2[\delta_2\left(\frac{K_*}{L}\right)^{-\rho_2} + (1 - \delta_2)]^{-\frac{(1+\rho_2)}{\rho_2}}$$

Whether σ_2 is greater or less than one, $F_L(K_*,L)$ can always be increased by reducing L. If σ_2 is less than one, as seems reasonable, $F_L(K_*,L)$ approaches a limit as L goes to zero. This implies that if the wage rate divided by the marginal revenue of output takes on an appropriate value, a profit-maximizing enterpreneur would employ a very small quantity of labor (perhaps just one worker) with his entire capital stock. This requires an unreasonably large degree of ex post substitutability; note that this property of the $F_L(K_*,L)$ curve is not altered by making σ_2 very small.

A reasonable condition to impose on the ex post function is that a certain minimum number of workers (L_{min}, less than L_*) be required to operate the capital stock K_*. This condition, which might be called the "limited substitutability condition", can be imposed by specifying that the $F_L(K_*,L)$ curve reach a maximum at $L = L_{min}$. A particular version of the VES function has this property, as will be explained below.

In the analysis of the profitability of shift-work it is particularly important to use a production function which sets reasonable limits on ex post substitution. With a CES function, the cost of fully utilizing the capital stock at night can always be reduced to insignificance by making the number of night workers very small. Night-time utilization of capital is never (significantly) unprofitable.[3] Such an absurd conclusion can be avoided by working with a production function which satisfies the "limited substitutability condition".

Consider now the VES production function proposed by Bruno (1962). (See also Lu and Fletcher (1968). A good exposition is in Nerlove (1967).) The Bruno function is

$$X = \gamma[\delta K^{-\rho} + (1 - \delta)(K/L)^{-\rho m} L^{-\rho}]^{-1/\rho} \qquad (5)$$

Again we assume constant returns to scale. If m = 0, this function collapses to the CES. It can be shown [see Nerlove (1967)] that σ, which is not constant, is given by

$$\sigma = \frac{1}{1 + \rho - \frac{\rho m}{S_k}} \qquad (6)$$

where S_k is the share of capital in combined labor and capital costs. The average products are

$$X/L = \gamma[\delta(K/L)^{-\rho} + (1 - \delta)(K/L)^{-\rho m}]^{-1/\rho} \qquad (7)$$

$$X/K = \gamma[\delta + (1 - \delta)(K/L)^{\rho(1 - m)}]^{-1/\rho}. \qquad (8)$$

The marginal products are

$$\frac{\partial X}{\partial L} = \frac{(1 - \delta)(1 - m)}{\gamma^{\rho}} \left(\frac{X}{L}\right)^{1+\rho} \left(\frac{K}{L}\right)^{-\rho m} \tag{9}$$

$$\frac{\partial X}{\partial K} = \frac{1}{\gamma^{\rho}} \left(\frac{X}{K}\right)^{1+\rho} [\delta + (1-\delta)m\left(\frac{K}{L}\right)^{\rho(1-m)}]. \tag{10}$$

(In equations (6) through (10), the corresponding CES formula is obtained by setting m = 0). The maximum of X/L can be found either by setting $\partial X/\partial K$ equal to zero or by setting $X/L = \partial X/\partial L$. In either case the critical capital-labor ratio, $(K/L)_{crit}$, is found to be

$$(K/L)_{crit} = [\delta/(1 - \delta)(-m)]^{1/\rho(1 - m)} \tag{11}$$

We shall assume that m is negative so that $(K/L)_{crit}$ will be defined. It is now clear that the Bruno function, with m negative,[4] satisfies the limited substitutability condition; for any given K_*, the minimum number of workers required to operate the entire capital stock can be found from (11).

III. Ex Post Substitution and the Profitability of Shift-Work

The putty-rubber production function, with a CES ex ante function and a Bruno-VES ex post function, will now be used to analyze the

the effects on the profitability of shift-work of ex post substitutability. Since our model does not permit an analytical solution, a numerical analysis will be provided.

We assume initial values of the factor-price ratio (w_1/r) and of the CES parameters δ_1 and σ_1. (With no loss in generality γ_1 is set equal to 1.) These imply initial values of the single-shift capital-labor ratio (K^1/L^1) and of the single-shift capital share (θ). (Alternatively we assume values for σ_1, θ and w_1/r and derive the corresponding δ_1 and K^1/L^1). The night-time wage differential (α, which equals $w_2/w_1 - 1$) is assumed to be given.

Let us define K^2 as the capital stock on the second shift and L_*^2 as the amount of labor at the point at which the ex ante and the ex post isoquants are tangent. We shall assume that σ_2 at the point (K^2, L_*^2) is known (remember that σ_2 changes as we move away from this point) and also that the parameter m is known. Then we shall find the parameters δ_2 and γ_2 by a more complicated version of the procedure described above in equations (3) and (4). Once all the parameters of both functions are known, then the amounts of labor and capital employed per unit of output on the first and second shifts of the double-shift system can be ascertained and the profitability of double-shift operation can be determined.

A rather complicated iterative procedure (Step 1) is used to find the parameters δ_2 and ρ_2 and the variables L_1^2/L_2^2, K^2/L_*^2, K^2/L_1^2, and

K^2/L^2_2. At the end of this procedure, we will know the information depicted in Fig. 2, but we will not know where on the rays for system 2 the factory will operate. That is, we will not know a variable such as L^2_1/L^1. This will be determined in Step 2. (The reader not interested in technical details may wish to skip to the end of Step 2 below.)

Step 1. We assume an initial value of L^2_1/L^2_2. Then the average wage rate on system 2, w_*, can be found from

$$\frac{w_*}{w_1} = \frac{1}{w_1}\,\frac{L_1^2\,(w_1) + L_2^2(w_2)}{L_1^2 + L_2^2} = \frac{L_1^2/L_2^2 + 1 + \alpha}{L_1^2/L_2^2 + 1} \tag{12}$$

At the point where the ex ante and ex post isoquants are tangent, $(K^2,\ L^2_*)$, the rate of technical substitution is equal to the average wage rate on system 2, w_*, divided by the system 2 price of capital services, r/2. Hence (see footnote 2 above) K^2/L^2_* can be found from

$$\frac{K^2}{L_*^2} = \left[\frac{\delta_1}{1-\delta_1}\;\frac{w_*}{w_1}\;\frac{w_1}{r/2}\right]^{\sigma_1} \tag{13}$$

Next the capital share (S_k) at the point of tangency can be determined from

$$S_k = \frac{(r/2)\,K^2}{(r/2)K^2 + w_*L_*^2} = \frac{\dfrac{r}{2w_1}\;\dfrac{K^2}{L_*^2}}{\dfrac{r}{2w_1} + \dfrac{w_*}{w_1}} \tag{14}$$

ρ_2 is then calculated from a modification of (6):

$$\rho_2 = \frac{1 - \sigma_2}{\sigma_2}\;\frac{S_k}{S_k - m} \tag{15}$$

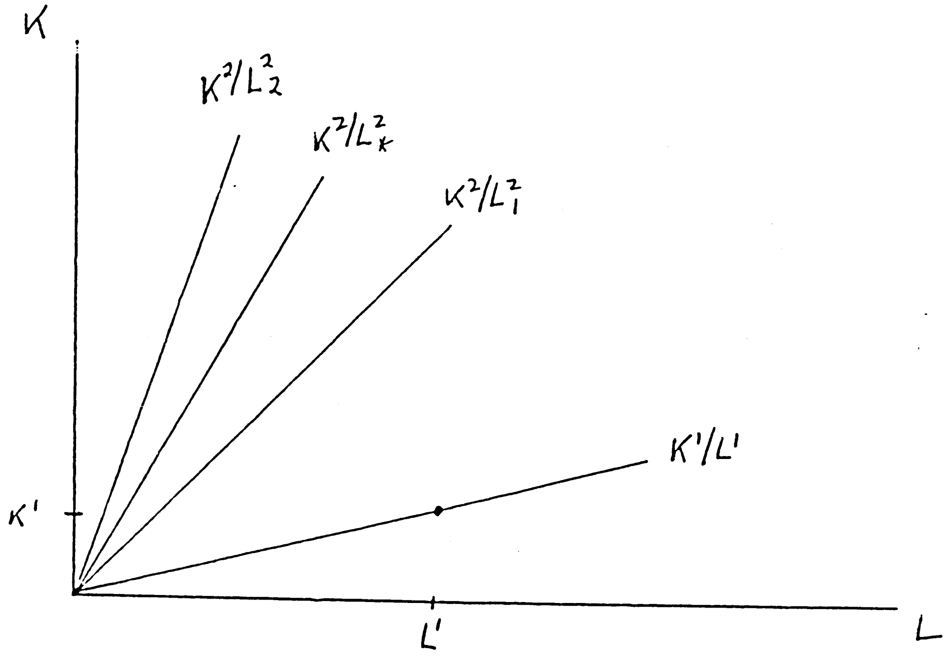

Fig. 2

In the ex post production function, an expression for the rate of technical substitution at the point (K^2, L_*^2) can be obtained by dividing (9) by (10). Setting this equal to $w_*/(r/2)$ yields

$$\frac{w_*}{r/2} = \frac{(1-m)(K^2/L_*^2)}{\frac{\delta_2}{1-\delta_2}\left(\frac{K^2}{L_*^2}\right)^{\rho_2(m-1)} + m} \tag{16}$$

This equation can be solved for δ_2.

Cost minimization under the double-shift system implies selection of K^2, L_1^2, and L_2^2 such that

$$\lambda \frac{\partial X_1^2}{\partial L_1^2} = w_1 ; \quad \lambda \frac{\partial X_2^2}{\partial L_2^2} = w_2 ; \quad \lambda \left[\frac{\partial X_1^2}{\partial K^2} + \frac{\partial X_2^2}{\partial K^2}\right] = r$$

where λ is the marginal cost of output. The first two of these conditions imply (see equation (9))

$$\frac{\partial X_2^2/\partial L_2^2}{\partial X_1^2/\partial L_1^2} = \left[\frac{Z_2^2}{Z_1^2}\right]^{-\frac{(1+\rho_2)}{\rho_2}} \left[\frac{K^2/L_2^2}{K^2/L_1^2}\right]^{1+\rho_2(1-m)} = \frac{w_2}{w_1} \tag{17}$$

where

$$Z_1^2 = \delta_2 + (1-\delta_2)(K^2/L_1^2)^{\rho_2(1-m)}$$

$$Z_2^2 = \delta_2 + (1-\delta_2)(K^2/L_2^2)^{\rho_2(1-m)}$$

The first and third of the cost minimization conditions imply

$$\frac{\dfrac{\partial X_1^2}{\partial K^2} + \dfrac{\partial X_2^2}{\partial K^2}}{\partial X_1^2/\partial L_1^2} = \frac{\delta_2 + (1-\delta_2)m\left[\dfrac{K^2}{L_1^2}\right]^{\rho_2(1-m)} + \left\{\delta_2+(1-\delta_2)m \;\dfrac{K^2}{L_2^2}^{\rho_2(1-m)}\right\}\left[\dfrac{Z_2^2}{Z_1^2}\right]^{-\frac{(1+\rho_2)}{\rho_2}}}{(1-\delta_2)m \;(K^2/L_1^2)^{1+\rho_2(1-m)}} = \frac{r_1}{w} \qquad (18)$$

Equations (17) and (18) can be solved for K^2/L_1^2 and K^2/L_2^2.

Equations (12) through (18) were used to calculate values of variables and parameters, on the assumption of a given value of L_1^2/L_2^2. This initial value can now be checked by

$$\frac{L_1^2}{L_2^2} = \frac{K^2/L_2^2}{K^2/L_1^2} \qquad (19)$$

If the value of L_1^2/L_2^2 determined in (19) is not equal to the initial value, another initial value can be selected and the calculations repeated until convergence is obtained. Step 1 has now been completed.

In the computer program implementing Step 1, K^2/L_2^2 was required to be less than $(K/L)_{crit}$. When the program attempted to select a value of K^2/L_2^2 greater than $(K/L)_{crit}$, the two values were set equal.[5] This ensures that L_2^2 will be greater than or equal to L_{min} and that the capital stock will be fully utilized on the second shift.

Step 2. In this step the variable L_1^2/L^1 will be determined. First the value of γ_2 is calculated from the equality of output per worker for the ex ante and the ex post functions at the point (K^2, L_*^2) (see (7) for the formula for X/L).

$$\gamma_2 = \frac{\left[\delta_1\left(\frac{K^2}{L_*^2}\right)^{-\rho_1} + (1 - \delta_1)\right]^{-1/\rho_1}}{\left[\delta_2\left(\frac{K^2}{L_*^2}\right)^{-\rho_2} + (1 - \delta_2)\left(\frac{K^2}{L_*^2}\right)^{-\rho m}\right]^{-1/\rho}} \tag{20}$$

Now X_1^2/L_1^2 and X_2^2/L_2^2 can be found (see (7) again). We require $X_1^2 + X_2^2$ to equal X^1. Hence

$$\frac{L_1^2}{L^1}\left[\frac{X_1^2}{L^1} + \frac{X_2^2}{L_2^2}\ \frac{L_2^2}{L_1^2}\right] = \frac{X^1}{L^1} \tag{21}$$

L_1^2/L^1 can be found from (21). This completes Step 2.

In Step 1 and Step 2 we have computed, for given values of (w_1/r), θ, σ_1, σ_2, α, and m, the values of L_1^2/L_2^2, K^2/L_1^2, and L_1^2/L^1 which are consistent with cost minimization and with the properties of the ex post production function (and with equal output for the two systems). These values can be substituted into the formula for the cost ratio (see Section I)[6]

$$CR = \left[\frac{rK^2}{wL_1^2} + 1 + (1 - \alpha)\frac{L_2^2}{L_1^2}\right]\frac{L_1^2}{L^1}(1 - \theta)$$

Table 1 shows the value of CR and L_1^2/L_2^2 for various values of the σ_1, σ_2, θ, and m. The two columns on the right of the table display the putty-clay model ($\sigma_2 = 0$). L_1^2/L_2^2 equals 1.0 in these cases. The two columns on the left give the CES ex post function. In the top panel of the table ($\sigma_1 = .4$) if $\theta = .30$, the cost ratio drops from .997 for $\sigma_2 = 0$ to .979 for $\sigma_2 = .2$. The reason the cost ratio falls is that the firm is able to economize on more expensive night-time labor. The middle columns show the Bruno-VES ex post function. The larger is (-m), the smaller is L_1^2/L_2^2, and this leads to a larger cost ratio.

Thus the Bruno-VES function constrains ex post substitution somewhat more than the CES ex post function. This difference becomes particularly important in those cases in Table 1 in which the cost ratio cannot be calculated for the CES function. The firm, if forced to work two shifts, will put the smallest possible number of workers on the night shift; L_1^2/L_2^2 goes toward infinity. With the Bruno-VES function, the firm must not let L_2^2 fall below L_{min}, the minimum number of workers required to operate the entire capital stock. Cases where $L_2^2 = L_{min}$ are marked with an asterisk in Table 1.

Table 1 also shows that an increase in σ_2, holding σ_1 constant, increases L_1^2/L_2^2 and decreases the cost ratio. (Compare the second and third panels in the table.)

The results can be summarized as follows. Ex post substitutability is increased by moving from a putty-clay model to a Bruno-VES ex post function (with $\sigma_2 > 0$) and then to a CES ex post function (with the same σ_2). Such substitutability is also increased by increasing σ_2. If the ex ante and ex post functions are both CES and $\sigma_2 = \sigma_1$,

Table 1. Shift-Work Profitability and

Ex Post Substitution

	$\sigma_1 = 0.4,\ \sigma_2 = 0.2$						$\sigma_1 = 0.4,\ \sigma_2 = 0$	
	CES(m=0)		VES(m=-.10)		VES(m=-.70)			
	C.R.	L_1^2/L_2^2	C.R.	L_1^2/L_2^2	C.R.	L_1^2/L_2^2	C.R.	L_1^2/L_2^2
θ = .10	n.a.	n.a.	1.119*	1.691*	1.122*	1.626*	1.163	1.0
.20	n.a.	n.a.	1.046*	1.780*	1.051*	1.649*	1.079	1.0
.30	.9791	1.660	.9800	1.570	.9807	1.518	.997	1.0
.40	.9079	1.365	.9081	1.350	.9082	1.341	.918	1.0
	$\sigma_1 = 1.4,\ \sigma_2 = 0.2$						$\sigma_1 = 1.4,\ \sigma_2 = 0$	
θ = .10	n.a.	n.a.	1.089*	1.787*	1.094*	1.651*	1.122	1.0
.20	n.a.	n.a.	.9993	1.368	.9995	1.371	1.011	1.0
.30	.9091	1.244	.9091	1.241	.9092	1.240	.915	1.0
.40	.8281	1.182	.8280	1.181	.8280	1.181	.832	1.0
	$\sigma_1 = 1.4,\ \sigma_2 = 0.6$						$\sigma_1 = 1.4,\ \sigma_2 = 0$	
θ = .10	n.a.	n.a.	1.025*	16.51*	1.030*	9.086*	1.122	1.0
.20	n.a.	n.a.	.9753	2.742	.9756	2.703	1.011	1.0
.30	.8967	1.942	.8967	1.939	.8968	1.933	.915	1.0
.40	.8209	1.660	.8209	1.659	.8209	1.658	.832	1.0

n.a. = not available because it could not be calculated.

*L_2^2 is set at the minimum level required to utilize the whole capital stock.

Notation: σ_1 = ex ante elasticity of substitution; σ_2 = ex post elasticity of substitution; CR = cost ratio (system 2 over system 1); L_1^2/L_2^2 = ratio of day-shift workers to night-shift workers on system 2. θ = share of capital on system 1. The night-shift wage differential (α) is 50 through out the table.

the model becomes putty-putty. All these increases in ex post substitutability increase the profitability of shift-work.

III. Summary

In this paper the theory of capital utilization has been extended from putty-clay to putty-rubber assumptions. The form of the ex post production function was found to be of some importance. If the ex post function is of CES form (with constant returns to scale), the degree of substitutability is so great, (even if σ_2 is low) that the firm can always reduce the cost of double-shift operation to insignificance by employing a very small amount of labor on the second shift. Thus double-shift operation can never be (significantly) unprofitable. When the ex post production function is of the Bruno-VES form, on the other hand, a certain minimum number of workers must be employed to operate the entire capital stock. Since double-shift operation is defined as the operation of the entire capital stock for two shifts, shift-work can be significantly unprofitable when the ex post function is Bruno-VES.

The profitability of shift-work is increased by an increase in the degree of ex post substitutability. This can be brought about either by an increase in σ_2, or with σ_2 constant by a switch from a Bruno-VES ex post production function to one of the CES form.

FOOTNOTES

1. It is clear from our discussion that we are referring to the instantaneous elasticity of substitution, rather than a concept of the elasticity which includes changes in the rate of utilization [Winston (1974 b), 155-156].

2. The rate of technical substitution for the CES is [see Henderson and Quandt (1971, p. 86)]:

$$\frac{\partial X/\partial L}{\partial X/\partial K} = \frac{1-\delta}{\delta}\left[\frac{K}{L}\right]^{1+\rho}$$

3. Recall Baily's theorem that when the ex post function is Cobb-Douglas with constant returns to scale, shift-work is always profitable. Her argument would also apply where the ex post function is CES with σ_2 greater than one.

4. The fact that there is some empirical evidence indicating that m is positive in ex ante production functions [e.g. Hildebrand and Liu (1965, pp. 36-39)] provides no argument against using a negative m in an ex post production function.

5. In this case $\lambda(\partial X_2^2/\partial L_2^2) = w_2$ no longer holds and hence neither does (17). The other equations (12) through (19) remain valid.

6. The cost ratio is independent of the assumed value of w_1/r. A formal proof of this proposition would be difficult. It was verified in our computer program by changing w_1/r. It is appropriate that the value of w_1/r should have no influence on the substantive results since w_1/r can be made equal to any value by an appropriate choice of units.

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