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THE THEORY OF CAPITAL UTILIZATION: SOME EXTENSIONS

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The Theory of Capital Utilization: Some Extensions

In an earlier paper (Betancourt and Clague, 1975) we developed a theory of the firm's joint decision on the size of the capital stock and its rate of utilization. A CES production function was used to show that the profitability of shift-work depends on the night-shift wage premium, the capital intensity of the production process, the prices of labor and capital, the elasticity of substitution between labor and capital, the magnitude of economies of scale, and the elasticity of demand. The present paper will discuss some controversies that have arisen about the theory of utilization and will extend our theory in some new directions.

In one section of our previous paper we assumed that the firm was constrained to produce the same output under both the single-shift and double-shift systems. This assumption was relaxed in a later section of the paper, where we provided a numerical analysis of the profitability of shift-work when output was free to vary. We pointed out that the condition for profitable shift-work was unaffected by the relaxation of the output restraint if the degree of homogeneity of the production function was constant, but we did not give an explicit analysis of this case. Here we shall provide such an analysis. In the process we respond to Millan's claim that the theory of shift-work is drastically altered by the relaxation of the output constraint.

The shift-work profitability condition presented in our earlier paper differs somewhat from the profitability conditions given by Winston (1974) and Millan (1974) in the treatment of the capital intensity variable. We have worked with the share of capital cost in value added under single-shift operation, while the other two authors worked with the distribution parameter of the CES production function and the factor-price ratio. Here we shall compare the merits of these two approaches, paying particular attention to the data requirements for empirical application of each approach.

Finally, our theory was based on the assumption that the maintenance requirements and depreciation of fixed capital did not depend on the number of shifts worked. Here we shall generalize our theory to incorporate a variety of assumptions about depreciation and maintenance.

Section I summarizes the basic model and discusses the two approaches to the treatment of capital intensity. Section II relaxes the output restraint in the case where the degree of homogeneity of the production function is constant, while in section III depreciation and maintenance are made to depend on the number of shifts worked.

I. The Basic Model and the Two Approaches to Capital Intensity.

The notation is the same as in our earlier paper. Let us define

K = stock of fixed capital

L = daily flow of labor services

r = cost of owning a unit of capital stock for a day

w_1 = wage rate for a day-time shift

A superscript refers to the system of operations (1 for one-shift system, 2 for double-shift system) and a subscript (1 or 2) refers to the day or night shift of system 2. Thus K^2/L_1^2 , for example, represents the capital-labor ratio on the first shift of the double-shift system. In the basic model the firm is constrained to produce the same output on the two systems. Shift-work is profitable, therefore, only when total costs of system 2 are less than total costs of system 1, or

$$rK^1 + w_1L^1 > rK^2 + w_1L_1^2 + w_2L_2^2$$

where w_2 refers to the night-time wage. Since *ex post* substitution between labor and capital is rules out, $L_1^2 = L_2^2$. Let us define α as the night-shift wage premium ($\alpha = w_2/w_1 - 1$) and θ as the share of capital costs in combined labor and capital costs under system 1 ($\theta = rK^1/(rK^1 + w_1L^1)$). Then, dividing the above equation by its left-hand side and simplifying yields:

$$1 > [\frac{rK^2}{w_1L^2} + (2+\alpha)]\frac{L_1^2}{L^1}(1-\theta) \qquad (1.1)$$

The R.H.S. of (1.1) is the ratio of costs of system 2 to those of system 1; it is called the cost ratio (CR). The cost of capital r equals

$P_k(i+d)$, where i is the interest rate, d is depreciation, and P_k the cost of a standard machine. Depreciation has been assumed independent of the system of operations. This assumption will be relaxed in section III.

The production function is assumed to be of the CES form. Output under system 1 is

$$X^1 = [\delta(S^1)^{-\rho} + (1-\delta)(L^1)^{-\rho}]^{-\beta/\rho}$$

where β is the degree of homogeneity, $\sigma = 1/(1+\rho)$ is the elasticity of substitution, and S^1 is the flow of capital services under system 1. We define the utilization rate u from the equation $S=uK$. The maximum rate of utilization within a shift is denoted u^*. Under our assumption of no wear-and-tear depreciation, the firm will utilize all its capital within a shift; hence $S^1=u^*K^1$. For system 2,

$$X^2_1 = [\delta(S^2_1)^{-\rho} + (1-\delta)(L^2_1)^{-\rho}]^{-\beta/\rho} = X^2_2$$

where $S^2_1 = u^*K^2$. Cost minimization implies selection of the following capital labor ratios (see Henderson and Quandt, p. 86):

$$\frac{K^1}{L^1} = \left[\frac{w_1}{r} \frac{\delta}{1-\delta}\right]^{\sigma} (u^*)^{\sigma-1} \tag{1.2}$$

$$\frac{K^2}{L^2_1} \left[\frac{(w_1+w_2)/2}{r/2} \frac{\delta}{1-\delta}\right]^{\sigma} (u^*)^{\sigma-1} = \frac{K^2}{L^2_2} \tag{1.3}$$

Winston (1974) and Millan (1974) presented their shift-work profitability condition in terms of δ and w_1/r. There is nothing logically incorrect about this procedure but it may give rise to confusion in empirical application because it is not obvious what value one should take for δ. δ of course is an economists' construct; it is estimated from data rather than observed directly. The value of δ, moreover, is arbitrary, for any value of δ is consistent with the same set of real world data. This is because δ depends on the units in which the factors are measured, and while labor comes in natural units, the same is not true of machines. Since the price of the standard machine, P_k, is set arbitrarily, the price of capital, r, is also arbitrary. δ is actually estimated from (1.2) or (1.3), after the value of σ has been estimated or assumed. The choice of units for P_k affects both r and the stock of capital K, and this choice of units affects δ (unless σ=1, in which case δ equals the share θ).[1/]

The problem of estimating δ can be avoided entirely by working with θ, as in Betancourt and Clague (1975). First divide (1.3) by (1.2) to obtain

$$\frac{K^2}{L_1^2} = \frac{K^1}{L^1} (2+\alpha)^\sigma$$

Since $rK^1/w_1L^1 = \theta/(1-\theta)$, we have

$$\frac{rK^2}{w_1L_1^2} = \frac{\theta}{1-\theta} (2+\alpha)^\sigma \qquad (1.4)$$

An expression can also be developed for L_1^2/L^1 in terms of θ, namely[2/]

$$\frac{L_1^2}{L^1} = 2^{-1/\beta}[\theta(2+\alpha)^{\sigma-1} + (1-\theta)]^{\sigma/(1-\sigma)} \tag{1.5}$$

Now substitute (1.4) and (1.5) into (1.1) to obtain

$$1 > 2^{-1/\beta}(2+\alpha)\,[\theta(2+\alpha)^{\sigma-1} + (1-\theta)]^{\frac{1}{1-\sigma}} \tag{1.6}$$

The R.H.S. of (1.6) is the cost ratio. Since neither r nor δ appear, none of the variables in the cost ratio depends on units of measurement.

We are frequently interested in the effects on shift-work of factor prices, since these prices can easily be influenced by governmental policy. In our analysis, factor prices affect the profitability of shift-work through their effects on θ. As is well known, a fall in w_1/r will raise θ if $\sigma<1$, but will lower θ if $\sigma>1$.

In the discussion up to this point, the choice between using θ or δ in the shift-work profitability condition is a matter of presentation but not of substance. There is another issue involved here, however, which is more substantive. We are interested in calculating the effects on the profitability of shift-work of a change in σ. Since there is a great deal of uncertainty about the values of σ in the real world, we would like to be sure that any conclusions we might reach about the profitability of shift-work would hold up under a variety of values for σ. The question arises as to what should be held constant when σ is changed. If δ is held constant, it can be seen from (1.2) that a change in σ implies a change in the capital-labor ratio at the initial set of factor prices. Such a procedure does not isolate the effects of a change in σ.

On the other hand, holding θ constant does hold K^1/L^1 constant at the initial set of factor prices and hence isolates the effects of the elasticity of substitution.

II. Relaxing the Output Restraint When the Degree of Homogeneity of the Production Function is Constant.

In the basic model the firm is constrained to produce the same output under single-shift and double-shift operation. This assumption might be rationalized by supposing that the firm is an oligopolist facing a kinked demand curve. If we assume instead that the firm faces a smooth demand curve, then normally output will be different under the two systems and the theory of shift-work must be generalized to accommodate this fact.

Patricio Millan has asserted that the conclusions derived from the basic model are radically altered when the output restraint is relaxed. He claims that it is no longer true that multiple-shift plants are always favored by an increase in capital intensity (δ), a decrease in the night-shift wage premium (α), and an increase in economies of scale (β) (Millan, pp. 89, 95, 98, 111). It will be shown here that his conclusions are wrong with respect δ and α ; that is, these variables have the same qualitative effects on the profitability of shift-work when the ou put restraint is relaxed in the manner indicated below. We do find that there a conditions under which an increase in β favors shift-work, and unlike Millan we provide an economic rationale for these results.[3/]

The analysis is greatly simplified if it is assumed that the demand curve has a constant elasticity of demand (n) and the production function has a constant degree of homogeneity (β). The latter assumption is not fully satisfactory, for it implies that the average cost curve approaches zero asymptotically as output expands, but it seems nevertheless to be a useful approximation for theoretical purposes.

The following notation will be employed:

Π^1, Π^2 = profits under systems 1 and 2

TC^1, TC^2 = total costs under systems 1 and 2

X^1, X^2 = optimal outputs under systems 1 and 2

$TC^2(X^1)$, for example, refers to total costs under system 2 at output level X^1

$TR(X)$ = total revenue at output X

We further define $m=\Pi^1/TR(X^1)$ as the profit margin under system 1. Now

$$m = \frac{TR(X^1) - TC^1(X^1)}{TR(X^1)} = \frac{AR(X^1) - AC^1(X^1)}{AR(X^1)}$$

where AR is average revenue and AC is average cost. It is well known that $MR=AR(1-1/n)$, where n is the (absolute value of the) elasticity of the demand curve and MR is marginal revenue. Similarly $MC=AC(1-1/\varepsilon)$, where ε is the (absolute value of the) elasticity of the average cost curve and MC is marginal cost. It can be shown that $\varepsilon = \beta/(\beta-1)$, where β is the (constant) degree of homogeneity of the production function. 4/

Therefore $AC=MC(\varepsilon-1)/\varepsilon = MC(\beta)$.

Hence

$$m = 1 - \frac{AC^1(X^1)}{AR(X^1)} = 1 - \frac{MC^1(X^1).\beta}{MR(X^1)n/(n-1)} = 1 - \frac{\beta(n-1)}{n} \qquad (2.1)$$

This is, the profit margin is reduced by an increase in β or an increase in n. Now the second-order condition for profit maximization is that the MR must cut the MC from above, or that the elasticity of the MR curve be less than the elasticity of the MC curve. The elasticity of the MR curve is n and that of the MC curve is $\beta/(\beta-1)$. Thus the second-order condition is that $n<\beta/(\beta-1)$ or $\beta<n/(n-1)$. In our simple model, then, with both β and n constant, the second-order conditions imply that the profit margin m be positive.

Next we develop an expression for $\Pi^2/TR(X^1)$.

$$\frac{\Pi^2}{TR(X^1)} = \frac{TR(X^2)-TC^2(X^2)}{TR(X^1)} = \frac{TR(X^2)}{TR(X^1)} - \frac{TC^2(X^2)}{TC^2(X^1)} \; \frac{TC^2(X^1)}{TC^1(X^1)} \; \frac{TC^1(X^1)}{TR(X^1)} \qquad (2.2)$$

We show in the appendix that

$$\frac{TR(X^2)}{TR(X^1)} = \left[\frac{X^2}{X^1}\right]^{1-\frac{1}{n}} \qquad (2.3)$$

$$\frac{TC^2(X^2)}{TC^2(X^1)} = \left[\frac{X^2}{X^1}\right]^{1/\beta} \qquad (2.4)$$

$$\frac{X^2}{X^1} = ICR^{\frac{\beta n}{n+\beta-\beta n}} \qquad (2.5)$$

where ICR is the inverse of the cost ratio, or $ICR = TC^1(X^1)/TC^2(X^1)$. Note that with β constant, ICR is independent of the level of output. Substituting these expressions into (2.2) gives

$$\frac{\Pi^2}{TR(X^1)} = ICR^{\frac{\beta(n-1)}{n-\beta-\beta n}} - ICR^{\frac{n}{n+\beta-\beta n}} \quad ICR^{-1} \ (1-m)$$

$$= ICR^{exp} - ICR^{exp} \ (1-m)$$

$$\frac{\Pi^2}{TR(X^1)} = m \ [ICR^{exp}] \tag{2.6}$$

where $exp=\beta(n-1)/(n+\beta-\beta n)$. Since $m=\Pi^1/TR(X^1)$, we have

$$\frac{\Pi^2 - \Pi^1}{TR(X^1)} = m[ICR^{exp} - 1] \tag{2.7}$$

$$\text{and} \quad \frac{\Pi^2}{\Pi^1} = ICR^{exp} \tag{2.8}$$

$(\Pi^2 - \Pi^1)/TR(X^1)$ and Π^2/Π^1 are two measures of the profitability of shift-work. We shall use (2.7) and (2.8) to show the effects of changes in the parameters. The two measures behave similarly with respect to most parameter changes, but there are some differences which will be noted below. We shall start with $(\Pi^2-\Pi^1)/TR(X^1)$.

The first point to note about (2.7) is that the condition $\Pi^2-\Pi^1>0$ is equivalent to the condition that the cost ratio (CR) be less than one. This follows immediately from the fact that the second-order conditions imply that both m and exp are positive. Hence $(\Pi^2-\Pi^1)$ will be positive if and only if ICR exceeds unity, which implies that the cost ratio be less than one.

The second point, also quite straightforward, is that the relaxation of the output restraint in no way alters the conclusion from the basic model that $(\Pi^2-\Pi^1)/TR(X^1)$ is increased by an increase in θ, an increase in σ, or a decrease in α. This follows from the fact that θ, σ, and α all enter (2.7) through ICR, and since m and exp are both positive, anything which increases ICR must increase the R.H.S. of (2.7). ICR, the relative cost of system 1, is increased by an increase in θ , an increase in σ, and a decrease in α (see equation (1.6)).[5/] (Note that the first two points about $(\Pi^2-\Pi^1)/TR(X^1)$ apply equally to Π^2/Π^1.)

The third point about (2.7) is that the effects of β on $(\Pi^2-\Pi^1)/TR(X^1)$, are no longer unambiguous. Under certain conditions an increase in β will increase $(\Pi^2-\Pi^1)/TR(X^1)$, rather than decrease it, as we are led to expect from the basic model. The reasons for this are somewhat complicated and will be taken up after the next paragraph.

For the analysis of the effects of n, it is more convenient to use Π^2/Π^1. An increase in n increases exp in (2.8). Therefore an increase in n pushes Π^2/Π^1 further from unity, increasing Π^2/Π^1 when that ratio exceeds one and reducing Π^2/Π^1 when that ratio falls short of one.

The effects of an increase in β on our two measures of profitability of shift-work can be seen in Table 1. We start with a given level of ICR*, or ICR under constant returns to scale. Since

$$ICR = 2^{\frac{1}{\beta}-1} \cdot 2 \cdot (2+\alpha)^{-1}[\theta(2+\alpha)^{\sigma-1} + (1-\theta)]^{\frac{1}{\sigma-1}} = 2^{\frac{1}{\beta}-1} \cdot ICR^*$$

TABLE 1

Effects of β on Π^2 and Π^1

	$\beta=1.05$	$\beta=1.15$	$\beta=1.25$	$\beta=1.35$	$\beta=1.45$
I. $ICR^* = .80$ 1/					
ICR 2/	.774	.731	.696	.668	.645
$\Pi^1/TR(X^1)=m$	.300	.233	.167	.100	.033
$\Pi^2/TR(X^1)$ 3/	.165	.083	.027	.003	.000
$(\Pi^2-\Pi^1)/TR(X^1)$	-.135	-.150	-.139	-.097	-.033
Π^2/Π^1	.550	.357	.164	.027	.000
II. $ICR^* = 1.00$ 1/					
ICR 2/	.968	.914	.871	.836	.806
$\Pi^1/TR(X^1)=m$	.300	.233	.167	.100	.033
$\Pi^2/TR(X^1)$ 3/	.278	.173	.083	.020	.000
$(\Pi^2-\Pi^1)/TR(X^1)$	-.022	-.060	-.083	-.080	-.033
Π^2/Π^1	.926	.743	.500	.198	.002
III. $ICR^*=1.20$ 1/					
ICR 2/	1.161	1.096	1.045	1.003	.968
$\Pi^1/TR(X^1)=m$	.300	.233	.167	.100	.033
$\Pi^2/TR(X^1)$ 3/	.425	.316	.207	.102	.013
$(\Pi^2-\Pi^1)/TR(X^1)$	.125	.082	.041	.002	-.020
Π^2/Π^1	1.417	1.353	1.244	1.024	.386
IV. $ICR^* = 1.40$ 1/					
ICR 2/	1.355	1.279	1.219	1.170	1.129
$\Pi^1/TR(X^1)=m$	.300	.233	.167	.100	.033
$\Pi^2/TR(X^1)$ 3/	.609	.524	.448	.410	1.125
$(\Pi^2-\Pi^1)/TR(X^1)$	.309	.290	.282	.310	1.092
Π^2/Π^1	2.030	2.245	2.689	4.100	33.76

Notes

1. $ICR^* = 2(2+\alpha)^{-1}[\theta(2+\alpha)^{\sigma-1} + (1-\theta)]^{1/(\sigma-1)}$
2. $m = 1 - \beta(n-1)/n$
3. $\Pi^2/TR(X^1) = m \cdot ICR^{\exp}$

$n=3$ throughout the table.

an increase in β with ICR* constant makes ICR fall. This result, familiar from the basic model, is shown in row 1 of Table 1. An increase in β also reduces the profit margin under system 1, since $m=1-\beta(n-1)/n$ (equation (2.1)), as shown in row 2. Normally as β rises, $\Pi^2/TR(X^1)$ $(=m.ICR^{exp})$ falls since both m and ICR fall. However, since

$$exp=\frac{\beta\ (n-1)}{n-\beta(n-1)}=\frac{1}{\frac{1}{\beta}\frac{1}{n-1}-1}$$

an increase in β increases exp, and in the bottom panel of Table 1 we see that when β goes from 1.35 to 1.45, $\Pi^2/TR(X^1)$ actually increases. What is the economics underlying this result? An increase in β makes the AC and MC curves steeper and increases the optimal levels of output X^1 and X^2. In the bottom panel of Table 1, AC^2 lies below AC^1 and although the increase in β makes the two curves closer for a given output, the steepening of the curves causes X^2/X^1 to rise (This can also been seen from (2.5)). When the increase in X^2/X^1 is large enough, it is quite possible for $\Pi^2/TR(X^1)$ to rise.

The value of $(\Pi^2-\Pi^1)/TR(X^1)$ is shown in the fourth row of Table 1. This expression rises when β increases under two different sets of circumstances. (a) In the top two panels, $\Pi^2$is less than Π^1; an increase in β reduces both $\Pi^1/TR(X^1)$ and $\Pi^2/TR(X^1)$, but the former is larger in absolute value and falls by a larger absolute amount. (b) In the bottom panel, where $\Pi^2>\Pi^1$, the fall in $\Pi^2/TR(X^1)$ as β increases is moderated by the increase in X^2/X^1, with the result that $(\Pi^2-\Pi^1)/TR(X^1)$ sometimes increases.

The effects of β on Π^2/Π^1 are not quite so complicated. In the top three panels Π^2/Π^1 always falls as β increases, while in the bottom panel Π^2/Π^1 uniformly rises with β. The reason for the rise is the same as (b) in the previous paragraph, namely, that as β increases X^2/X^1 increases when ICR is large. The precise conditions under which this will occur can be obtained by differentiating (2.8).[6/]

$$\frac{\partial(\Pi^2/\Pi^1)}{\partial\beta} = ICR^{\exp} \quad \frac{(n-1)}{[n-\beta(n-1)]^2} \{n \ln ICR^* - \ln 2\}$$

All the terms in the derivative are necessarily positive except the term { }. Thus the sign of the derivative depends on this term, which will be negative if

$$(ICR^*)^n < 2$$

This condition is always satisfied if ICR* is less than one. It will be violated only if ICR* and n are both large enough. If ICR*=1.26, this occurs when n exceeds 3 and if ICR* = 1.41, this occurs when n exceeds 2.

In summary, the relaxation of the output restraint in the manner of this section leaves a number of the conclusions of the basic model intact. It remains true that the profitability of shift-work, whether measured by $(\Pi^2-\Pi^1)/TR(X^1)$ or Π^2/Π^1, is increased by an increase in θ, an increase in σ, and a decrease in α. It also remains true that the condition $\Pi^2>\Pi^1$

holds if and only if the cost ratio is less than one. What does change is that $(\Pi^2-\Pi^1)/TR(X^1)$ and Π^2/Π^1 may be increased by an increase in β under certain circumstances. For Π^2/Π^1 to increase with β, ICR* must be fairly large ($[ICR^*]^{n}>2$, to be exact). The conditions under which $(\Pi^2-\Pi^1)/TR(X^1)$ is increased by β are not easily summarized, but this much can be said: An increase in β can never change a situation in which $\Pi^1>\Pi^2$ into a situation in which $\Pi^2>\Pi^1$.[7/] Finally, the relaxation of the output constraint brings the elasticity of demand explicitly into the analysis; the effects of n can be summarized by saying that an increase in n pushes Π^2/Π^1 farther from unity.

III. A More General Treatment of Depreciation and Operating Costs.

In our previous analysis the cost of capital r, defined as the cost of owning and operating a unit of capital stock for year, was assumed to be the same for the single-shift and double-shift systems. This assumption could be justified by supposing that depreciation was due entirely to obsolescence (and not at all to wear and tear), and by ignoring machine-operating costs, such as maintenance, repair, and fuel.

Baily (1974, p. 35) has introduced an "hourly use-related cost of capital", which is the same for single-shift and double-shift operation. Here we shall assume that hourly operating costs are a rising function of total hours of machine use. Machine life will become endogenous, since machines will be replaced when their operating costs make them more expensive than new ones. Under either our assumptions or Baily's, the cost of capital for system 2 (r_2) will be higher than the cost for system 1 (r_1).

In section A below we shall make some specific assumptions about operating costs, depreciation, and the interest rate and calculate some illustrative values of r_2/r_1. Then in section B the cost ratio will be generalized to incorporate the term r_2/r_1. Finally, we shall see how much this changes the cost ratio.

A. Illustrative Values of $r_2/r1$.

One approach to the firm's decision problem is to assume maximization of the present value of profits over the life of the asset. An equivalent approach, which is more convenient for our purposes, is to convert all revenues and costs into constant annual flows. The constant annual equivalent f of a variable level of costs (F_t) can be computed by equating the present value of the two flows over the life of the asset (n):

$$\sum_{t=1}^{n} f(1+i)^{-t} = \sum_{t=1}^{n} F_t(1+i)^{-t}$$

Hence $$f = (\Sigma F_t(1+i)^{-t}) / \Sigma(1+i)^{-t}$$

The annual cost of capital r is defined as $P_k(i+d+c)$, where P_k is the price of a standard machine, i is the interest rate, d is the depreciation rate, and c is the rate of operating costs. The annual depreciation change on a machine costing \$1000 with life of 10 years would be \$1000 (d) where d is computed from

$$\sum_{t=1}^{10} d\,(1+i)^{10-t} = 1.00$$

Here d equals .076, which means that if \$76 is set aside every year and invested at 6%, it will cumulate to \$1000 in 10 years. The combined interest and depreciation costs would be \$1000(i+d)=\$1000(.06+.076) =\$136.

Operating costs include maintenance, repair, and fuel. We shall include under this rubric the costs associated with the breakdown of machines.[8/] It seems quite reasonable to assume that operating costs, especially the component related to machine failure, will increase sharply with cumulated hours of operation. For purposes of our illustration, we shall assume that operating costs for a standard machine (OC) are a quadratic function of cumulated hours of operation (H):

$$OC = b\left(\frac{H}{2000}\right)^2$$

where b is a constant which will be allowed to take on various values. H is divided by 2000 to simplify the arithmetic. We shall assume that a machine is normally operated 2000 hours a year under system 1 and 4000 hours a year under system 2. In this case, operating costs in year t for system 1 (OC_t^1) and for system 2 (OC_t^2) become $OC_t^1 = bt^2$ and $OC_t^2 = b(2t)^2 = 4bt^2$.

The average annual operating cost over the life of the asset (\$1000.c, where \$1000 is the price of the machine) is computed by equating the present value of the actual stream of operating costs with the present value of a constant stream of costs:

$$\sum_{t=1}^{n} \frac{1000c}{(1+i)^t} = \sum_{i=1}^{n} \frac{OC_t}{(1+i)^t}$$

Average operating costs for system 1 are a function of i and of asset life n:

$$1000.\ c_1(i,n) = b\left[\sum_{t=1}^{n} \frac{t^2}{(1+i)^t}\right] \Big/ \sum_{t=1}^{n} \frac{1}{(1+i)^t}$$

Similarly, average operating costs for system 2 are

$$1000.\ c_2(i,n) = 4b\left[\sum_{t=1}^{n} \frac{t^2}{(1+i)^t}\right] \Big/ \sum_{t=1}^{n} \frac{1}{(1+i)^t}$$

In Table 2 we show values of c_1 and c_2 as functions of n. i is set at .06 and b is set at 1.0. The table shows that as n increases, both c_1 and c_2 increase, and that at each n, c_2 is greater than c_1. Now for each system of operations, the life of the asset is selected so as to minimize the average annual cost of capital, $r=P_k(i+d+c)$. An increase in n lowers d but raises c. The optimal n_1 is 12 years (r_1=\$164.7) and the optimal n_2 is 7 years (r_2=\$251.8). (See Table 2). r_2/r_1 thus equals 1.53.

Table 3 shows the values of r_1, r_2 and r_2/r_1, for various values of i and b. A higher interest rate increases the optimal life of assets, because it lowers the present value of high operating costs at the end of the asset's life. This increases the relative weight of i in (i+d+c), which raises r_1 more than r_2; thus r_2/r_1 falls as i increases.

TABLE 2

The Optimal Life of Assets Under System 1 and System 2

n	$1000c_1(n)$	$1000c_2(n)$	$1000d(n)$	$r1(n)$	$r2(n)$
6	13.99	55.96	143.4	217.4	259.3
7	18.16	72.64	119.1	197.3	$\boxed{251.8}$
8	22.79	91.17	101.0	183.8	252.2
9	27.86	111.43	87.0	174.9	258.5
10	33.33	133.32	75.9	169.2	269.2
11	39.19	156.75	66.8	166.0	283.5
12	45.40	181.60	59.3	$\boxed{164.7}$	300.9
13	51.95	207.78	53.0	164.9	320.7

Note: $i = .06$ throughout.

c = operating costs, d= depreciation, $r=1000(i+d+c)$

TABLE 3

Cost of Capital Under Various Values of i and b

A. i=.06

	r1	n1	r2	n2	r2/r1
b=.6	143	15	214	9	1.50
b=1.0	165	11	252	7	1.53
b=1.8	196	10	304	6	1.55

B. i=.10

	r1	n1	r2	n2	r2/r1
b=.6	165	16	235	9	1.42
b=1.0	186	13	272	8	1.46
b=1.8	217	10	325	6	1.50

C. i=.14

	r1	n1	r2	n2	r2/r1
b=.6	191	17	257	10	1.35
b=1.0	211	14	294	8	1.40
b=1.8	240	11	348	6	1.45

A higher value of b increases operating costs in every year, shortens the optimal life of assets and raises r_2 more than r_1.

In our scheme, it is not optimal for the firm to operate the second shift at less than full capacity. The argument proving this assertion is rather complicated and will be relegated to a footnote. 9/

Summary. In this section we have calculated illustrative values for r_2/r_1, on the assumption that operating costs are proportional to the square of cumulated hours of machine use. With $i=.06$, r_2/r_1 is about 1.5. Let us note the effects of two changes in assumptions.

(a) In our scheme, no allowance was made for technical improvement in machines; incorporation of this phenomenon would reduce r_2/r_1. In the extreme, if machines were replaced solely because of obsolescence and operating costs were negligible, r_2/r_1 would equal 1.0.

(b) If operating costs were proportional to the third or fourth power (rather than the square) of cumulated hours of machine use, n_2 would approach one-half of n_1 and r_2/r_1 would increase. In the extreme, machines might be assumed to operate perfectly until a certain number of hours of use had been reached, at which point they fall apart. Under this assumption, with $i=.06$, r_2/r_1 would equal 1.728,

B. The Cost Ratio When r_2 Differs from r_1.

The condition that system 1 costs be greater than system 2 costs, for a given level of output, can be written

$$r_1 K^1 + w_1 L^1 > r_2 K^2 + w_1 L_1^2 (2+\alpha)$$

$$1 > \left[\frac{r_2 K^2}{w_1 L_1^2} + (2+\alpha) \right] \frac{L_1^2}{L^1} (1-\theta) \tag{3.1}$$

where $\theta = r_1 K^1 / (r_1 K^1 + w_1 L^1)$. This derivation parallels that of section I above.) The capital-labor ratios are

$$\frac{K^1}{L^1} = \left[\frac{\delta}{1-\delta} \frac{w_1}{r_1} \right]^{\sigma} (u^*)^{\sigma-1} \tag{3.2}$$

$$\frac{K^2}{L_1^2} = \left[\frac{\delta}{1-\delta} \frac{w_1 (2+\alpha)}{r_2} \right]^{\sigma} (u^*)^{\sigma-1} = \frac{K^2}{L_2^2} \tag{3.3}$$

Since $r_1 K^1 / w_1 L^1 = \theta/(1-\theta)$, we have

$$\frac{r_2 K^2}{w_1 L_1^2} = \frac{\theta}{1-\theta} \left[\frac{r_1}{r_2} \right]^{\sigma-1} (2+\alpha)^{\sigma} \tag{3.4}$$

The condition $X^2 = X^1$ implies (see footnote 2 above)

$$\frac{L_1^2}{L^1} = 2^{-1/\beta} \left[\theta (2+\alpha)^{\sigma-1} (r_1/r_2)^{\sigma-1} + (1-\theta) \right]^{\sigma/(1-\sigma)} \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.1) yields

$$1 > 2^{-1/\beta}(2+\alpha)[\theta(2+\alpha)^{\sigma-1}(r_2/r_1)^{1-\sigma}+(1-\theta)]^{1/(1-\sigma)} \tag{3.6}$$

Table 4 shows the effect on the cost ratio of changes in r_2/r_1 The table illustrates the case of constant returns to scale, a night-shift premium of 20 per cent and an elasticity of substitution of 0.5. With $r_2/r_1 = 1.0$, shift-work is profitable as long as θ is greater than about 0.14. When r_2/r_1=1.50, shift-work is not profitable until θ reaches about 0.27.

TABLE 4

Effect of r_2/r_1 on the Cost Ratio

	Cost Ratio		
	r_2/r_1=1.0	r_2/r_1=1.50	r_2/r_1=1.728
θ= .10	1.0295	1.0621	1.0751
θ= .20	.9614	1.0247	1.0506
θ= .30	.8955	.9880	1.0348
θ= .40	.8321	.9521	1.0022

Note. β=1, α=.50, σ=.5 throughout

In summary, the introduction of use-related operating costs raises r_2/r_1 and thereby reduces the profitability of shift-work. Shift-work also involves changes in the capital-labor ratio and in labor and capital productivity. These changes are strongly influenced by the substitution of capital services for labor services. Anything which raises r_2/r_1 will diminish the extent of such substitution.

Finally, let us mention that all our models have been based on the assumptions of perfect foresight, the absence of risk, and the impossibility of ex post substitution between labor and capital. These assumptions might be fruitfully relaxed in subsequent research.

APPENDIX

Derivation of Equations (2.3), (2.4), and (2.5)

We can write $X=[a\ TC^2(X)]^{\beta}$ where a is a constant which depends on factor prices. Hence

$$\frac{TC^2(X^2)}{TC^2(X^1)} = \left[\frac{X^2}{X^1}\right]^{1/\beta} \qquad (2.4)$$

This is equation (2.4) in the text. For future reference, note that

$$\frac{MC^2(X^2)}{MC^2(X^1)} = \left[\frac{X^2}{X^1}\right]^{\frac{1}{\beta}-1} \qquad (A\ 1)$$

The demand curve may be written $X = AP^{-n}$. Hence

$$TR(X) = \left(\frac{1}{A}\right)^{-1/n} X^{1-(1/n)};\ MR(X) = \left(\frac{1}{A}\right)^{-1/n} \left(1-\frac{1}{n}\right)X^{-1/n}$$

Hence $$\frac{TR(X^2)}{TR(X^1)} = \left[\frac{X^2}{X^1}\right]^{1-\frac{1}{n}} \qquad (2.3)$$

This is equation (2.3) in the text. Note that

$$\frac{MR(X^1)}{MR(X^2)} = \left[\frac{X^1}{X^2}\right]^{-1/n} = \left[\frac{X^2}{X^1}\right]^{1/n} \qquad (A2)$$

With β constant, the cost ratio (CR) is independent of the level of output; that is, $TC^2(X)=CR\ .\ TC^1(X)$. Let us define ICR as 1/CR, or the inverse of the cost ratio. Thus $TC^1(X)=ICR\ .\ TC^2(X)$. Hence $MC^1(X)=ICR\ .\ MC^2(X)$. This holds for any level of output, including X^1. Thus

$$\frac{MC^1(X^1)}{MC^2(X^2)} = \frac{MC^1(X^1)}{MC^2(X^1)} \quad \frac{MC^2(X^1)}{MC^2(X^2)} = ICR\left[\frac{X^2}{X^1}\right]^{1-\frac{1}{\beta}} \qquad (A3)$$

where (A1) was used in the second step.

Profit maximization implies that $MR(X^1)=MC^1(X^1)$ and $MR(X^2)=MC^2(X^2)$. Thus (A2) can be set equal to (A3), yielding

$$\left[\frac{X^2}{X^1}\right]^{1/n} = ICR \left[\frac{X^2}{X^1}\right]^{1-\frac{1}{\beta}}$$

Hence

$$\frac{X^2}{X^1} = ICR^{\frac{\beta n}{n+\beta-\beta n}} \qquad (2.5)$$

This is equation (2.5) in the text.

Footnotes

1 The dependence of δ on the units of measurement is not always realized. Winston (1974,p. 541) incorrectly states that $\delta/(1-\delta)$ is relative shares.

2 Since $X^1 = 2X^2_1$,

$$\left[\frac{[\delta\,(\frac{S^1}{L^1})^{-\rho} + (1-\delta)]\,(L^1)^{-\rho}}{[\delta\,(\frac{S^2}{L^2_1})^{-\rho} + (1-\delta)]\,(L^2_1)^{-\rho}}\right]^{-\frac{\beta}{\rho}} = 2$$

$$\left[\frac{\delta\{u^*\,\frac{w_1}{r}\,\frac{\delta}{1-\delta}\}^{\sigma-1} + (1-\delta)}{\delta\{u^*\,\frac{w_1(2+\alpha)}{r}\,\frac{\delta}{1-\delta}\}^{\sigma-1} + (1-\delta)}\right]^{-\frac{\beta}{\rho}} \frac{L^1}{L^2_1}^{\beta} = 2$$

But $\frac{rK^1}{w_1L^1} = \frac{\theta}{1-\theta} = [u^*\,\frac{w_1}{r}]^{\sigma-1}\,(\frac{\delta}{1-\delta})^{\sigma}$

Hence

$$\frac{L^2_1}{L^1} = 2^{-1/\beta}\left[\frac{\delta(\frac{\theta}{1-\theta})(\frac{1-\delta}{\delta}) + (1-\delta)}{\delta(\frac{\theta}{1-\theta})(\frac{1-\delta}{\delta})(2+\alpha)^{\sigma-1} + (1-\delta)}\right]^{\frac{\sigma}{\sigma-1}}$$

$$\frac{L^2_1}{L^1} = 2^{-1/\beta}\,[\theta(2+\alpha)^{\sigma-1} + (1-\theta)]^{\frac{\sigma}{1-\sigma}}$$

[3] We plan to give a detailed critique of Millan's work in a comment on the final version of his paper.

[4] We can write $TC(X) = (\mu X)^{1/\beta}$ where μ is a constant which depends on factor prices. Now

$$\varepsilon = -\frac{dX}{dAC}\ \frac{AC}{X} = \frac{-\mu^{1/\beta}\ X^{\frac{1}{\beta}-1}}{(\frac{1}{\beta}-1)\mu^{\frac{1}{\beta}}\ X^{\frac{1}{\beta}-2}.X} = \frac{\beta}{\beta-1}$$

[5] These results contradict those of Millan (1974), who stated that under certain conditions an increase in δ would favor single-shift plants. But an increase in δ with factor prices and σ constant implies an increase in θ, and (2.7) and (2.8) imply that an increase in θ must always increase $(\Pi^2-\Pi^1)/TR(X^1)$ and Π^2/Π^1. These two equations also make clear that Millan is wrong in stating that an increase in α might favor multiple-shift plants.

[6] Recall that if $y=a^{f(x)}$, then $dy/dx = f'(x)\ .\ \ell n a.y$. Define

$$ex=\frac{(1-\beta)(n-1)}{n-\beta(n-1)} \quad \text{and} \quad exp=\frac{\beta(n-1)}{n-\beta(n-1)}$$

$$\frac{d(ex)}{d\beta} = \frac{-(n-1)}{[n-\beta(n-1)]^2} \quad \text{and} \quad \frac{d(exp)}{d\beta} = \frac{n(n-1)}{[n-\beta(n-1)]^2}$$

$$\frac{\partial(\Pi^2/\Pi^1)}{\partial\beta} = 2^{ex}\ \frac{n(n-1)}{[n-\beta(n-1)]^2}\ \ell n(ICR^*).(ICR^*)^{exp} + (ICR^*)^{exp}.\ \frac{-(n-1)}{[n-\beta(n-1)]^2}\ \ell n2.2^{ex}$$

$$\frac{\partial(\Pi^2/\Pi^1)}{\partial\beta} = 2^{ex}(ICR^*)^{exp}\ .\ \frac{(n-1)}{[n-\beta(n-1)]}\{n\ \ \ell n\ ICR^* - \ \ell n\ 2\}$$

[7] This follows directly from the facts that $\Pi^2 \gtrless \Pi^1$ as $ICR \gtrless 1$ and that ICR is reduced by an increase in β.

[8] We are taking the flow of revenue to be constant. When machine breakdown reduces actual production, we shall treat this as an increase in operating costs rather than a loss in revenue.

[9] If utilization on the second shift of the double-shift system (u_2^2) is set less than u^*, the firm must buy a somewhat larger capital stock in order to produce the same output with the same labor input. The new K^2 must be $2.0/(1+u_2^2/u^*)$ times the old. Now the new operating costs per year would be

$$OC_t^2 = b[1+u_2^2/u^*)t]^2$$

and the new annual equivalent operating costs would be

$$c_2(i,n,u_2^2) = b(1+u_2^2/u^*)^2 \left(\sum_{t=1}^{n} t^2/(1+i)^t \right) / \sum_{t=1}^{n} 1/(1+i)^t$$

This formula was used to determine $r_2(i, u_2^2)$.

The problem now is to see whether a decline in u_2^2 below u^* would lower r_2 by more than it raises K^2. Clearly a reduction in u_2^2 would not be worthwhile if the percentage increase in K^2 exceeded the percentage reduction in r_2. Moreover, for each successive reduction in u_2^2 the gains diminish (since u_2^2 is squared in the expression for $c_2(i, n_2, u_2^2)$ and the costs increase (the percentage increase in K^2 increases as u_2^2 declines); therefore if the first reduction in u_2^2 is not profitable, no subsequent reduction will be. Changing u_2^2/u^* from 1.0 to .99 was unprofitable in every case in Table 1. It follows that the optimal u_2^2/u^* must be greater than .99, or for practical purposes must be 1.0.

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