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by

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ABSTRACT

The widely used mean-variance approach to decisions under uncertainty requires estimates of the parameters of the joint distribution of returns. When optimal behavior is determined using estimates, rather than the true values, the decision is a random variable.

We examine the usefulness of mean-variance analysis by deriving the bias and variance-covariance matrix for the decision vector. The latter shows that decisions based on estimated parameters can have a large variance around the true optimum. The results show that optimal decisions can differ substantially from those based on mean-variance analysis.

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1. INTRODUCTION

Since Markowitz examined the portfolio diversification problem using mean-variance analysis, this method has been used extensively to model choices from investment alternatives with uncertain returns. In addition, the technique has been applied to a wide variety of economic decisions--examples include hedging (Berck (1981)); adoption of new technologies (Just and Zilberman (1983)); corporate financial decisions (Rubinstein (1973)); the demand for money (Tobin (1958)); and the allocation of fixed assets to uncertain production processes, particularly agricultural land allocation (Freund (1956)).

The mean-variance approach is consistent with the widely accepted von Neumann-Morgenstern expected utility paradigm when either utility is quadratic or utility is negative exponential and the returns from the relevant alternatives are jointly normally distributed.[1] Since quadratic utility implies increasing absolute risk aversion, it is the latter which usually serves as the justification for mean-variance analysis. However, the method requires a vector of means and an associated variance-covariance matrix for the joint distribution of returns from the uncertain prospects being considered; these are almost always unknown to the decision-maker (Markowitz (1952), p. 91). Thus, the unknown parameters must be estimated using sample or other information, making the optimal decision vector random. Commonly, sample estimates replace population parameters and allocations are based on the resulting estimate of optimal behavior. For obvious reasons, this approach has become known as the parameter certainty equivalent (PCE) (e.g., Bawa, Brown, and Klein

(1979)) or "plug-in" approach (Pope and Ziemer (1984)). This estimation procedure leads to an additional source of uncertainty, estimation risk, that itself is the subject of a considerable body of literature.

In the context of the portfolio choice problem, this literature includes examinations of a broad range of problems and is well summarized in Bawa, Brown, and Klein (1979). Discussions of estimation risk in the financial economics literature have focused on Monte Carlo simulations of its importance for individual investors (Frankfurter, Phillips, and Seagle (1971), Brown (1979)); for financial market equilibrium (Bawa and Brown (1979), Alexander and Resnick (1985)); and on the derivation of optimal (Bayesian) estimators in the presence of estimation risk (Klein and Bawa (1979a, b)). Despite this attention to the problem, no one has established the sampling properties of the widely-used mean-variance estimators of optimal behavior.

In this paper we examine the implications of estimation risk for the usefulness of mean-variance analysis by exploring the sampling properties of the mean-variance decision vector. In doing so, we consider a number of factors important to determining optimal behavior and drawing inferences from mean-variance analysis. We show first that the decision vector is a biased but consistent estimator of the choices which maximize expected utility, given knowledge of the unknown parameters. An unbiased decision vector is easily obtained from this result. We then derive a variance-covariance matrix for both decision vectors, finding that there may be an unacceptably large amount of variation in estimates of the optimal decision. We explore the factors that determine the reliability of such estimates of expected utility maximizing behavior and then derive the ex ante expected utility from using such estimators. From the derivations, it is clear that the bias and variance in an optimal decision depend on the risk attitudes of the decision-maker, the number of prospects, the amount of historical information available, the underlying distribution, and the total amount of fixed resources to be allocated. Finally, we examine how one might use these results to move from a point estimate of optimal behavior toward interval estimation.

The paper proceeds as follows. The next section outlines the portfolio choice problem and a particular case which leads to the widely used linear (in mean and variance) objective function. We use the expected utility moment-generating function approach suggested by Freund (1956) and Hammond (1974) and reformulate the problem to recognize explicitly the estimation of unknown parameters which is involved. In the third section, we derive the mean, bias, and variance of the parameter certainty equivalent estimator of the optimal decision vector and suggest an unbiased estimator. We also discuss the factors that determine the magnitude of bias and variance in this section. Section four contains expressions for the expected utility of the decision-maker in using each of these estimators. In section five, we demonstrate the importance of recognizing estimation risk for some simple portfolio allocation problems which have appeared in the finance and agricultural economics literature. The last section contains a summary and conclusions about the implications of estimation risk for both positive and normative analyses using mean-variance analysis.

2. THE MODEL AND ESTIMATION PROBLEM

In single-period portfolio choice problems, the decision-maker must allocate a fixed resource, such as land or wealth, among risky investment alternatives to maximize the expected utility from end-of-period wealth. The problem can be formulated as

$$\max_{\underline{y} \epsilon C_o} EU[W_o(1 + \underline{y}'\underline{\tilde{R}})]$$

subject to

$$\underline{y}'\underline{i} = 1$$

where EU(•) denotes the expected value of the decision-maker's utility function; W_o is initial wealth; $\underline{y}$ is a vector of choices from the set of alternatives, C_o; $\underline{\tilde{R}}$ denotes the random vector of returns; and $\underline{i}$ is a vector of ones. The elements in $\underline{y}$ are the shares of W_o allocated to each alternative in the vector $\underline{\tilde{R}}$.

Equivalently, the problem can be expressed as

$$\max_{\underline{\ell} \epsilon C} EU(\underline{\ell}'\underline{\tilde{\pi}})$$

subject to

$$\underline{\ell}'\underline{i} = W_o$$

and

$$\underline{\tilde{\pi}} = \underline{i} + \underline{\tilde{R}}.$$

Under a set of quite restrictive but frequently imposed assumptions, the above problems reduce to the maximization of an objective function which is linear in the mean and variance of portfolio returns. Possibilities include quadratic utility or the combination of normal returns and a utility function of the negative exponential form. Below, we develop the solution for the latter case by applying mean-variance analysis to the land allocation problem that originated with Freund (1956) and has been expanded upon in recent papers by Collender and Zilberman (1985) and Collender and Chalfant (1986).[2]

2.1. The Land Allocation Decision

The decision-maker's problem is to allocate L acres of land to k crops, where returns per acre $\underline{x}$ are distributed as $N_k(\underline{\mu}, \Sigma)$. We assume that the decision-maker maximizes the expected value of an exponential utility function

$$U(\pi) = -\exp(-r\pi)$$

where r is the Arrow-Pratt measure of risk aversion and π denotes profits:

$$\pi = \sum_{i=1}^{k} \ell_i x_i,$$

and ℓ_i is the acreage planted to crop i. We assume that per-acre returns are net of production costs, and we treat the technologies as predetermined and consider only the acreage decision.

The first-order conditions for maximizing expected utility involve the derivatives of the moment-generating function, M, of the random vector $\underline{x}$ with respect to each $t_i = -r\ \ell_i$. They are shown by Collender and Zilberman (1985) to be

$$\frac{M_1}{M} = \frac{M_i}{M}, \qquad i = 2, \ldots, k.$$

These conditions are then equivalent to

$$M^{-1} A \nabla M = \underline{0},$$

∇M being the k-vector of derivatives of M and A being a (k - 1) x k matrix of the form

$$A = \left[\underline{i} \;\middle|\; -I_{k-1}\right]$$

where $\underline{i}$ is used throughout the paper to denote a vector of ones. For a multivariate normal moment-generating function with $\underline{t} = -r\,\underline{\ell}$, this condition gives us

$$M^{-1} A \cdot M[\underline{\mu} - r\,\Sigma\,\underline{\ell}] = \underline{0}$$

or

$$A\,\Sigma\,\underline{\ell} = \frac{1}{r} A\,\underline{\mu}.$$

Note that $A\,\Sigma$ is not a square matrix so it cannot be inverted to solve for $\underline{\ell}$. It is only $(k - 1) \times k$, and we need one more restriction on $\underline{\ell}$, so we add that the farm size is L:

$$\underline{i}_k'\,\underline{\ell} = L.$$

Then, the system of k restrictions on $\underline{\ell}$ for maximization of expected utility can be solved, once estimates of $\underline{\mu}$ and Σ are available:

$$\underline{\ell} = \begin{bmatrix} A\,\hat{\Sigma} \\ \underline{i}_k' \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{r} A\,\hat{\underline{\mu}} \\ L \end{bmatrix}.$$

where $\hat{\Sigma}$ estimates Σ and $\hat{\underline{\mu}}$ estimates $\underline{\mu}$.

2.2. The Estimation Problem

If the true population parameters were known, then $\underline{1}$ would be the optimal decision (hereafter $\underline{1}^*$) in the sense of maximizing expected utility. Estimation risk exists when parameter estimates must be used in place of population parameters. The result is that the solution $\underline{\ell}$ is only an estimate of $\underline{\ell}^*$, call it $\hat{\underline{\ell}}$. The decision will be suboptimal if $\hat{\underline{1}}$ differs from $\underline{1}^*$ in the sense that $EU(\pi|\hat{\underline{\ell}}) < EU(\pi|\underline{\ell}^*)$. Furthermore, $\hat{\underline{l}}$ is random (as it is a function of past realizations of returns) through $\hat{\underline{\mu}}$ and $\hat{\Sigma}$. Its sampling behavior is critical for evaluating its use as a substitute for $\underline{\ell}^*$. In this section we formalize the random nature of $\hat{\underline{\ell}}$.

We assume that data are available on the k different returns per acre observed over n periods and are collected in a k x n matrix X. Column t of X is a draw, at time t, from $N_k(\mu, \Sigma)$, and we assume timewise independence in these draws. Our estimates $\hat{\underline{\mu}}$ and $\hat{\Sigma}$ are obtained from

$$\hat{\underline{\mu}} = \frac{1}{n} X \, \underline{i}_n$$

and

$$\hat{\Sigma} = (n - 1)^{-1} (X - \hat{\underline{\mu}} \, \underline{i}_n') (X - \hat{\underline{\mu}} \, \underline{i}_n')'.$$

If we let $Z = X - \underline{\mu} \, \underline{i}_n'$ be deviations from population means so that the columns of Z, $Z_{\cdot i}$ are independent draws from $N_k(0, \Sigma)$, we can then write

$$\hat{\underline{\mu}} = \frac{1}{n} Z\underline{i}_n + \underline{\mu}.$$

Also, our estimator for the variance matrix, Σ, can be expressed as

$$\hat{\Sigma} = (n - 1)^{-1} (X - \hat{\underline{\mu}} \, \underline{i}_n') (X - \hat{\underline{\mu}} \, \underline{i}_n')' = (n - 1)^{-1} X \, P_n \, X'$$

$$= (n - 1)^{-1} Z \, P_n \, Z' = (n - 1)^{-1} VV'$$

where $P_n = I - i_n(i_n' \, i_n)^{-1} \, i_n' = I - \frac{1}{n} \, i_n \, i_n'$ is a symmetric, idempotent matrix with rank n - 1 and V is a k x (n - 1) matrix formed by using the eigenvalue-eigenvector factorizaton of P_n. Then, the expression for $\hat{\underline{\ell}}$ becomes

$$\hat{\underline{\ell}} = \begin{bmatrix} (n-1)^{-1} A Z P_n Z' \\ \underline{i}_k' \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{r} A(\underline{\mu} + \frac{1}{n} Z\underline{i}_n) \\ L \end{bmatrix}.$$

The inverse above is shown in the appendix (A.1) to equal the partitioned expression

$$A'[AVV'A']^{-1} (n-1) \left[I_{k-1} \;\vdots\; -(n-1)^{-1} AVV'\underline{e}_1 \right] + \left[\underline{\underline{0}} \;\vdots\; \underline{e}_1 \right]$$

where the $\underline{\underline{0}}$ matrix is k x (k - 1) and $\underline{e}_1$ denotes the first elementary vector of length k.

The solution vector, $\hat{\underline{\ell}}$, can therefore be written as

$$\hat{\underline{\ell}} = \left\{ \left[A'(AVV'A')^{-1} (n-1) + \underline{\underline{0}} \;\vdots\; -A'(AVV'A')^{-1} AVV'\underline{e}_1 + \underline{e}_1 \right] \right\}_k \left[\begin{matrix} \frac{1}{r} A\left(\underline{\mu} + \frac{1}{n} Z\underline{i}_n\right) \\ L \end{matrix} \right]$$

or, upon multiplication,

$$\hat{\underline{\ell}} = A'(AVV'A')^{-1} \cdot \frac{n-1}{r} A(\underline{\mu} + \frac{1}{n} Z\underline{i}_n) + L\underline{e}_1 - LA'(AVV'A')^{-1} AVV'\underline{e}_1.$$

It simplifies later results if the first column of V', $V'\underline{e}_1$, is separated into two parts, one orthogonal to AV. We write T = AV and note that AVV'A' = TT':

$$V'\underline{e}_1 = V_{1.}' = (T)' \underline{\alpha} + \underline{u}.$$

The first part is a linear combination of the columns of T', the second a normally distributed random vector, $\underline{u}$, which has a zero expectation. This vector $\underline{u}$ is independent of the elements of T and, to make this true, we set

$$\underline{\alpha} = (A\Sigma A')^{-1} A \Sigma_{.1}$$

where $\Sigma_{\cdot 1}$ denotes the first column of Σ. This result is shown in the appendix (A.2).

We can express $\hat{\underline{\ell}}$ in terms of $\underline{\alpha}$ and $\underline{u}$ by making use of this equality:

$$T(V'e_1) = TV_{1\cdot}' = T[T'\underline{\alpha} + \underline{u}]$$

$$= TT'\underline{\alpha} + T\underline{u}.$$

This gives us

$$\hat{\underline{\ell}} = A'(TT')^{-1} \frac{n-1}{r} A(\underline{\mu} + \frac{1}{n} Z\underline{i}_n) + L\underline{e}_1 - LA'(TT')^{-1} \{TT'\underline{\alpha} + T\underline{u}\}$$

$$= A'(TT')^{-1} \frac{n-1}{r} A(\underline{\mu} + \frac{1}{n} Z\underline{i}_n) + L\underline{e}_1 - LA'\underline{\alpha} - LA'(TT')^{-1} T\underline{u}.$$

3. SAMPLING PROPERTIES OF THE MEAN-VARIANCE DECISION VECTOR

We are now ready to derive the mean and variance of $\hat{\underline{\ell}}$, the mean-variance decision vector for the PCE case. Before proceeding, however, we note some key independence results which simplify the derivations which follow. First, T = AV and u are independent by construction. It is also the case that $Z\underline{i}_n$ and V are independent, as shown in the appendix (A.3). Finally, since $\underline{u} = V'(\underline{e}_1 - A'(A\Sigma A')^{-1} A\Sigma_{\cdot 1})$ is a linear combination of elements in V, it is also independent of $Z\underline{i}_n$.

3.1. The Mean and Bias in $\hat{\underline{\ell}}$ and an Unbiased Alternative

The expected value of $\hat{\underline{\ell}}$ consists of three terms:

$$E(\hat{\underline{\ell}}) = A' E(TT')^{-1} A \frac{n-1}{r} \underline{\mu} - LA'\underline{\alpha} + L\underline{e}_1.$$

The terms in our expression for $\hat{\underline{\ell}}$ which involve $\underline{u}$ and $Z\underline{i}_n$ each have zero expectations because those terms do; independence of T from each of them lets us take expectations of $(TT')^{-1}$ over T and the expectations of $\underline{u}$ and $Z\underline{i}_n$ separately. Note that $(TT') \sim$ Wishart $(A\Sigma A', n - 1)$, which implies that

$$E(TT')^{-1} = \frac{(A\Sigma A')^{-1}}{(n-1)-(k-1)-1} = \frac{(A\Sigma A')^{-1}}{n-k-1}$$

(Anderson (1984), p. 270) and, upon substitution of this result,

$$E(\hat{\underline{\ell}}) = (n-k-1)^{-1} A'(A\Sigma A')^{-1} A \frac{n-1}{r} \underline{\mu} - LA'\alpha + Le_1$$

$$= \frac{n-1}{n-k-1} \cdot \frac{1}{r} A'(A\Sigma A')^{-1} A \underline{\mu} - LA'\alpha + Le_1.$$

The optimal decision when $\underline{\mu}$ and Σ are known can be shown to be

$$\underline{\ell}^* = \frac{1}{r} A'(A\Sigma A')^{-1} A\underline{\mu} + L\underline{e}_1 - LA'\underline{\alpha},$$

so an expression for the bias can be obtained by subtraction:

$$\text{Bias } (\hat{\underline{\ell}}) = E(\hat{\underline{\ell}}) - \underline{\ell}^* = \left[\left(\frac{n-1}{n-k-1}\right) - 1\right] A'(A\Sigma A')^{-1} A \cdot \frac{1}{r} \underline{\mu}.$$

The factor $(n - 1)/(n - k - 1)$ in $E(\hat{\underline{\ell}})$ is responsible for the bias in $\hat{\underline{\ell}}$. This term is due to the uncertainty about Σ; with only $\underline{\mu}$ uncertain, $\hat{\underline{\ell}}$ would be unbiased. As is evident, $\hat{\underline{\ell}}$ is asymptotically unbiased, keeping the number of alternatives k fixed. Also note that the bias vector, hereafter $\underline{b}$, must satisfy

$$\underline{b}'\underline{i} = 0$$

since the elements in $\hat{\underline{\ell}}$ and $\underline{\ell}^*$ both sum to L--the total amount of land to be allocated. Thus, some elements in $\hat{\ell}$ will be biased upward and others biased downward. However, the effect of this bias on the portfolio mean and variance is unambiguous.

The portfolio mean is $\hat{\underline{\ell}}'\mu$, for a given $\hat{\underline{\ell}}$; the expectation over $\hat{\underline{\ell}}$, for fixed $\underline{\mu}$, is

$$E(\hat{\underline{\ell}}'\underline{\mu}) = (\underline{\ell}^* + \underline{b})'\ \underline{\mu} = \underline{\ell}^{*\prime}\underline{\mu} + \underline{b}'\underline{\mu}.$$

Note that

$$\underline{b}'\underline{\mu} = c\ \underline{\mu}'A'(A\Sigma A')^{-1}\ A\underline{\mu}$$

where c is a positive scalar. This expression is nonnegative, then, since $A'(A\Sigma A')^{-1} A$ is positive semidefinite. Thus, the portfolio chosen using the PCE method will, on average, have a higher mean than the optimal choice $\underline{\ell}^*$.

An unbiased estimator, $\tilde{\underline{\ell}}$, can be obtained by rescaling $\hat{\Sigma}$ by the offending constant:

$$\tilde{\underline{\ell}} = \begin{bmatrix} A\hat{\Sigma} \cdot \left(\frac{n-1}{n-k-1}\right) \\ \underline{i}_k' \end{bmatrix}^{-1} \begin{pmatrix} \frac{1}{r} A\hat{\underline{\mu}} \\ L \end{pmatrix} = \begin{bmatrix} (n-k-1)^{-1} AVV' \\ i_k' \end{bmatrix}^{-1} \begin{pmatrix} \frac{1}{r} A\hat{\underline{\mu}} \\ L \end{pmatrix}.$$

By repeating the steps in the Appendix, the inverse above can be shown to differ from the previous case only in that (n - k - 1) replaces (n - 1), so that the unbiased decision vector is

$$\tilde{\underline{\ell}} = \frac{n-k-1}{r}\ A'(TT')^{-1}\ A(\underline{\mu} + \frac{1}{n}\ Z\underline{i}_n) - LA'\underline{\alpha} - LA'(TT')^{-1}\ T\underline{u} + L\underline{e}_1.$$

3.2. Sampling Variance of the Decision Vectors

In this section, we derive the sampling variance for the unbiased decision vector, $\underline{\tilde{\ell}}$, obtained above. It is easy to adjust the result to find a similar expression for $\underline{\hat{\ell}}$. Our goal is most easily accomplished by finding an expression for the characteristic function of the difference between $\underline{\tilde{\ell}}$ and its expectation.

The deviation of $\underline{\tilde{\ell}}$ from $E(\tilde{\ell})$, shown previously to be the same as the optimal choice, $\underline{\ell}^*$, can be written as

$$\tilde{\ell} - E(\tilde{\ell}) = \frac{n-k-1}{r} A'[(TT')^{-1} - E(TT')^{-1}] A\underline{\mu}$$

$$+ \frac{n-k-1}{nr} A' E(TT')^{-1} A Z\underline{i}_n - LA' (TT')^{-1} T\underline{u}.$$

Again, the matrix T and the vectors $\underline{z} = Z\underline{i}_n$ and $\underline{u}$ are normally distributed with zero expectations.

We turn now to deriving the characteristic function for $\underline{\tilde{\ell}} - E(\underline{\tilde{\ell}})$. In doing so, we will use some results for $Z\underline{i}_n$ and $\underline{u}$ established in the appendix (A.4).

The characteristic function of the random vector $\underline{\tilde{\ell}} - E(\underline{\tilde{\ell}})$ is given by

$$\phi(\underline{t}) = E\{\exp[i\underline{t}(\tilde{\ell} - E(\underline{\tilde{\ell}}))]\}$$

$$= E\left\{\exp\left[i\underline{t}'\left\{\frac{n-k-1}{r} A'\left[(TT')^{-1} - E(TT')^{-1}\right]A\underline{\mu}\right\}\right]\right.$$

$$\cdot \exp\left[i\underline{t}'\left(\frac{n-k-1}{nr} A' (TT')^{-1} A\underline{z}\right)\right]$$

$$\left.\cdot \exp\left[- iL'\underline{t}'A\ (TT')^{-1} T\underline{u}\right]\right\}.$$

Observe that, by properties of normal random vectors,

$$E[\exp(i\,\underline{k}'\underline{z})] = \exp(-\tfrac{1}{2}\, n\, \underline{k}'\, \Sigma\, \underline{k})$$

and

$$E[\exp(i\,\underline{k}'\,\underline{u})] = \exp(-\tfrac{1}{2}\,\bar{\Sigma}\,\underline{k}'\,\underline{k}),$$

where $\bar{\Sigma}$ is the scalar defined in the appendix (A.4). Then, taking expectations over $\underline{z}$ and $\underline{u}$, we obtain

$$\varphi(\underline{t}) = E_W\left\{\exp\, i\underline{t}'\left(\frac{n-k-1}{r}\, A'\,\bar{W}\,A\underline{\mu}\right)\right.$$
$$\cdot \exp\left[-\frac{1}{2}\,\frac{(n-k-1)^2}{nr^2}\,\underline{t}'\,A'\,W\,A\,\Sigma\,A'\,W\,A\underline{t}\right]$$
$$\left.\cdot \exp\left(-\frac{1}{2}\,L^2\,\bar{\Sigma}\,\underline{t}'\,A'\,W\,A\underline{t}\right)\right\}$$

where $W = (TT')^{-1}$ and $\bar{W} = W - E(W)$.

One convenient substitution in the above expression involves the term $W\,A\,\Sigma\,A'\,W$. This can be replaced using the identity

$$W\,A\,\Sigma\,A'\,W = \bar{W}\,A\,\Sigma\,A'\,\bar{W} + \frac{2}{n-k-1}\,\bar{W} + (n-k-1)^{-1}\,E(W)$$

or

$$\bar{W}\,A\,\Sigma\,A'\,\bar{W} + \frac{2}{n-k-1}\,\bar{W} + (n-k-1)^{-2}\,(A\,\Sigma\,A')^{-1}.$$

When this is substituted into the characteristic function for $\tilde{\ell} - E(\tilde{\ell})$, the result is

$$\varphi(\underline{t}) = E_W\left\{\exp\left[i\underline{t}'\,\frac{(n-k-1)}{r}\,A'\,\bar{W}\,A\underline{\mu}\right]\exp\left[-\frac{1}{2}\,\frac{(n-k-1)^2}{nr^2}\,\underline{t}'\,A'\,\bar{W}\,A\,\Sigma\,A'\,\bar{W}\,A\underline{t}\right]\right.$$
$$\cdot \exp\left[-\frac{1}{2}\,\frac{(n-k-1)^2}{nr^2}\,\underline{t}'\,A'\left(\frac{2}{n-k-1}\,\bar{W}\right)A\underline{t}\right]\exp -\left[\frac{1}{2}\,\frac{(n-k-1)^2}{nr^2}\,\underline{t}'\,A'\,\frac{(A\,\Sigma\,A')^{-1}}{(n-k-1)^2}\,A\underline{t}\right]$$
$$\left.\cdot \exp\left(-\frac{1}{2}\,L^2\,\bar{\Sigma}\,\underline{t}'\,A'\,W\,A\underline{t}\right)\right\};$$

the middle three exponentials are the result of the substitution. The first two terms will remain, but a few substitutions make the last three exponentials more convenient. We rearrange until one term involving $\underline{t}'\ A'\ \bar{W}\ A\underline{t}$ and one involving $\underline{t}'\ A'(A\ \Sigma\ A')^{-1}\ A\underline{t}$ are obtained.

We can write

$$\varphi(\underline{t}) = E_W\Bigg\{\exp\left(i\underline{t}'\ \frac{n-k-1}{r}\ A'\ \bar{W}\ A\underline{\mu}\right) \exp\left[-\frac{1}{2}\frac{(n-k-1)^2}{nr^2}\ \underline{t}'\ A'\ \bar{W}\ A\ \Sigma\ A'\ \bar{W}\ A\underline{t}\right]$$
$$\cdot \exp\left[-\frac{1}{2}\frac{2(n-k-1)}{nr^2}\ \underline{t}'\ A'\ \bar{W}\ A\underline{t}\right] \exp\left[-\frac{1}{2}\frac{1}{nr^2}\ \underline{t}'\ A'(A\ \Sigma\ A')^{-1}\ A\underline{t}\right]$$
$$\cdot \exp\left(-\frac{1}{2}L^2\ \bar{\Sigma}\ \underline{t}'\ A'\ W\ A\underline{t}\right) \cdot \exp\left[\pm\frac{1}{2}L^2\ \bar{\Sigma}\ \underline{t}'\ A'\ E(W)\ A\underline{t}\right]\Bigg\}$$

by simplifying the third and fourth exponentials and multiplying by the last one which changes nothing. Combining term 5 and the positive part of term 6, we obtain

$$\exp\left(-\frac{1}{2}L^2\ \bar{\Sigma}\ \underline{t}'\ A'\ \bar{W}\ A\underline{t}\right)$$

and the negative part in term 6 is

$$\exp\left[-\frac{1}{2}(n-k-1)^{-1}\ L^2\ \bar{\Sigma}\ \underline{t}'\ A'(A\ \Sigma\ A')^{-1}\ A\underline{t}\right].$$

Hence, the former combines with term 3 above and the latter with term 4; the result is

$$\varphi(\underline{t}) = E_W\Bigg\{\exp\left(i\underline{t}'\ \frac{n-k-1}{r}\ A'\ \bar{W}\ A\underline{\mu}\right) \exp\left[-\frac{1}{2}\frac{(n-k-1)^2}{nr^2}\ \underline{t}'\ A'\ \bar{W}\ A\ \Sigma\ A'\ \bar{W}\ A\underline{t}\right]$$

$$\cdot \exp\left(-\frac{1}{2}\left[\bar{\Sigma} L^2 + \frac{2(n-k-1)}{nr^2}\right] \underline{t}' A' \bar{W} A\underline{t}\right)$$

$$\cdot \exp\left[-\frac{1}{2}\left(\frac{\bar{\Sigma} L^2}{n-k-1} + \frac{1}{nr^2}\right) \underline{t}' A'(A \Sigma A')^{-1} A\underline{t}\right]\Bigg\}.$$

This expression can now be differentiated twice with respect to $\underline{t}$ to find the variance-covariance matrix for $\tilde{\underline{\ell}}$, $E\{[\tilde{\underline{\ell}} - E(\tilde{\underline{\ell}})] [\tilde{\underline{\ell}} - E(\tilde{\underline{\ell}})]'\}$. Since the expected value of $[\tilde{\underline{\ell}} - E(\tilde{\underline{\ell}})]$ is zero, the first derivative serves only to obtain the second derivative:

$$\frac{\partial \phi}{\partial \underline{t}} = E_W \Bigg\{ \exp(\cdot) \Bigg[i \frac{n-k-1}{r} A' \bar{W} A\underline{\mu} - \frac{(n-k-1)^2}{nr^2} A' \bar{W} A \Sigma A' \bar{W} A\underline{t}$$

$$- \left[\bar{\Sigma} L^2 + \frac{2(n-k-1)}{nr^2}\right] A' \bar{W} A\underline{t} - \left[\frac{\bar{\Sigma} L^2}{n-k-1} + \frac{1}{nr^2}\right] A'(A \Sigma A')^{-1} A\underline{t} \Bigg]\Bigg\}.$$

where exp(•) denotes the set of four exponentials inside the expectation above. Differentiating once again, with respect to $\underline{t}'$, we obtain first the expression above times the transpose of the exponent's derivative and then a term due to the dependence of the bracketed expression above on $\underline{t}$:

$$\frac{\partial^2 \phi}{\partial \underline{t}\, \partial \underline{t}'} = E_W \left\{ \exp(\cdot)\, [\cdot]\, [\cdot]' \right\} - E_W \Bigg\{ \exp(\cdot) \Bigg[\frac{(n-k-1)^2}{nr^2} A' \bar{W} A \Sigma A' \bar{W} A$$

$$+ \left[\bar{\Sigma} L^2 + 2\frac{(n-k-1)}{nr^2}\right] A' \bar{W} A + \left(\frac{\bar{\Sigma} L^2}{n-k-1} + \frac{1}{nr^2}\right) A'(A \Sigma A')^{-1} A \Bigg]\Bigg\}.$$

The exponentials vanish at $\underline{t} = \underline{0}$, leaving

$$V(\tilde{\underline{\ell}}) = -\frac{\partial^2 \phi}{\partial \underline{t}\, \partial \underline{t}'}\Bigg|_{\underline{t}=\underline{0}} = -E_W \left[i^2 \frac{(n-k-1)^2}{r^2} A' \bar{W} A\underline{\mu}\, \underline{\mu}' A' \bar{W} A \right]$$

$$+ E_W\left\{\frac{(n-k-1)^2}{nr^2} A'\,\bar{W}\,A\,\Sigma\,A'\,\bar{W}\,A + \left[\bar{\Sigma}\,L^2 + 2\,\frac{(n-k-1)}{nr^2}\right] A'\,\bar{W}\,A\right\}$$

$$+\left(\frac{\bar{\Sigma}\,L^2}{n-k-1} + \frac{1}{nr^2}\right) A'(A\,\Sigma\,A')^{-1}\,A.$$

The first term reduces to

$$\frac{(n-k-1)^2}{r^2} E_W[A'\,\bar{W}\,A\underline{\mu}\,\underline{\mu}'\,A'\,\bar{W}\,A],$$

and the last term is a constant with respect to W. From the middle two terms, only the first remains since the second has a zero expectation over W. Thus,

$$V(\tilde{\underline{\ell}}) = \frac{(n-k-1)^2}{r^2} E_W\left[A'\,\bar{W}\,A\left(\underline{\mu}\,\underline{\mu}' + \frac{1}{n}\,\Sigma\right) A'\,\bar{W}\,A\right] + \left[\frac{\bar{\Sigma}\,L^2}{(n-k-1)} + \frac{1}{nr^2}\right] \cdot \left[A'(A\,\Sigma\,A')^{-1}\,A\right].$$

The first term is difficult to simplify, requiring still more substitutions.

It is shown in the appendix (A.5) that the variance matrix reduces to

$$V(\tilde{\underline{\ell}}) = \frac{(n-k-1)}{r^2(n-k)^2} A'(A\,\Sigma\,A')^{-1}\,A\underline{\mu}\,\underline{\mu}'\,A'(A\,\Sigma\,A')^{-1}\,A$$

$$+\left(\frac{(n-k-1)}{nr^2(n-k)^2} + \frac{\bar{\Sigma}\,L^2}{(n-k-1)} + \frac{1}{nr^2}\right) A'(A\,\Sigma\,A')^{-1}\,A.$$

A similar series of steps can be used to show that the PCE decision vector, $\hat{\underline{\ell}}$, has a larger sampling variance. The only difference is that (n - 1) replaces (n - k - 1) when the latter appears in $V(\tilde{\underline{\ell}})$; hence

$$V(\hat{\underline{\ell}}) = \frac{(n-1)}{r^2(n-k)^2} A'(A\,\Sigma\,A')^{-1}\,A\,\underline{\mu}\,\underline{\mu}'\,A'(A\,\Sigma\,A')^{-1}\,A$$

$$+ \left(\frac{(n - 1)}{nr^2(n - k)^2} + \frac{\bar{\Sigma}\, L^2}{(n - k - 1)} + \frac{1}{nr^2} \right) A'(A\ \Sigma\ A')^{-1}\ A.$$

This shows that the unbiased decision vector $\tilde{\underline{\ell}}$ has a smaller variance matrix, in the sense that $V(\hat{\underline{\ell}}) - V(\tilde{\underline{\ell}})$ is positive semidefinite.

3.3. Factors Affecting the Bias and Variance of the PCE Estimator

From inspection of the terms for the bias and variance of the PCE estimator, it is clear that several factors affect their magnitudes. For the bias, these factors include the underlying population parameters as well as the sample size, number of alternative enterprises (investments), and the absolute measure of risk aversion. The variance is affected by the same factors plus the initial wealth or fixed resource constraint. Table 1 presents rates of change for the bias and variance with respect to each of these factors, with the exception of the number of alternative enterprises. The effect of k is more difficult to determine since it affects the dimensions of the matrices Σ and A and the vector $\underline{\mu}$.

It is clear on inspection that both the bias and variance converge to zero as either the sample size or the measure of absolute risk aversion gets large. It is also clear that as L increases the variance increases. An increase in μ_i will increase both the variance and the bias of the estimated allocation of L to alternative i but will have an ambiguous effect on the bias of other alternatives. An increase in the variance of a particular alternative will increase both the bias and variance of the PCE estimator, but an increase in a covariance will have ambiguous effects. These last results are important for analyzing the effects of technological or other changes on the impact of estimation risk, all else held constant.

TABLE 1

Factors Affecting the Bias and Variance of $\hat{\underline{\ell}}$

Partial with respect to	Bias	Variance
n (sample size)	$\frac{-k}{r(n-k-1)^2} A'(A \Sigma A')^{-1} A\underline{\mu} < 0$	$\left[\frac{-(n+k-2)}{r^2(n-k)^3}\right] A'(A \Sigma A')^{-1} A\underline{\mu}\,\underline{\mu}' A'(A \Sigma A')^{-1} A + \left\{\left[\frac{-(n+k-2)}{nr^2(n-k)^3}\right] - \left[\frac{n-1}{n^2r^2(n-k)^2}\right] - \frac{\bar{\Sigma} L^2}{(n-k-1)^2} - \frac{1}{n^2r^2}\right\} A'(A \Sigma A')^{-1} A < 0$
r (absolute risk aversion)	$\frac{-k}{r^2(n-k-1)} A'(A \Sigma A')^{-1} A\underline{\mu} < 0$	$\left[\frac{-2(n-1)}{r^2(n-k)^2}\right] A'(A \Sigma A')^{-1} A\underline{\mu}\,\underline{\mu}' A'(A \Sigma A')^{-1} A - \left[\frac{2(n-1)}{nr^3(n-k)^3} + \frac{2}{nr^3}\right] A'(A \Sigma A')^{-1} A < 0$
L (initial portfolio size)		$2\,\frac{\bar{\Sigma} L}{(n-k-1)} A'(A \Sigma A')^{-1} A \underset{>}{\leq} 0$ as $\bar{\Sigma} \underset{>}{\leq} 0$
μ_i	$\frac{k}{r(n-k-1)} A'(A \Sigma A')^{-1} A\underline{e}_i$	$\left[\frac{(n-1)}{r^2(n-k)^2}\right] A'(A \Sigma A')^{-1} A\underline{e}_i\,\underline{e}_i' A'(A \Sigma A')^{-1} A > 0$
σ_{ij}	$\frac{-k}{r(n-k-1)} A'(A \Sigma A')^{-1} A\underline{e}_i\,\underline{e}_j' A'(A \Sigma A')^{-1} A\underline{\mu}$ > 0 if $i = j$	$\left[\frac{(n-1)}{r^2(n-k)^2}\right] A'(A \Sigma A')^{-1} A\underline{e}_i\,\underline{e}_j' A'(A \Sigma A')^{-1} A\underline{\mu}\,\underline{\mu}' A'(A \Sigma A')^{-1} A\underline{e}_i\,\underline{e}_j' \cdot A'(A \Sigma A')^{-1} A + \left[\frac{n-1}{nr^2(n-k)^2} + \frac{\bar{\Sigma} L^2}{(n-k-1)} + \frac{1}{nr^2}\right] A'(A \Sigma A')^{-1} A\underline{e}_i\,\underline{e}_j' A'(A \Sigma A')^{-1} A$ > 0 if $i = j$

Note: e_i = $\underline{i}$th column of I_k.

4. EX ANTE EXPECTED UTILITY FROM PCE ESTIMATORS

To summarize the results so far, we have considered the effect of estimation risk on the statistical properties of the commonly used PCE approach to estimating optimal mean-variance decisions. With returns following a multivariate normal $N_k(\underline{\mu}, \Sigma)$ and μ and Σ unknown, the decision vector $\hat{\underline{\ell}}$ obtained using sample estimates is biased as an estimator of the unknown optimum ℓ^*. It also has greater variation than the unbiased vector we derived, $\tilde{\underline{\ell}}$, making the latter an improved rule for mean-variance decisions in terms of estimating $\underline{\ell}^*$. Of necessity, both $EU(\pi|\hat{\underline{\ell}})$ and $EU(\pi|\tilde{\underline{\ell}})$ are less than $EU(\pi|\underline{\ell}^*)$ so estimation risk must reduce average (ex ante) welfare.

It is possible to show that $\tilde{\underline{\ell}}$ dominates $\hat{\underline{\ell}}$ in terms of ex ante expected utility. To see this, consider the certainty equivalent associated with each estimator of $\underline{\ell}$.[3] For the case of multivariate normal returns and a negative exponential utility function, the certainty equivalent is

$$CE(\underline{\ell}) = E[\underline{\ell}'\underline{\mu} - \frac{1}{2} r\underline{\ell}'\Sigma\underline{\ell}]$$

with knowledge of $\underline{\mu}$ and Σ. This reduces to

$$CE(\ell^*) = \underline{\ell}^{*\prime}\underline{\mu} - \frac{1}{2} r\ell^{*\prime}\Sigma\ell^*.$$

Consider now the certainty equivalent associated with the unbiased decision vector, $\tilde{\underline{\ell}}$.

$$CE(\tilde{\underline{\ell}}) = E[\tilde{\underline{\ell}}'\underline{\mu} - \frac{1}{2} r\tilde{\underline{\ell}}'\Sigma\tilde{\underline{\ell}}]$$

$$= \underline{\ell}^{*\prime}\underline{\mu} - \frac{1}{2} r \cdot E[tr\tilde{\underline{\ell}}'\Sigma\tilde{\underline{\ell}}]$$

$$= \underline{\ell}^{*\prime}\underline{\mu} - \frac{1}{2} r \cdot \text{tr}\Sigma E(\underline{\tilde{\ell}\tilde{\ell}'})$$

$$= \underline{\ell}^{*\prime}\underline{\mu} - \frac{1}{2} r \cdot \text{tr}\Sigma[V(\underline{\tilde{\ell}}) + \underline{\ell}^*\underline{\ell}^{*\prime}]$$

$$= \underline{\ell}^{*\prime}\underline{\mu} - \frac{1}{2} r[\text{tr}\Sigma V(\underline{\tilde{\ell}}) + \underline{\ell}^{*\prime}\Sigma\underline{\ell}^*]$$

$$= CE^* - \frac{1}{2} r\ \text{tr}[\Sigma V(\underline{\tilde{\ell}})] < CE^*.$$

The inequality holds because $\text{tr}[\Sigma V(\tilde{\ell})] > 0$, which can be proven using this fact: for A positive semidefinite (PSD) and P nonsingular, P'AP is PSD. Write Σ as PP' and note that $\text{tr}(PP'V(\tilde{\ell})] = \text{tr}[P'V(\tilde{\ell})P]$. The latter term is the sum of nonzero eigenvalues which are all positive since the matrix is PSD.

Now consider using the biased estimator, $\underline{\hat{\ell}}$:

$$CE(\underline{\hat{\ell}}) = E\left[\underline{\hat{\ell}}'\underline{\mu} - \frac{1}{2} r\underline{\hat{\ell}}'\Sigma\underline{\hat{\ell}}\right]$$

Recall that $E(\underline{\hat{\ell}}) = \underline{\ell}^* + \underline{b}$, where $\underline{b}$ denotes Bias $(\underline{\hat{\ell}})$. Thus,

$$CE(\underline{\hat{\ell}}) = \ell^{*\prime}\mu + [\text{Bias}(\hat{\ell})]'\mu - \frac{1}{2} r\ \text{tr}\Sigma[V(\hat{\ell}) + E(\hat{\ell})\ E(\hat{\ell})']$$

$$= \ell^{*\prime}\mu + \underline{b}'\mu - \frac{1}{2} r\ \text{tr}\Sigma V(\hat{\ell}) - \frac{1}{2} r\ \text{tr}\Sigma E(\hat{\ell})\ E(\hat{\ell})'$$

$$= \ell^{*\prime}\mu + b'\mu - \frac{1}{2} r\ \text{tr}\Sigma V(\hat{\ell}) - \frac{1}{2} r\ \text{tr}\Sigma[\ell^* + b]\ [\ell^* + b]'$$

$$= \ell^{*\prime}\mu + b'\mu - \frac{1}{2} r\ \text{tr}\Sigma V(\hat{\ell}) - \frac{1}{2} r[\ell^* + b]'\ \Sigma[\ell^* + b]$$

$$= CE^* + b'\mu - \frac{1}{2} r\ \text{tr}\Sigma V(\hat{\ell}) - \frac{1}{2} rb'\Sigma b - \frac{1}{2} r \cdot 2b'\Sigma\ell^*.$$

It is easy to see that

$$\underline{b}'\underline{\mu} = r\underline{b}'\Sigma\underline{\ell}^*,$$

using earlier results. Thus, $CE(\hat{\underline{\ell}})$ becomes

$$CE(\hat{\underline{\ell}}) = CE^* - \frac{1}{2}\, r\, tr[\Sigma V(\hat{\underline{\ell}})] - \frac{1}{2}\, r\underline{b}'\Sigma\underline{b}.$$

The certainty equivalent associated with $\hat{\underline{\ell}}$ is less than that associated with $\underline{\ell}^*$, since the last two terms are negative.

It is also the case that

$$tr[\Sigma V(\hat{\underline{\ell}})] > tr[\Sigma V(\tilde{\underline{\ell}})]$$

since $V(\hat{\underline{\ell}}) - V(\tilde{\underline{\ell}})$ was shown earlier to be positive semidefinite. Hence

$$CE(\hat{\underline{\ell}}) < CE(\tilde{\underline{\ell}}).$$

Thus, the commonly used PCE estimator $\hat{\underline{\ell}}$ is biased, inefficient, and leads to a lower expected utility than does the unbiased estimator.

5. EXAMPLES FROM FINANCE AND AGRICULTURAL LAND ALLOCATION

To illustrate the importance of these findings, we performed some calculations using our results and parameters from published papers in agricultural economics and finance. We proceed as follows. Assume that the reported sample estimates of the mean vector and covariance matrix are in fact the population parameters of the joint normal distribution of returns to various enterprises (investments). Using results from section three of this paper, we calculate the mean, variance, and certainty equivalent of the PCE estimator as well as the mean (also equal to the true optimum $\underline{\ell}^*$), variance, and certainty equivalent of the unbiased estimator. We repeat this process for several levels of risk aversion and varying sample sizes for each example. Results are presented in Tables 2 and 3.

TABLE 2

Effects of Risk Aversion and Sample Size on Reliability of PCE Estimates of Land Allocation and on Certainty Equivalents

Sample size	Expected values of PCE allocations and certainty equivalents				
	Acres in				
	Carrots	Celery	Cucumbers	Peppers	CE
			r = 0.002924		
6	68.14 (20.25)	28.33 (7.37)	88.29 (20.43)	15.24 (8.80)	-474.63
30	68.61 (6.72)	28.27 (2.45)	88.24 (6.78)	14.89 (2.91)	27854.15
100	68.65 (3.54)	28.26 (1.29)	88.23 (3.58)	14.86 (1.54)	30372.70
Optimal decision	68.66	28.26	88.23	14.85	31343.34
			r = 0.00029		
6	56.18 (32.89)	30.00 (11.34)	89.67 (31.17)	24.16 (15.87)	50680.40
30	60.90 (8.64)	29.34 (3.10)	89.13 (8.56)	20.64 (3.89)	59642.12
100	61.32 (4.50)	29.28 (1.62)	89.08 (4.47)	20.32 (2.02)	60078.62
Optimal decision	61.49	29.26	89.06	20.20	60238.95
			r = 0.0000355		
6	-39.00 (213.77)	43.24 (71.2)	100.67 (194.3)	95.10 (108.75)	16203.81
30	0.44 (45.17)	37.87 (15.77)	96.21 (43.43)	66.36 (21.33)	67296.04
100	3.04 (23.08)	37.39 (8.09)	95.81 (22.29)	63.76 (10.82)	68866.00
Optimal decision	4.38	37.20	95.65	62.76	69407.08

Note: Figures in parentheses are standard deviations of PCE estimators of land allocation.

TABLE 3

Effects of Risk Aversion and Sample Size on Reliability of PCE Estimates of Stock Portfolio Allocation and on Certainty Equivalent

Sample size	Chrysler	New York Shipping	Bulova	CE
	Expected values of PCE allocations and certainty equivalents — Dollars in			
	r = 0.0001			
6	2242.00	480.00	7277.00	
	(14723)	(11113)	(15167)	-3013.00
	(11942)	(9005)	(12294)	-1573.00
30	2346.00	1249.00	6405.00	
	(4676)	(3477)	(4768)	742.00
	(4526)	(3365)	(4615)	768.00
100	2358.00	1341.00	6301.00	
	(2467)	(1831)	(2512)	1039.00
	(2444)	(1814)	(2489)	1041.00
Optimal decision	2363.00	1377.00	6259.00	1153.00
	r = 0.00001			
6	-3206.00	-39882.00	53088.00	
	(143576)	(108573)	(148092)	-35735.00
	(114877)	(86870)	(118490)	-21333.00
30	-2168.00	-32194.00	44362.00	
	(45107)	(33595)	(46043)	151.00
	(43553)	(32438)	(44456)	414.00
100	-2045.00	-31279.00	43323.00	
	(23770)	(17675)	(24236)	2923.00
	(23530)	(17496)	(23992)	2944.00
Optimal decision	-1995.00	-30913.00	42908.00	3984.00
	r = 0.000001			
6	-57689.00	-443507.00	511196.00	
	(1435387)	(1085474)	(1480557)	-368202.00
	(1148311)	(868380)	(1184447)	-224182.00
30	-47311.00	-366626.00	423937.00	
	(450899)	(335834)	(460262)	-9505.00
	(435351)	(324254)	(444391)	-6876.00
100	-46076.00	-357473.00	413549.00	
	(237612)	(176683)	(242272)	18188.00
	(235212)	(174898)	(239825)	18404.00
Optimal decision	-45581.00	-353812.00	409394.00	28795.00

Note: Figures in parentheses are standard deviations of estimates of optimal allocations. For each level of risk aversion and sample size, the first figures in parentheses are the standard deviations of the PCE estimators and the second line of figures in parentheses are the standard deviations of the unbiased estimators.

The first example uses data from Hazell's (1971) article introducing Minimization of Total Absolute Deviations (MOTAD). The example he uses is the allocation of 200 acres of land among four vegetable crops (carrots, celery, cucumbers, and peppers) with sample moments

$$\hat{\underline{\mu}} = [253 \quad 443 \quad 284 \quad 516]'$$

and

$$\hat{\Sigma} = \begin{bmatrix} 11264 & -20548 & 1424 & -15627 \\ -20548 & 125145 & -27305 & 29297 \\ 1424 & -27305 & 10585 & -10984 \\ -15627 & 29297 & -10984 & 93652 \end{bmatrix}.$$

We consider three levels of r (.002924, .00029, .0000355) and three sample sizes (6, 30, and 100). The range of risk attitudes can be characterized as extreme to moderate for the gamble under consideration. This range was chosen for a purely practical consideration--at lower levels of risk aversion, the solution is not an interior one, but the derivations in this paper only apply to interior solutions. Noninterior solutions involve either truncations or negative allocations to some crops; the latter are, of course, impossible. Results are reported in Table 2. In the context of this example, it is interesting to note that the sample size used in the original article was six.

The second example is based on the experiment performed by Frankfurter et al. (1971). They examined the effect of estimation risk on efficient portfolios of $10,000 using three assets (securities of Chrysler, New York Shipping, and Bulova) with returns and variance of returns per dollar invested given by

$$\hat{\underline{\mu}} = [.1664 \quad .0664 \quad .2135]'$$

and

$$\hat{\Sigma} = \begin{bmatrix} .2101 & -.0115 & .1115 \\ -.0115 & .1664 & -.0037 \\ .1115 & -.0037 & .2223 \end{bmatrix}.$$

Unlike the case of land allocation, negative allocations are reasonable and constitute short sales, which we assume are costlessly made. For this example, we use the same sample sizes but allow the measure of absolute risk aversion to take the values (.0001, .00001, and .000001). Results are presented in Table 3.

The reader should note that r, the measure of absolute risk aversion, cannot be chosen arbitrarily by the researcher. Ideally, the individual decision-maker would be able to provide information about his risk preferences. More likely, the researcher will have other indications about the appropriate risk attitudes for the decision-maker (or group of decision-makers). For example, Blume (1980) argues that the appropriate measure of relative risk aversion in U. S. financial markets appears to be about 2. This would imply $r \cong 3*10^{-5}$ for the land allocation example or $r \cong 2*10^{-4}$ for the stock portfolio example. If this characterization is indeed appropriate, our examples illustrate the degree to which recommendations based on the PCE estimators of optimal portfolio allocations should be hedged given limited historical data. Consider, for instance, the land allocation example and a decision-maker with r = 0.0000355. Even with 30 observations, for the crop with the highest mean return, the ratio of the expected amount of land to be allocated to its standard deviation is 3.11. For the stock portfolio example, with r = 0.0001, the same ratio is only 1.34. Thus, a large interval of possible decisions cannot be excluded from consideration on the basis of the PCE estimator.

7. SUMMARY AND CONCLUSIONS

In this paper we have developed expressions for the sampling properties of the widely used PCE estimator of optimal portfolio allocation under uncertainty. Since it ignores an important source of uncertainty, that due to unknown parameters in the distribution of returns from risky investment opportunities, the sampling properties of the PCE estimator--especially with small sample sizes--can be quite poor. We have demonstrated that the PCE estimator is, indeed, biased and inefficient and have suggested an unbiased alternative with a lower variance.

Another important result in this paper is the distinction between the expected utility from allocating ones's portfolio according to the true optimum decision vector and the expected utility of using an estimator of that vector--the latter being necessarily less than the former. However, we showed that the difference is less for the unbiased alternative than for the PCE decision vector.

Although the results in this paper strictly apply to the commonly assumed, but special case of normally distributed investment returns and negative exponential utility, several points are of general concern. First, uncertainty about the nature of the distribution of returns, including uncertainty about the true population parameters of that distribution, is an important source of uncertainty above and beyond any risk recognized in the data. The presence of this uncertainty makes the optimal decision vector random and, therefore, suggests that any prescriptions made should include some acknowledgment of this uncertainty such as confidence intervals or standard deviations of the estimates. To do otherwise implies greater certainty as to the proper course of action than is actually possible.

A second point of general interest is the implication of this research for generating information on the statistical behavior of returns from risky activities. Our expression for $V(\underline{\tilde{I}})$ suggests that researchers can determine the value of collecting more information to lower the uncertainty of estimates of optimal behavior. It would be worthwhile to use these results to develop better estimates of optimal behavior. At a minimum, they show that, in many cases, it will be important to report results as interval estimates, rather than treating as certain what are in fact only estimates of optimal behavior.

APPENDIX

A.1. Verification of the inverse of $\begin{bmatrix} (n-1)^{-1} AVV' \\ i_k' \end{bmatrix}$.

I_k must equal

$$\begin{bmatrix} (n-1)^{-1} AVV' \\ i_k' \end{bmatrix} \left[(n-1) A'(AVV'A')^{-1} \left[I_{k-1} \;\vdots\; \frac{-AVV'e_1}{n-1} \right] + (0 \;\vdots\; e_1) \right]$$

$$\begin{bmatrix} (n-1)^{-1} AVV' \\ i_k' \end{bmatrix} \left[(n-1) A'(AVV'A')^{-1} I_{k-1} + \underline{\underline{0}} \;\vdots\; -A'(AVV'A')^{-1} AVV'e_1 + e_1 \right]$$

which is of the form

$$\begin{bmatrix} \underset{k-1\ k}{A} \\ \underset{1\ k}{B} \end{bmatrix} \begin{bmatrix} \underset{k\ k-1}{C} & \underset{k\ 1}{D} \end{bmatrix} = \begin{bmatrix} \underset{k-1\ k-1}{AC} & \underset{k-1\ 1}{AD} \\ \underset{1\ k-1}{BC} & \underset{1\ 1}{BD} \end{bmatrix}.$$

For the product to equal I_k, we must find that $AC = I_{k-1}$, $BD = 1$, $BC = \underline{0}'$ [of dimension $1 \times (k-1)$], and $AD = \underline{0}$ [of dimension $(k-1) \times 1$].

$$AC = (n-1)^{-1} (n-1) AVV'A'(AVV'A')^{-1} I_{k-1} = I_{k-1}$$

$$BD = i_k' \left[-A'(AVV'A')^{-1} AVV'e_1 + e_1 \right]$$

$$= i_k' e_1 - i_k' A'(AVV'A')^{-1} AVV'e_1$$

$$= 1.$$

The term in BD involving $i_k' A'$ vanishes since the rows of A (columns of A') each sum to 1. This also holds for BC making that product $\underline{0}'$.

Finally,

$$AD = (n-1)^{-1} AVV' - A'(AVV'A')^{-1} AVV'e_1 + e_1$$

$$= -(n-1)^{-1} AVV'A'(AVV'A')^{-1} AVV'e_1 + (n-1)^{-1} AVV'e_1 = 0.$$

Q.E.D.

A.2. Derivation of $\underline{\alpha}$ and $\underline{u}$.

$$E[AVV'_{1\cdot}] = E[AVV'e_1] = E\{AV[(AV)'\underline{\alpha} + u]\}$$

$$= E[AVV'A'\underline{\alpha} + AVu]$$

$$= A\,E(VV')\,A'\underline{\alpha} + 0.$$

Since $E(VV') = E(Z\,P_n\,Z') = (n-1)\,\Sigma$, $E(AVV'_{1\cdot}) = E(AVV'e_1) = A[(n-1)\,\Sigma]\,e_1 = (n-1)\,A\Sigma_{\cdot 1}$, where $\Sigma_{\cdot 1}$ is the first column of Σ. Therefore, $\underline{\alpha}$ must solve

$$(n-1)\,A\Sigma_{\cdot 1} = A[(n-1)\,\Sigma]\,A'\underline{\alpha}$$

or

$$\underline{\alpha} = (A\Sigma A')^{-1} A\Sigma_{\cdot 1}.$$

This defines the linear combination of the columns of $(AV)'$ that separates $V'e_1$ into two parts, one independent of AV.

The desired result, $E(AV\underline{u}) = \underline{0}$, is obtained:

$$\underline{u} = V'_{1\cdot} - V'A'\underline{\alpha}$$

$$= V'(e_1 - A'\underline{\alpha})$$

and

$$E(AV\underline{u}) = E[AVV'(e_1 - A'\underline{\alpha})]$$

$$= E(AVV'e_1) - E(AVV'A'\underline{\alpha})$$

$$= (n-1)\, A\Sigma_{\cdot 1} - A\, E(VV')\, A'\, (A\Sigma A')^{-1}\, A\Sigma_{\cdot 1}$$

$$= (n-1)\, A\Sigma_{\cdot 1} - (n-1)\, (A\Sigma A')\, (A\Sigma A')^{-1}\, A\Sigma_{\cdot 1}$$

$$= (n-1)\, A\Sigma_{\cdot 1} - (n-1)\, A\Sigma_{\cdot 1} = \underline{0}.$$

A.3. Independence of V and $Z\underline{i}_n$:

We have

$$Z\, P_n\, Z' = Z\, U\, D\, U'\, Z' = VV'$$

since P_n is of rank n - 1 and has eigenvalues 0 or 1. D can be taken to be the matrix

$$\begin{pmatrix} I_{n-1} & \underline{0} \\ \underline{0}' & 0 \end{pmatrix}.$$

Hence,

$$V = Z\, U\, \Lambda = Z\, U \begin{pmatrix} I_{n-1} \\ \underline{0}' \end{pmatrix}$$

so that

$$V_{ij} = \sum_{p=1}^{n-1} Z_{ip}\, U_{pj}.$$

A typical element of $Z\underline{i}_n$, the $\underline{k}$th one, is

$$\sum_{\ell=1}^{n} Z_{k\ell}.$$

We now show independence of typical elements of V and $Z\underline{i}_n$:

$$E[V_{ij} \cdot (Z\underline{i}_n)_k] = E[\sum_{p=1}^{n-1} \sum_{\ell=1}^{n} Z_{ip} U_{pj} Z_{k\ell}]$$

$$= \sum_{p=1}^{n-1} U_{pj} E[Z_{ip} Z_{kp}]$$

$$= \sum_{p=1}^{n-1} U_{pj} \Sigma_{ik}$$

(since $E[Z_{ip} Z_{k\ell}] = 0$ unless $p = \ell$)

$$= \Sigma_{ik} \sum_{p=1}^{n-1} U_{pj}.$$

Now, since $\underline{i}_n' P_n = \underline{0}'$, we are guaranteed that $\Sigma_{p=1}^{n-1} U_{pj} = 0$:

$$\underline{0}' = \underline{i}_n' P_n = \underline{i}_n U D U' = \underline{i}_n' U \begin{bmatrix} I_{n-1} & \underline{0} \\ \underline{0}' & 0 \end{bmatrix} U'$$

$$= \underline{i}_n' U \begin{pmatrix} I_{n-1} & \underline{0} \\ \underline{0}' & 0 \end{pmatrix}$$

$$\Rightarrow \sum_{p=1}^{n-1} U_{pj} = 0 \qquad \text{for all } j = 1, 2, \ldots, n-1.$$

$$E[\sum_{\ell} \sum_{p} Z_{i\ell} U_{\ell p} \Lambda_{pj} \cdot \sum_{n} Z_{kn}] = E[\sum_{\ell} \sum_{p} \sum_{n} Z_{i\ell} U_{\ell p} \Lambda_{pj} \cdot Z_{kn}]$$

$$= E[\sum_{\ell} \sum_{n} Z_{i\ell} U_{\ell j} \text{ (0 unless } p = j) \cdot Z_{kn}].$$

Now, unless $\ell = n$, then $E(Z_{i\ell} Z_{kn}) = 0$, by timewise independence. Hence, we obtain

$$E[V_{ij} \cdot (Zi_n)_k] = E[\sum_{\ell} Z_{i\ell} U_{\ell j} Z_{k\ell}]$$

$$= \Sigma_{ik} \cdot \sum_{\ell} U_{\ell j} = \Sigma_{ik} \cdot \underline{i}_n' \underline{u}_j.$$

The inner product of i_n and any characteristic vector of P_n, such as $\underline{u}_j$, must be 0, which gives the desired result. Every element in V and in Zi_n is normally distributed; we have now shown that every element in V has a zero correlation with every element in Zi_n, implying independence.

A.4. Expectations Involving $Z\underline{i}_n$ and $\underline{u}$.

A first step in simplifying is to evaluate $E_Z(Zi_n i_n' Z')$. Consider a typical element, the $\underline{i},\underline{j}$th one; its expectation is

$$E\left[\sum_{\ell} Z_{i\ell}\, \underline{i}_{n_\ell} \sum_{m} \underline{i}_{n\,m}' \, Z_{mj}'\right] = E\left[\sum_{\ell} Z_{i\ell} \sum_{m} Z_{jm}\right]$$

which is the expectation of the $\underline{i}$th row sum in Z times the $\underline{j}$th row sum in Z. The first term is the sum of observations through time on the random variable Z_i, which is the $\underline{i}$th return X_i minus its mean μ_i. The second term is the sum of observations through time on Z_j.

For any i,j combination, there are n terms in the product of these sums with expectations equal to the $\underline{i},\underline{j}$th element of Σ, the rest vanish by time-wise independence. Hence, the term above has expectation n Σ_{ij}, and

$$E[Z\underline{i}_n\underline{i}_n' Z'] = n\, \Sigma.$$

Of course, $E(Z\underline{i}_n) = 0$. When $E(Z\underline{i}_n)$ appears alone, it vanishes.

We also need $E(\underline{u}\underline{u}')$. Recall that

$$\underline{u} = V'[\underline{e}_1 - A'\underline{\alpha}]$$

and that V can be written as

$$V = Z\ U\ \Lambda$$

where V is k x (n - 1) and $\Lambda = \begin{bmatrix} I_{n-1} \\ \underline{0}' \end{bmatrix}$.

Then

$$\begin{aligned} E(uu') &= E[V'(e_1 - A'\alpha)\ (e_1 - A'\alpha)'\ V] \\ &= E[\Lambda'\ U'\ Z'(e_1 - A'\alpha)\ (e_1 - A'\alpha)'\ Z\ U\ \Lambda] \\ &= \Lambda'\ U'\ E[Z'(e_1 - A'\alpha)\ (e_1 - A'\alpha)'\ Z]\ U\ \Lambda. \end{aligned}$$

Consider any expectation of the form

$$E[Z'b\ b'Z].$$

The expectation of a typical element i,j is the expectation of the $\underline{i}$th element in the column vector $Z'\underline{b}$ times the $\underline{j}$th element in the row vector $\underline{b}'Z$.

$$\begin{aligned} E[Z'b\ b'Z]_{i,j} &= E(\sum_m Z'_{im}\ b_m)\ (\sum_k b_k\ Z_{kj}) \\ &= E\left[\sum_m \sum_k b_m\ b_k\ Z'_{im}\ Z_{kj}\right] \\ &= \sum_m \sum_k b_m\ b_k\ E(Z'_{im}\ Z_{kj}). \end{aligned}$$

Here, i and j denote sample periods while m and k denote particular investment opportunities or land uses. Unless i = j, the Z terms are independent by timewise independence, hence, we obtain

$$\sum_m \sum_k b_m b_k \cdot \Sigma_{mk}$$

and

$$E[Z'b\, b'Z] = \underline{b}'\, \Sigma\, \underline{b}\, I_n.$$

Thus,

$$E(\underline{uu}') = \Lambda'\, U'\, E[Z'(e_1 - A'\underline{\alpha})\, (\underline{e}_1 - A'\underline{\alpha})'\, Z]\, U\, \Lambda$$

$$= \Lambda'\, U'\, (e_1 - A'\underline{\alpha})'\, \Sigma\, (e_1 - A'\underline{\alpha})\, U\, \Lambda$$

$$= \Lambda'\, U'[\underline{e}_1'\, \Sigma\, \underline{e}_1 - \underline{e}_1'\, \Sigma\, A'(A\Sigma A')^{-1}\, A\Sigma_{\cdot 1} - \Sigma_{\cdot 1}' A'(A\Sigma A')^{-1}\, A\Sigma\underline{e}_1$$

$$+ \Sigma_{\cdot 1}'\, A'(A\Sigma A')^{-1}\, A\Sigma A'(A\Sigma A')^{-1}\, A\Sigma_{\cdot 1}]\, U\, \Lambda$$

$$= \Lambda'\, U'[\Sigma_{11} - \Sigma_{\cdot 1}'\, A'(A\Sigma A')^{-1}\, A\Sigma_{\cdot 1}]\, U\, \Lambda$$

$$= [\Sigma_{11} - \Sigma_{\cdot 1}'\, A'(A\Sigma A')^{-1}\, A\Sigma_{\cdot 1}] \cdot \Lambda'\, U'\, U\, \Lambda$$

$$= [\Sigma_{11} - \Sigma_{\cdot 1}'\, A'(A\Sigma A')^{-1}\, A\Sigma_{\cdot 1}] \cdot \Lambda'\, \Lambda.$$

Since $\Lambda'\, \Lambda = I_{n-1}$, we obtain the diagonal matrix $\bar{\Sigma} \cdot I_{n-1}$, where

$$\bar{\Sigma} = \Sigma_{11} - \Sigma_{\cdot 1}'\, A'(A\Sigma A')^{-1}\, A\Sigma_{\cdot 1}\, .$$

A.5. Simplification of $V(\tilde{\underline{x}})$.

Note that the matrix

$$A \left[\underline{\mu}\, \underline{\mu}' + \frac{1}{n}\, \Sigma \right] A'$$

is positive definite and so can be expressed as C C', where C is full rank and lower triangular. Now let $S = C^{-1}\, T$. A series of manipulations using these matrices is necessary to simplify the expectation which remains in the expression for the variance matrix for $\tilde{\underline{\gamma}}$:

$$E_W \left[A' \bar{W} A \left(\underline{\mu}\, \underline{\mu}' + \frac{1}{n} \Sigma \right) A' \bar{W} A \right]$$

$$= E_W \; [A' \bar{W} C\, C' \bar{W} A]$$

$$= E_T \{A' [(T\,T')^{-1} - E(T\,T')^{-1}]\; C\, C' [(T\,T')^{-1} - E(T\,T')^{-1}]\; A\}$$

$$= A'\, E_T \left[\left\{ [C\, C^{-1} (T\,T')\, C'^{-1}\, C']^{-1} - E[C\, C^{-1} (T\,T')\, C'^{-1}\, C']^{-1} \right\} C\, C' \left\{ \cdot \right\} \right] A$$

$$= A'\, C'^{-1}\, E_S \{[(S\,S')^{-1} - E(S\,S')^{-1}]\; [(S\,S')^{-1} - E(S\,S')^{-1}]\}\; C^{-1} A.$$

Now, the matrix $S = C^{-1}\, T = C^{-1}\, A\, V$ has dimension $(k - 1) \times (k - 1)$; furthermore, $S\,S'$ follows the Wishart distribution because $(T\,T')$ does so. It has an expectation given by

$$E(S\,S') = C^{-1}\, E(T\,T')\, C'^{-1} = (n - 1)\, C^{-1} A\, \Sigma\, A'\, C'^{-1} = (n - 1)\, \Sigma_S.$$

To evaluate the expectation

$$E\{[(S\,S')^{-1} - E(S\,S')^{-1}]\; [(S\,S')^{-1} - E(S\,S')^{-1}]\},$$

we use a result due to Shaman (1980):

$$\mathrm{Cov}[\mathrm{vec}(S\,S')^{-1}] = \frac{\Sigma_S^{-1} \otimes \Sigma_S^{-1} + (n - k)^{-1}\, (\mathrm{vec}\; \Sigma_S^{-1})\, (\mathrm{vec}\; \Sigma_S^{-1})'}{(n - k + 1)\, (n - k)\, (n - k - 1)}$$

provided that $n > k + 1$.

Suppose we are interested in the $\underline{i,j}$th element in our expectation.

$$E \left\{ \left[(SS')^{-1} - E(SS')^{-1} \right] \left[(SS')^{-1} - E(SS')^{-1} \right] \right\}_{i,j}$$

$$= \sum_{\ell} E\left\{ \left[(SS')^{-1} - E(SS')^{-1}\right]_{i\ell} \left[(SS')^{-1} - E(SS')^{-1}\right]_{\ell j} \right\}.$$

The expectations in that sum can be found by making use of Shaman's result. Note that the vec of $(SS')^{-1}$ is a $(k - 1)^2 \times 1$ vector, making the variance matrix $(k - 1)^2 \times (k - 1)^2$. The elements $i\ell$ and ℓj of $(SS')^{-1}$ can be found in that vector as elements $i + (\ell - 1)(k - 1)$ and $\ell + (j - 1)(k - 1)$, respectively.

To find the expected value of their product then, we find element $i + (\ell - 1)(k - 1)$, $\ell + (j - 1)(k - 1)$ in Shaman's expression. That element involves the scalar $[(n - k + 1)(n - k)(n - k - 1)]^{-1}$, the appropriate element of $\Sigma_S^{-1} \times \Sigma_S^{-1}$, and the corresponding element from $(n - k)^{-1} (\text{vec } \Sigma_S^{-1}) (\text{vec } \Sigma_S^{-1})'$. Recall that the Kronecker product of a $(k - 1) \times (k - 1)$ matrix with itself produces a $(k - 1)^2 \times (k - 1)^2$ matrix in which the $\underline{i,j}$th block here will be $(\Sigma_S^{-1})_{ij} \cdot \Sigma_S^{-1}$. If we find row $i + (\ell - 1)(k - 1)$, we are in row i of the $\underline{\ell}$th row of blocks; moving over to column $\ell + (j - 1)(k - 1)$ means we are in column ℓ of the $\underline{j}$th column of blocks. Thus, since we are in block-row ℓ, block-column j, we obtain $(\Sigma_S^{-1})_{\ell j}$. That element is multiplied by Σ_S^{-1} to make up that $\underline{\ell,j}$th block, but we are interested only in the $\underline{i,\ell}$th element, so we obtain

$$\left(\Sigma_S\right)^{-1}_{\ell j} \left(\Sigma_S\right)^{-1}_{i\ell}$$

from the Kronecker product.

By similar operations, extracting that element from vec Σ_S^{-1} times its transpose produces $(\Sigma_S^{-1})_{i\ell}$ from the vec and $(\Sigma_S^{-1})_{\ell j}$ from its transpose. Hence, our expectation is

$$[(n - k + 1)(n - k)(n - k - 1)]^{-1}[(\Sigma_S^{-1})_{\ell j}(\Sigma_S^{-1})_{i\ell} + (n - k)^{-1}(\Sigma_S^{-1})_{i\ell}(\Sigma_S^{-1})_{\ell j}]$$

and it simplifies to

$$\frac{\left(\Sigma_S^{-1}\right)_{\ell j}\left(\Sigma_S^{-1}\right)_{i\ell}}{(n - k - 1)\,(n - k)^2}$$

which, when summed over ℓ, yields the $\underline{i,j}$th element of

$$\frac{\Sigma_S^{-1} \cdot \Sigma_S^{-1}}{(n - k - 1)\,(n - k)^2}.$$

This characterizes every element of

$$E\left\{[(SS')^{-1} - E(SS')^{-1}]\,[(SS')^{-1} - E(SS')^{-1}]\right\}$$

so that we can rewrite the first term in our expression for $V(\tilde{\ell})$ as follows:

$$\left(\frac{n - k - 1}{r}\right)^2 A'\,C'^{-1} \cdot \frac{\Sigma_S^{-1}\;\Sigma_S^{-1}}{(n - k - 1)\,(n - k)^2}\,C^{-1} A$$

by substituting Shaman's result. Now, replace Σ_s^{-1} by the inverse of $C^{-1} A\,\Sigma\,A'\,C'^{-1}$:

$$=\left(\frac{n - k - 1}{r}\right)^2 \frac{1}{(n - k - 1)\,(n - k)^2} A'\,C'^{-1}\left[C^{-1}\,A\Sigma A'C'^{-1}\right]^{-1}\left[C^{-1}\,A\Sigma A'C'^{-1}\right]^{-1} C^{-1} A$$

$$=\left(\frac{n - k - 1}{r}\right)^2 \frac{1}{(n - k - 1)\,(n - k)^2} A'\,(A\Sigma A')^{-1}\,C\,C'\,(A\Sigma A')^{-1}\,A$$

$$=\left(\frac{n - k - 1}{r}\right)^2 \frac{1}{(n - k - 1)\,(n - k)^2} A'(A\Sigma A')^{-1} \cdot A(\underline{\mu\mu}' + \frac{1}{n}\,\Sigma)\,A'(A\Sigma A')^{-1}\,A.$$

$$=\left(\frac{n - k - 1}{r}\right)^2 \frac{1}{(n - k - 1)\,(n - k)^2} A'(A\Sigma A')^{-1}\,A\underline{\mu\mu}'A'(A\Sigma A')^{-1}\,A$$

$$+\left(\frac{n - k - 1}{r}\right)^2 \frac{1}{(n - k - 1)\,(n - k)^2} A'(A\Sigma A')^{-1}\,(A\,\frac{1}{n}\,\Sigma A')\,(A\Sigma A')^{-1}\,A$$

$$= \left(\frac{n - k - 1}{r}\right)^2 \frac{1}{(n - k - 1)(n - k)^2} A'(A\Sigma A')^{-1} A\underline{\mu}\underline{\mu}' A'(A\Sigma A')^{-1} A$$

$$+ \left(\frac{n - k - 1}{r}\right)^2 \frac{1}{n(n - k - 1)(n - k)^2} A'(A\Sigma A')^{-1} A.$$

If we combine our terms, we find that

$$V(\tilde{\underline{\ell}}) = \frac{(n - k - 1)^2}{r^2} \frac{1}{(n - k - 1)(n - k)^2} A'(A \Sigma A')^{-1} A\underline{\mu}\, \underline{\mu}' A'(A \Sigma A')^{-1} A$$

$$+ \frac{(n - k - 1)^2}{r^2} \frac{1}{n(n - k - 1)(n - k)^2} A'(A \Sigma A')^{-1} A$$

$$+ \left(\frac{\bar{\Sigma} L^2}{n - k - 1} + \frac{1}{nr^2}\right) A'(A \Sigma A')^{-1} A$$

or

$$V(\tilde{\underline{\ell}}) = \frac{(n - k - 1)}{r^2(n - k)^2} A'(A \Sigma A')^{-1} A\underline{\mu}\, \underline{\mu}' A'(A \Sigma A')^{-1} A$$

$$+ \left(\frac{(n - k - 1)}{nr^2(n - k)^2} + \frac{\bar{\Sigma} L^2}{(n - k - 1)} + \frac{1}{nr^2}\right) A'(A \Sigma A')^{-1} A.$$

FOOTNOTES

[1]Fama (1965) and others have demonstrated that this condition is overly restrictive. Returns must be drawn from a probability distribution belonging to a "location-scale" family.

[2]The use of the negative exponential utility function simplifies the derivations which follow, and they hold, strictly speaking, only for that utility function. However, the qualitative results we establish concerning the problem of estimation risk are likely to carry over to other sets of risk attitudes. Analytic results for those utility functions will be complicated by the fact that the Arrow-Pratt measure of absolute risk aversion, r, will itself become random since it depends on expected end-of-period wealth. The widely used negative exponential utility function is convenient since r is constant for all levels of expected wealth.

[3]Recall that expected utility and the certainty equivalent are related monotonically--maximizing one is, therefore, equivalent to maximizing the other.

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