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Normative Economics under Uncertainty and Risk Aversion:
The Land Allocation Problem Revisited

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1. Introduction

The land allocation problem is a cornerstone of farm management and farm management education. The question of how the farm manager can best allocate land among competing enterprises has been solved under a wide variety of maintained hypotheses and degrees of complexity. Approaches range from linear programming to stochastic dominance and risk programming techniques. Under conditions of risk aversion and uncertainty, continuous choice techniques for solving the land allocation problem include quadratic programming, minimization of total absolute deviations (MOTAD), mean -- semi-variance programming, th expected utility -- moment generating function (EUMGF) technique and various incarnations of the single index model.

While economists have been concerned with the relative performance of the various techniques (cf. Collins and Barry, Collender and Chalfant (1986b), and Thompson and Hazell) and with the sensitivity of the objective function to "quasi-optimal" solutions (cf. Collins and Barry, Schurle and Erven), they have as a rule ignored the overall question of the ability of these techniques to provide reliable estimators of the full information optimum allocations, although it has been widely recognized that solutions under any of these techniques will be suboptimal given less than perfect knowledge about the statistical distributions of returns. This paper explores the nature of this suboptimality in more detail for the quadratic programming solution to the land allocation problem for both the utility maximizing objective and the description of the efficient frontier. The reader should note that under validating assumptions, e.g., normality of returns, the quadratic programming estimator of the optimal land allocation will necessarily outperform the MOTAD estimators (Hazell, p. 58) and be equivalent to the EUMGF estimators (Collender and Zilberman).

We show in this paper that estimation risk can have significant implications for the optimal behavior of the farm manager and severely limits the normative role economists can hope to play without devoting considerable time and effort toward assuring the generation of data sufficient to guarantee the performance of these techniques.

The paper proceeds as follows. In the next section we outline the portfolio choice problem and reformulate it to explicitly recognize the estimation involved. In the following section we discuss various types of confidence intervals that can be constructed around the efficient frontier and around the estimators of the optimal allocation vectors. In the fourth section we illustrate these confidence intervals using data from the Imperial Valley of California. Finally we draw some conclusions and suggest possibilities for further research.

2. The Model

In the context of these approaches to the land allocation problem, normative economists have taken two general tacks. The first is to elicit risk attitudes and find a utility maximizing land allocation given these attitudes. The second, and more common, is to develop an efficient frontier and allow individual decision makers to choose points on the frontier themselves. In this section we formally develop the effects of estimation risk on each of these approaches.

Maximizing Expected Utility

The single period portfolio choice problem asks the farm manager to allocate his land among risky alternatives to maximize the expected utility of end of period profit. In general this problem can be formalized as follows:

$$(1) \qquad \max_{\underline{l} \epsilon C} EU(\underline{l}'\tilde{\underline{x}})$$

$$\text{subject to } \underline{l}'\underline{i} = L \quad \text{and } A\underline{l} \leq \underline{b}$$

where L is the initial endowment of land, $\tilde{\underline{x}}$ is the random vector of returns from possible investments or production choices, $\underline{l}$ is the decision vector measured in absolute level, **C** is the choice set, A is a matrix of technical coefficients, $\underline{b}$ is a vector of resource constraints and $\underline{i}$ is a vector of ones.

Under a set of quite restrictive, but commonly imposed assumptions, such as quadratic utility or negative exponential utility and normality of returns from all investment opportunities, the problem in (1) reduces to a problem that is linear in the mean and variance of portfolio returns. We develop the solution to (1) assuming normality of returns and negative exponential utility, following the land allocation literature that originated with Freund and has been expanded by recent papers by Collender and Zilberman and Collender and Chalfant (1986a).

The problem is to allocate L acres of land to K crops, where returns per acre $\tilde{\underline{x}}$ are distributed $N_K(\underline{\mu}, \Sigma)$. We assume that the decision-maker maximizes the expected value of an exponential utility function

$u(\tilde{\pi}) = -\exp(-r\tilde{\pi})$

where r is the Arrow-Pratt measure of absolute risk aversion and $\tilde{\pi}$ denotes single period profits:

$$\tilde{\pi} = \underline{l}'\tilde{\underline{x}}$$

and l_i is the acreage planted to crop i. We assume that per acre returns are net of production costs, and we treat the technologies as predetermined and consider only the acreage decision.

With exponential utility the expected utility (1) is the moment generating function of the random variable $\tilde{\pi}$. The first-order conditions for maximizing expected utility are

$$\frac{M_1}{M} = \frac{M_i}{M}, \quad i = 2, \ldots, K,$$

where M is the moment generating function and M_i is its derivative with respect to $t_i = -r\, l_i$. Chalfant, Collender, and Subramanian show for the special case of normality of returns and no binding restrictions, the optimal land allocation vector, $\underline{l}^*$, solves

$$(1')\quad \hat{\underline{l}} = \begin{bmatrix} A\hat{\Sigma} \\ \underline{i}_k' \end{bmatrix}^{-1} \begin{bmatrix} 1/r\ A\hat{\mu} \\ L \end{bmatrix}$$

If true population parameters exist and are known and used in (2) then $\hat{\underline{l}}$ would be the optimal decision (hereafter $\underline{l}^*$) in the sense of maximizing expected utility. Estimation risk exists if parameter estimates are used in place of population parameters. A common practice is to calculate $\hat{\underline{l}}$ using the parameter certainty equivalent (PCE) or plug-in method, i.e., to treat sample parameters as if they were population parameters in solving (1′). Thus, this practice ignores the sampling distribution of parameter estimates. The decision will be suboptimal if $\hat{\underline{l}}$ differs from $\underline{l}^*$ in the sense that $EU(\tilde{\pi}|\hat{\underline{l}}) <$

$EU(\tilde{\pi}|\underline{l}^*)$. Under this practice, or many other estimation procedures, $\hat{\underline{l}}$ is random, as it is a function of past realizations of returns, through $\hat{\underline{\mu}}$ and $\hat{\Sigma}$.

Estimating the Efficient Frontier

Under perfect information concerning the distributions of returns the efficient frontier is obtained by minimizing the variance of income subject to a certain level of expected return and a set of technological and resource constraints as follows.

(2) $\min \underline{l}'\Sigma\underline{l}$

subject to $\underline{l}'\underline{\mu} = \gamma$ and $A\underline{l} \leq \underline{b}$,

where $0 \leq \gamma \leq \infty$ and $A\underline{l} \leq \underline{b}$ describes resource and technological constraints. If sample estimates are substituted for Σ and $\underline{\mu}$, then the coordinates describing the estimated efficient frontier will themselves be random variables. We proceed in the next section to discuss appropriate confidence intervals for estimates of the utility maximizing points obtained by solving (1) and for estimates of the efficient frontier obtained by solving (2).

3. Statistical Measures of Performance

The point estimates that are typically reported as the end product of land allocation modeling are seriously flawed since they ignore the estimation problems described in section 2 and thus the variability of the estimator around the true optimal allocation. One step that can be taken to illuminate the reliability associated with quadratic programming is to report confidence intervals rather than point estimates. There are a number of possible confidence intervals that can be reported. Farm managers will be interested in the confidence intervals around the efficient frontier (in mean, standard deviation space) and around the allocation vector (in land allocation space).

In addition, it is crucial to recognize the effects of estimation risk on the ex ante expected utility or certainty equivalent.

Thus, the statistical reliability of land allocation rules can be measured in several dimensions in the presence of estimation risk. In some of the few published studies that explicitly recognized estimation risk, Blume and Collender and Zilberman both measured for statistical differences in expected utility of using different portfolio strategies. In a more methodological work, Chalfant, Collender and Subramanian derive the variance of the PCE estimator of $\underline{1}^*$ as well as measuring its ex ante expected utility when the only constraint is on total land allocated.

It is well known that the confidence interval around a sample mean drawn from a sample with unknown variance takes the form (see Hogg and Craig, pp. 212-227 for derivations of confidence intervals):

(3) $[\hat{\mu} - bS/(n - 1)^{.5} , \hat{\mu} + bS/(n - 1)^{.5}]$

where $\bar{x}$ is the sample mean, S is the sample standard deviation, n is the sample size and b is the critical value for the student's t distribution at the desired confidence level. Similarly, the confidence interval around the sample variance takes the form:

(4) $[nS /d, nS /c]$,

where c and d are the upper and lower critical values for the appropriate χ at the desired confidence level. Confidence intervals around the standard deviation reported in the next section are simply the positve square root of the confidence intervals around the variance.

It is also well known that the sample mean and the sample variance are independent of each other (cf. Theil, p. 91). Thus, the confidence intervals around points on the efficient frontier will be rectangular boxes, and the confidence interval around the whole frontier will simply be the area between the lines connecting the northwest and southeast corners of these boxes as illustrated in figure 1.[1] Similarly, confidence intervals around the certainty equivalent of the gamble represented by a particular land allocation take the form

$$[\mu_1 - r/2\ \sigma_u^2,\ \mu_u - r/2\ \sigma_1^2]$$

where the subscripts u and l refer to the upper and lower bounds on the portfolio mean and variance respectively.

Of course, these confidence intervals apply to comparisons of the mean and variance to a constant. Often the comparison is made to stochastic alternatives. In this case, the desired confidence interval will be that for the difference in two means or the difference in two variances. These confidence intervals are also well known in the statistics literature. The confidence interval for the difference between two means is

(5) $$(\hat{\mu}_1 - \hat{\mu}_2) - bR,\ (\hat{\mu}_1 - \hat{\mu}_2) + bR,$$

where $R = \{[(nS_1^2 + mS_2^2)/(n + m - 2)]*(1/n + 1/m)\}^{.5}$, n is the sample size for the first mean, m is the sample size for the second mean, and b is as defined in (3). The confidence interval for the difference in two variances is

(6) $$f[nS_1^2/(n - 1)]/[mS_2^2/(m - 1)],\ g[nS_1^2/(n - 1)]/[mS_2^2/(m - 1)],$$

where f and g are the inverse F cumulative distribution function for the upper and lower critical values associated with a given confidence level.

4. An Example

Data from Hazell's article introducing MOTAD will serve to illustrate the importance of considering confidence intervals. The example Hazell examines is the allocation of 200 acres of land among four vegetable crops with sample moments

	(x1) Carrots	(x2) Celery	(x3) Cucumbers	(x4) Peppers
$\hat{\mu}$ =	253	443	284	516

and

$$\hat{\Sigma} = \begin{bmatrix} 11264 & -20548 & 1424 & -15627 \\ -20548 & 125145 & -27305 & 29297 \\ 1424 & -27305 & 10585 & -10984 \\ -15627 & 29297 & -10984 & 93652 \end{bmatrix}.$$

The allocation problem is also subject to additional constraints on labor hours and on rotation. These additional constraints, however are not binding at the levels of risk aversion discussed in this paper.

We proceed as follows. Assume that the sample estimates of the mean vector and covariance matrix are in fact the population parameters of the joint normal distribution of returns to various enterprises. Using results from section three of this paper, we calculate the PCE efficient frontier and the confidence intervals around the portfolio mean and variance. To compute the efficient frontier, the γ was allowed to vary from \$62,609 to \$77,140. This range corresponds to levels of absolute risk aversion from .002924 to .0000355, a range encompassing risk attitudes that can be characterized as extreme to moderate for the gamble under consideration. For each level of risk aversion reported in Table 1, we report the mean and standard deviation of the efficient land allocation, and the 90% confidence intervals around these sample estimates. Confidence intervals were computed based on 6, 30 and 100 observations, six observations being the sample size used by Hazell. These results are reported in Table 1. Figure 2 illustrates the effect of sample size on width of confidence interval for 6 and 100 observations.

Several important conclusions can be drawn from the information in Table 1. First, as risk aversion decreases the confidence intervals around the portfolio mean and standard deviation both increase holding everything else constant. This happens because as risk aversion decreases the variance of the optimal portfolio increases. This increase in variance leads directly to an increase in the width of the confidence intervals. Second, for the small sample size, these confidence intervals are so wide that it is statistically impossible to distinguish among any of the points on the frontier with a 90% confidence level. As sample size increases, confidence intervals shrink, but even with one hundred observations they are not so trivial as to justify ignoring them.

A closely related topic is explored in papers by Collins and Barry and by Schurle and Erven. A major focus of these papers is the degree to which sub-optimal land allocation (e.g., land allocation based on a less than full scale Markowitz model) leads to deterioration in the value of the objective function. Both sets of authors argue that the loss from "quasi-optimal" behavior is often too small to justify the costs of further optimizing. This situation calls for a comparison between stochastic means and variances. Therefore, the proper confidence intervals to use take the form of (5) and (6), and will necessarily be wider than those presented in Table 1, holding all else constant. In fact, for most agricultural applications data is sparse enough that it is statistically difficult to distinguish among a wide range of allocations.

Chalfant, Collender and Subramanian have shown that the PCE estimates of optimal land allocation are biased in small samples. They also have derived expressions for the variance of the land allocation vector and the certainty equivalent of using PCE estimates when the problem is unconstrained. We use these results to calculate the expectation of the PCE efficient frontier, the expectation and standard deviation of the PCE estimates of the optimal land allocation, and the certainty equivalent of the expectation of the PCE estimate. In Table 2 we present this information for samples of size 6, 30 and 100 observations. The results are compared to the full information optimum land allocation, its variance and its certainty equivalent.

Several conclusions can be drawn from the data in Table 2. First, as risk aversion increases or sample size decreases, estimation risk increases the bias and variance of the PCE estimates. Not recognizing estimation risk also leads decision makers to choose higher mean, higher variance land allocations than would be optimal given their risk preferences. In addition, the presence of estimation risk reduces the certainty equivalent of the gamble -- in some cases enough to make the certainty equivalent negative, which would cause the decision maker to plant less than his total land or forgo farming altogether. Interestingly, in small samples the certainty equivalent under estimation risk is not a monotonically decreasing function of the absolute risk aversion. At some point, the increase in bias and variance of the PCE estimates of land allocation caused by the decrease in risk aversion combined with the bias toward choosing higher variance portfolios increases the risk premium more than the reduction in risk aversion decreases it.

5. Summary and Conclusions

In this paper we have presented some results concerning the reliability of normative conclusions drawn from quadratic programming in the presence of estimation risk. We have developed confidence intervals around the mean-variance efficient frontier and around mean-variance utility maximizing points when the land allocation is taken as given. We illustrate these confidence intervals and the results of Chalfant, Collender and Subramanian with data from Hazell's pioneering paper on MOTAD. This illustration clearly demonstrates the degree to which the presence of estimation risk can call into question the reliability of normative recommendations based on less than complete information regarding the distributions of net returns from risky agricultural enterprises. Even with one hundred observations, at moderate levels of risk aversion the ratio of the mean to the standard deviation of PCE estimates of the optimal land allocation may be low enough to call any recommendations into question.

The punchline of this research is straightforward. Without devoting considerable energy and resources to the generation of data, the ability of economists to make reliable normative recommendations based on the application of models of decision making under uncertainty and risk aversion to important agricultural resource allocation problems is severely limited. This is especially true for the most intriguing problems -- those involving development or technological change -- where data is necessarily sparse.

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Table 1: 90% Confidence Intervals for Selected Points on the Efficient Frontier and Various Sample Sizes

			90% Confidence Intervals			
N = 6			Mean		Standard Deviation	
risk aversion	mean	standard deviation	lower bound	upper bound	lower bound	upper bound
0.0029240	62609	4624	58442	66777	3400	10563
0.0009000	63011	4687	58787	67235	3446	10706
0.0002900	64231	5247	59503	68960	3858	11986
0.0000355	77141	20873	58331	95950	15346	47677
N = 30			Mean		Standard Deviation	
risk aversion	mean	standard deviation	lower bound	upper bound	lower bound	upper bound
0.0029240	62609	4624	61150	64068	3881	6021
0.0009000	63011	4687	61532	64490	3933	6102
0.0002900	64231	5247	62576	65887	4403	6831
0.0000355	77141	20873	70555	83726	17516	27174
N = 100			Mean		Standard Deviation	
risk aversion	mean	standard deviation	lower bound	upper bound	lower bound	upper bound
0.0029240	62609	4624	61845	63374	4147	5239
0.0009000	63011	4687	62236	63786	4203	5310
0.0002900	64231	5247	63364	65099	4706	5944
0.0000355	77141	20873	73690	80591	18719	23645

Table 2: Effects of Risk Aversion and Sample Size on Reliability of PCE Estimates of Land Allocation, Portfolio Mean and Variance And Certainty Equivalent*

Risk Aversion	Acres in: x1	x2	x3	x4	Portfolio Mean	Portfolio Variance	CE
0.002924	68.66	28.26	88.23	14.85	62609.28	21385730	31343.34
N	Expectation of PCE Estimates						
6	68.14	28.33	88.29	15.24	62728.34	21494310	-474.63
	(20.25)	(7.37)	(20.43)	(8.80)			
30	68.61	28.27	88.24	14.89	62622.50	21395110	27854.15
	(6.72)	(2.45)	(6.78)	(2.91)			
100	68.65	28.26	88.23	14.86	62612.96	21388270	30372.70
	(3.54)	(1.29)	(3.58)	(1.54)			
0.00090	66.89	28.51	88.43	16.17	63010.91	21969340	53124.70
N	Expectation of PCE Estimates						
6	65.18	28.74	88.63	17.45	63397.72	23115450	41444.97
	(21.77)	(7.83)	(21.67)	(9.70)			
30	66.70	28.53	88.46	16.31	63053.89	22068390	51976.86
	(6.92)	(2.52)	(6.97)	(3.02)			
100	66.84	28.51	88.44	16.21	63022.87	21996200	52807.07
	(3.64)	(1.32)	(3.67)	(1.59)			
0.00029	61.49	29.26	89.06	20.20	64231.37	27533910	60238.95
N	Expectation of PCE Estimates						
6	56.18	30.00	89.67	24.16	65431.83	38572590	50680.40
	(32.89)	(11.34)	(31.17)	(15.87)			
30	60.90	29.34	89.13	20.64	64364.75	28487870	59642.12
	(8.64)	(3.10)	(8.56)	(3.89)			
100	61.32	29.28	89.08	20.32	64268.50	27792600	60078.62
	(4.50)	(1.62)	(4.47)	(2.02)			
0.00009	43.79	31.72	91.10	33.39	68232.89	85793620	64372.18
N	Expectation of PCE Estimates						
6	26.67	34.10	93.08	46.15	72101.03	200405100	42576.14
	(86.32)	(28.87)	(78.88)	(43.63)			
30	41.89	31.98	91.32	34.81	68662.69	95698320	63449.46
	(18.85)	(6.61)	(18.22)	(8.82)			
100	43.26	31.79	91.16	33.79	68352.53	88479540	64134.02
	(9.67)	(3.40)	(9.39)	(4.49)			
0.0000355	4.38	37.20	95.65	62.76	77140.51	435685800	69407.08
N	Expectation of PCE Estimates						
6	-39.00	43.24	100.67	95.10	86947.05	1172328000	16203.81
	(213.77)	(71.2)	(194.3)	(108.75)			
30	.44	37.87	96.21	66.36	78230.12	499346300	67296.04
	(45.17)	(15.77)	(43.43)	(21.33)			
100	3.04	37.39	95.81	63.76	77443.80	452949000	68866.
	(23.08)	(8.09)	(22.29)	(10.82)			

*The first line for each level of risk aversion contains the true or full information optimal allocation and corresponding portfolio mean, variance and certainty equivalent. Figures in parentheses are standard deviations of PCE estimators of land allocation.

ENDNOTES

1. Actually, the confidence interval around the estimated frontier described is for the usual case when the corners of each box are northeast of the corresponding corners of the proceeding box. This will not always be true as in the example below for a sample size of six. In this case, the confidence intervals around the efficient frontier will be lines connecting the northwest and southwest corners of the boxes around each point for some portion of the range.

Figure 1. Construction of 90% C. I.

Around Estimated Efficient Frontier

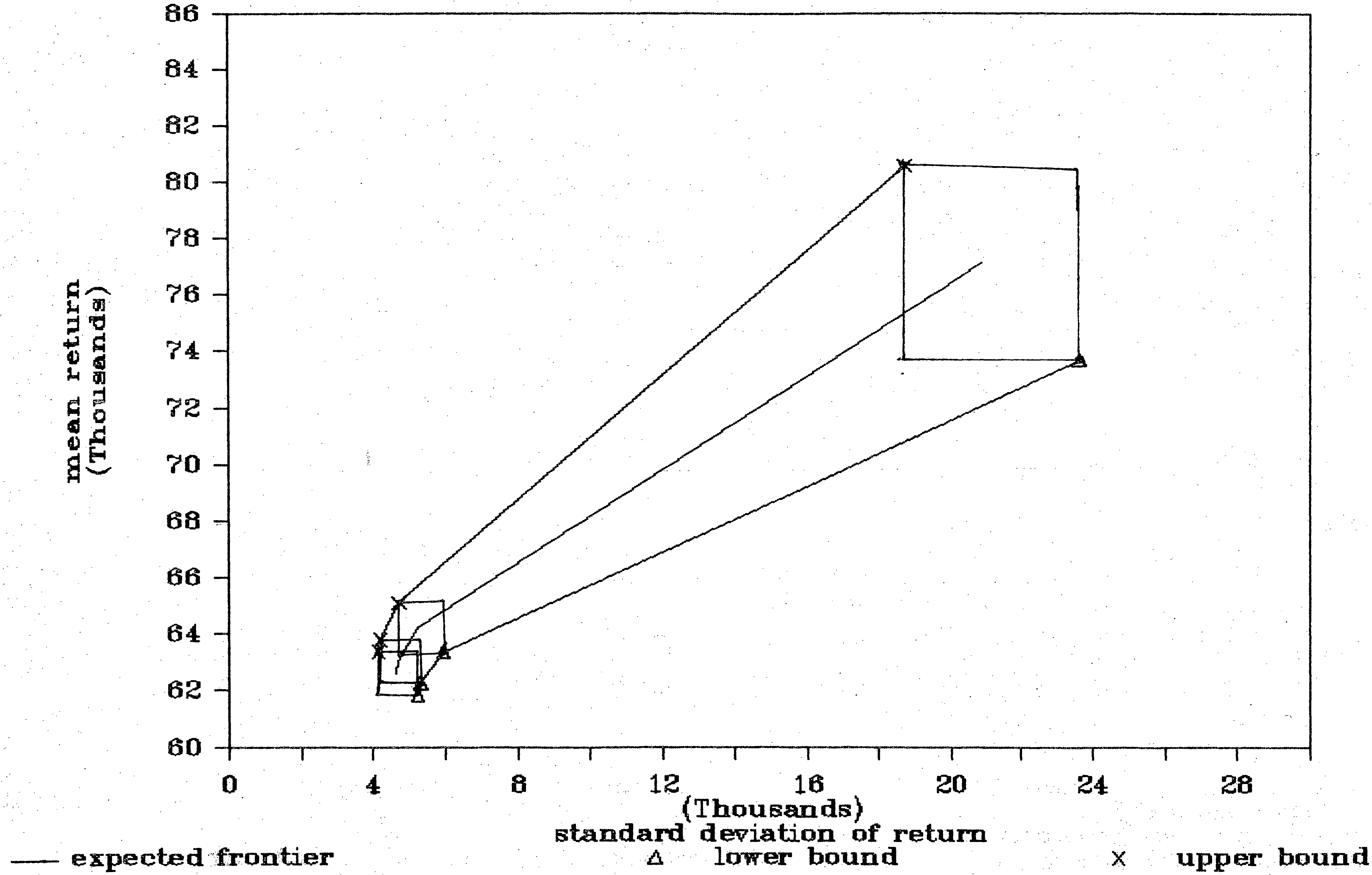

Figure 2. Effects of Sample Size

on 90% Confidence Interval