Working Papers Series

Working Paper WP90/15 July 1990

A PEDAGOGIC REVIEW OF THE EFFICIENT SET MATHEMATICS
APPLIED TO FARM OPERATING RISK

by

Calum G. Turvey
and
Timothy G. Baker

UNIVERSITY of GUELPH

Department of Agricultural Economics and Business

University of Guelph
Guelph, Ontario
Canada
N1G 2W1

A PEDAGOGIC REVIEW OF THE EFFICIENT SET MATHEMATICS APPLIED TO FARM OPERATING RISK

by

Calum G. Turvey*
and
Timothy G. Baker**

WORKING PAPER WP90/15

Department of Agricultural Economics and Business
University of Guelph
July 1990

* Calum Turvey is an assistant professor, Department of Agricultural Economics and Business, University of Guelph.

** Tim Baker is an associate professor, Department of Agricultural Economics, Purdue University.

A PEDAGOGIC REVIEW OF THE EFFICIENT SET MATHEMATICS

APPLIED TO FARM OPERATING RISK

Abstract

Efficient set mathematics is used to derive simple formulas for teaching EV efficient portfolio selection in agriculture. The formulas allow computation of portfolios along the EV frontier without the use of quadratic programming software. An example illustrating the use of the formulas is provided. Additionally, the separation theorem is discussed and illustrated with an example.

Key words: risk, EV, separation theorem, efficient set mathematics

A PEDAGOGIC REVIEW OF THE EFFICIENT SET MATHEMATICS APPLIED TO FARM OPERATING RISK

Despite the prominence of the mean-variance (EV) framework in agricultural economics there is a gap in the mathematical treatment of the EV model as taught to undergraduate and graduate students. For instance, most textbook discussions of this subject tend to present the EV criteria using graphs (Barry, Hopkin, and Baker; Penson and Lins; Robison and Barry). Most advanced discussions, usually found in journal articles, cast the EV framework as a whole farm quadratic program which requires some knowledge of computer software and programming. There is a need, therefore, for an intermediate exposition of EV analysis which avoids the need for specific knowledge of quadratic programming software while providing greater insight into the properties of the EV frontier than the graphical approach. One device suitable for meeting this exposition is the efficient set mathematics found in the finance literature (Fama; Roll, 1977, 1980; Black).

The purpose of this paper is to provide efficient set formulas suitable for intermediate level teaching of the concepts important in examining EV efficient farm operating plans. Since the intent of this article is pedagogical it is not meant to replace graphical or computerized QP approaches to teaching EV analysis, but to fill the gap left by using only these methods. The efficient set formulas provide solutions to the primal and dual risk- minimizing problems and relate these to concepts which educators commonly address when teaching EV analysis. The trade-off between expected profits and the variance of profits, the shape of the EV frontier, and the separation theorem are some of the concepts that are easily explained within the efficient set framework.[1]

Formulas similar to those found in Roll (1977, 1980) are derived which require only a basic knowledge of matrix algebra. Examples are provided to demonstrate the formulas and illustrate the concepts. In the first section the efficient set formulas are derived and examples of their use are presented. This is followed by a discussion of the separation theorem when cash-rent land is included as a riskless farm enterprise.

The Efficient Set Mathematics

The EV model used here follows the original Markowitz approach of minimizing the variance of revenues for a given level of expected revenue. The formulation is for a cash crop farm and requires only two constraints. The first constraint restricts expected revenue to a minimum value and the second restricts the use of land to the amount available. Nonnegativity restrictions on the decision variables are not imposed. In addition, the familiar minimization calculus with Lagrangian multipliers can be applied directly.

Derivation of the efficient set formulas begins by minimizing the following Lagrangian:

$$\underset{X,\lambda_1,\lambda_2}{\text{Minimize}} \quad Z = 1/2\ X'\ Q\ X + \lambda_1\ (K_p - C'\ X) + \lambda_2\ (L - e'\ X),$$

where X is a n x 1 vector of farm activity levels (acres); Q is a n x n positive

[1]The efficient set mathematics as outlined in this paper was presented as an introduction to risk programming in graduate level mathematical programming and agricultural finance courses at the University of Guelph. The students first solved EV problems by hand and then solved them on the computer using MINOS (Murtagh and Saunders). The students found the assignment useful. In particular they felt very comfortable with understanding the properties of the EV frontier with and without non-negativity imposed, interpretation of the dual varaibles, and the effect of a riskless asset on the portfolio choice problem.

definite variance-covariance matrix; λ_1 and λ_2 are scalar Lagrangian multipliers for expected net revenue and land, respectively; K_p is the level of expected revenue of the portfolio; C is a n x 1 vector of activity net revenues; L is the amount of land available; and e is a n x 1 vector of ones.

The first order conditions are:

$$\frac{\underline{\nabla X}}{X} = \begin{bmatrix} \partial Z/\partial x_1 \\ \partial Z/\partial x_2 \\ . \\ . \\ . \\ \partial Z/\partial x_n \end{bmatrix} = QX - \lambda_1 C - \lambda_2 e = 0, \tag{1}$$

$$\frac{\partial Z}{\partial \lambda_1} = K_p - C'X = 0, \text{ and} \tag{2}$$

$$\frac{\partial Z}{\partial \lambda_2} = L - e'X = 0. \tag{3}$$

In what follows, the first order conditions are manipulated to derive formulas for the optimum values of X^*, λ^*_1 and λ^*_2. Since Q is positive definite its inverse exists, and equation (1) can be rewritten in terms of the optimum solution vector X^*.

$$X^* = Q^{-1} \quad [C \ e] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}. \tag{4}$$

Premultiplying (4) by [C e]′ gives

$$[C \ e]' \ X^* = [C \ e]' \ Q^{-1} \ [C \ e] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}. \tag{4a}$$

Equation 4a can be arranged in terms of λ_1 and λ_2 as follows:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1} [C\ e]' X^*, \tag{4b}$$

where A is the following 2 x 2 matrix which results from expanding

$[C\ e]'\ Q^{-1} [C\ e]$:

$$A = \begin{bmatrix} C'Q^{-1}C & C'Q^{-1}e \\ e'Q^{-1}C & e'Q^{-1}e \end{bmatrix}. \tag{5}$$

Roll (1977) calls matrix A the "fundamental matrix of information" since it contains all the information about the basic data contained in the means, variances, and covariances of farm activities. This information is sufficient to prove all the important results of the efficient set mathematics. The scalar elements of A are called the "efficient set constants".

By defining

$C'\ Q^{-1} C$ as a, $e'\ Q^{-1} C$ and $C'\ Q^{-1}\ e$ as b, and $e'\ Q^{-1} e$ as c, the matrix A can be simplified to:

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{5a}$$

The fundamental matrix is an integral part of the development of the efficient set formulas. In the following sections formulas for the EV efficient solution vector X^*, the variance of the net revenue associated with X^*, and the values of the dual variables λ^*_1 and λ^*_2 are derived.

The EV Efficient Solution Vector X^* and Portfolio Variance

The most common aspect of EV analysis taught at universities is the trade-off between expected income and risk. Often a figure, such as the curved graph in figure 1, is presented to illustrate how expected income from combined farm enterprises is related to the risk (standard deviation) of the farm portfolio. To illustrate EV analysis beyond this graphical approach requires the use and knowledge of quadratic programming (QP) software. Use of the efficient set mathematics eliminates the need to program computers since simple EV models can be solved by hand. This section first derives formulas for calculating the vector of activities X^*, which is the efficient portfolio for a given level of expected income, and the portfolio variance. Then an example is presented to illustrate the use of these formulas.

Solving (2) and (3) in terms of K_p and L and substituting into (4b) results in:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1} \begin{bmatrix} C'X^* \\ e'X^* \end{bmatrix} = A^{-1} \begin{bmatrix} K_p \\ L \end{bmatrix} \tag{6}$$

Substituting (6) into (4) provides a formula for determining the optimal solution vector X^* at given levels of expected net revenue and available land.

$$X^* = Q^{-1} \ [C \ e] \ A^{-1} \begin{bmatrix} K_p \\ L \end{bmatrix} \tag{7}$$

Revenue variance for the portfolio X^* is found by substituting equation (7) into the following formula for variance:

$$\sigma_p^2 = X^{*\prime} Q \ X^* \ . \tag{8}$$

This substitution (see appendix 1) yields:

$$\sigma_p^2 = \frac{1}{ac-b^2} \left[K_p^2 c - 2 K_p L b + L^2 a \right] \quad (9)$$

Equation (9) is the variance of an EV efficient portfolio with expected net revenues K_p earned on L acres of land. The variance is described fully in terms of the efficient set constants, (a, b, and c), land, (L), and expected profits, K_p.

Equations (7) and (9) determine the EV efficient portfolio and variance for a given level of expected net revenues and land. Risk-return trade-offs are easily illustrated using these two formulas for alternative levels of expected income. Since the solutions are solved by hand, they are a practical teaching tool since they do not require the use of QP software. For more complex problems, perhaps those with four or more activities, solutions can be generated using the matrix inversion feature of a spreadsheet program such as Lotus 1-2-3 or SAS PROC MATRIX. Alternatively, for classroom illustration, one might set the covariance elements in Q equal to zero which greatly simplifies the calculation of Q^{-1}.

The use of equations (7) and (9) are illustrated below using the variance-covariance matrix (Q) and its inverse (Q^{-1}) presented in Table 1 for gross revenues of corn, soybeans, and wheat for Wellington County in Ontario.[2]

The expected net revenue vector (C′) is [126.16 114.59 87.09] for corn, soybeans, and wheat, respectively. Using this information the fundamental matrix

---

[2]The variance-covariance matrix was calculated from historical prices and yields for Wellington County, Ontario. Yields were detrended and prices were deflated using the CPI (1986 - 100)). Variable costs were compiled from Ontario Ministry of Agriculture and Food (1986) budget estimates.

is:

$$A = \begin{bmatrix} C'Q^{-1}C & C'Q^{-1}e \\ e'Q^{-1}C & e'Q^{-1}e \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 3.7168 & .0377599 \\ .037759 & .000404654 \end{bmatrix},$$

and its inverse is:

$$A^{-1} = \begin{bmatrix} 5.19089 & -484.383 \\ -484.383 & 47,671.00 \end{bmatrix}.$$

Using equation (7) and assigning L a value of 100 acres, the optimal portfolio of activities as a function of expected net revenue is:

$$X = \begin{bmatrix} 0.0300196 & -3.14716 \\ -0.00629469 & 1.30538 \\ -0.0237249 & 2.84178 \end{bmatrix} \begin{bmatrix} K_p \\ 100 \end{bmatrix}$$

Values for K_p can be entered into the equation once the above relation has been identified, for example:

if K_p = 10,800 : X* = [9.5 62.6 27.9], and

if K_p = 12,800 : X* = [69.5 50.0 -19.5].

The variance of expected revenues can be determined by using only the efficient set constants from the matrix A, a value for K_p and the number of acres. For example, using equation (9) with expected revenues of \$12,800 the variance can be determined as follows:

$$\sigma_p^2 = \frac{1}{(3.7168)(.000404654)-(.0377599)^2} [(12,800)^2(.000404654) -2(12,800)(.0377599) + (100)^2(3.7168)]$$

$$\sigma_p^2 = 87,164,920 \quad \text{and} \quad \sigma_p = 9336.22.$$

The EV frontier can be mapped out by using equation (9) with alternative levels of K_p, and the farm optimum portfolios at each point on the EV frontier can be determined using equation (7).

An Economic Interpretation of Negative Activity Levels

Negative activity values are the result of excluding non-negativity constraints from the problem. The economic interpretation of these results relate to an alternative land tenure mechanism similar to shortselling stocks in capital market theory. For example, at expected net revenue of $12,800 wheat enters the solution with a negative value of 19.5 acres. The tenure arrangement involves renting 19.5 acres of wheat land from another farmer and allocating this to corn and soybeans. Once crop revenues have been realized the farmer pays the 'lessor' an amount equal to that which would have been earned had wheat been grown. Based on the expected crop revenues (C), $14,498 could be earned from corn and soybean production (69.5 acres * $126.16/acre + 50.0 acres * $114.59/acre) of which $1,698 (19.5 acres * $87.09/acre) is to be paid to the other farmer ($14,498 - $1,698 = $12,800). The advantage to the lessee is that the risk associated with $12,800 income is less than that which would have occurred had non-negativity been imposed.

Another aspect of this tenure relationship is that the lessor is presumably willing to accept an uncertain rental payment whose expected value is $87.09/acre, but with a 50% probability of actually receiving more or less than this amount. This is markedly different from the classical land tenure system whereby the rental rate is negotiated and certain. Because the payment is certain, the cash rental rate will tend to be less than the risky rate. In later sections the efficient set mathematics is used to explain the separation theorem with cash rental land being included as a riskless enterprise.

Solving the Dual Solution for λ_1 and λ_2

The Lagrangian multipliers provide important risk opportunity cost information. The value of λ_1, is the marginal change in the variance of the optimal portfolio mix ($\sigma^2{}_p$) if K_p is increased or decreased. Similarly, λ_2 is the marginal change in $\sigma^2{}_p$ if the amount of land available changes. Note that from (6) both λ_1 and λ_2 can be determined by knowing a, b, c, K_p and L. Solving equation (6) we find that:

$$\lambda_1 = \left[\frac{cK_p - bL}{ac - b^2} \right] \quad \text{and} \quad \lambda_2 = \left[\frac{La - bk_p}{ac - b^2} \right] \qquad (10)$$

Substituting the values for a, b, c and L from the above example with K_p equal to \$12,800, the value for λ_1 is 15,928 and the value for λ_2 is -1,239,252. (The units of λ_1 are dollars squared per dollar of expected income and the units of λ_2 are dollars squared per acre.)

The dual variable λ_1 is the change in the objective function (1/2 of variance) per unit of change in expected income, i.e.

$$\lambda_1 = d\sigma^2{}_p/2dK_p$$

Therefore, 2 λ_1 is the inverse of the slope of the EV frontier at expected income K_p. If income is distributed normally and utility is negative exponential, then the slope of the EV frontier at an optimal solution is one half of the coefficient of absolute risk aversion. Under these conditions, individuals who prefer the portfolio corresponding to expected income K_p have absolute risk aversion equal to $1/\lambda_1$ (Turvey and Driver, 1986). In our example, a farmer choosing the portfolio with an expected income of \$12,800 would have an implied coefficient of absolute risk aversion of about .000063 (1/15928).

The values obtained in the above sections can be used to determine the certainty equivalent of expected net revenues. The certainty equivalent is defined as a certain level of income which would provide the same level of utility as the uncertain amount. Assuming negative exponential utility and joint multivariate normal returns the certainty equivalent is defined as (Robinson and Barry).

$$CE = K_p - \frac{\alpha}{2} \sigma_p^2 \tag{11}$$

where α (equal to $1/\lambda_1$) is the implied coefficient of absolute risk aversion. For expected revenues of \$12,800 the certainty equivalent is

$$CE = 12,800 - \frac{.000063}{2} (9336.22)^2 = \$10,054$$

Thus the farmer would be indifferent to receiving \$10,054 of income without risk, to a risky income of \$12,800.

The dual variable λ_2 is the change in portfolio variance with respect to a change in the land resource, i.e.

$$\lambda_2 = d\sigma^2{}_p/2dL$$

In general the shadow price of one constraint can be converted into units of a second constraint, rather than units of the objective function, by dividing by the shadow price of the second constraint (Preckel, Featherstone and Baker). If λ_2 is divided by λ_1 useful units result as follows:

$$\frac{\lambda_2}{\lambda_1} = (d\sigma^2{}_p/2dL)/(d\sigma^2{}_p/2dK_p) = dK_p/dL$$

Thus, the ratio λ_2/λ_1 which is the marginal value per acre of land in dollars of expected income is expressed in dollars per acre. In the example the marginal value of land is \$77.80 per acre (1,239,252/15,928).

The above example illustrates how the efficient set formulas can be used to solve for the primal and dual solutions of a risk minimizing problem, as well as the variance of the solution and the certainty equivalent. The following sections extend the use of these formulas to determine the minimum variance portfolio and to study the separation theorem.

The Global Minimum Variance Portfolio

The portfolio which has the least risk attainable, given the variance-covariance structure described by Q and using all of the fixed resource given by L, is defined as the global minimum variance portfolio (see Figure 1). Risk programming applications in agriculture usually consider only portfolios above the global minimum variance. Portfolios lying below the global minimum variance portfolio have decreasing expected revenue for increasing variance and, therefore, would not be selected by a rational risk averse farmer. Since nonnegativity restrictions on the choice variables are not imposed in the efficient set mathematics, portfolios below the global minimum variance portfolio are attainable. It is, therefore, useful to derive expressions for the global minimum variance, $\sigma^2{}_o$, the global minimum variance portfolio, X_o, and the expected revenue associated with X_o, K_o.

The expected revenue of the global minimum-variance portfolio is found by taking the partial derivative of $\sigma^2{}_p$ (equation (9)) with respect to K_p, setting this equal to zero, and solving for the K_p associated with the minimum-variance portfolio, which we label K_o.

$$\frac{\partial \sigma_p^2}{\partial K_p} = K_p c - Lb = 0. \tag{12}$$

Solving for K_p gives the expected revenue of the global minimum-variance portfolio (K_o):

$$K_o = \frac{Lb}{c} . \tag{13}$$

Substituting K_o for K_p in (9) gives the global minimum variance $\sigma^2{}_o$,

$$\sigma^2{}_o = \frac{L^2}{c} . \tag{14}$$

The solution vector, X_o, is found by substituting K_o into equation (7) (see appendix 2).

$$X_o = Q^{-1} e \frac{L}{c} . \tag{15}$$

To illustrate the use of these formulas, consider the efficient set constants used in the previous example (a = 12.245, b = .057, c = .00034), the inverse variance-covariance matrix in table 1, and a land base of 100 acres. Applying these data to equations (13), (14) and (15) yields:

$$K_o = \frac{(100)(.0377599)}{.000404654} = 933.14$$

$$\sigma_o^2 = \frac{(100)^2}{.000404654} = 24712444. \quad \text{and}$$

$$X_o = \begin{bmatrix} .000393908 & -.000381076 & -.000152806 \\ -.000381076 & .000563106 & .000108511 \\ -.000152806 & .000108511 & .000298381 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left[\frac{100}{.000404654} \right] = \begin{bmatrix} -34.6 \\ 71.8 \\ 62.8 \end{bmatrix}$$

In Figure 1 the minimum variance portfolio is located at the point (K_o = 9331.4, σ_o = 4971). The selection of portfolios below the global minimum variance portfolio would not be rational since an alternative portfolio exists with the same amount of risk but with greater expected revenue. The calculation of K_o, σ_o, and X_o is therefore used as a benchmark for defining a minimum expected income within the feasible set.

Separation Theorem

The above discussion provided formulas for determining the primal and dual solutions as well as the variances of the risk minimizing solution. This section describes the separation theorem as it applies to farm operating decisions. The concept is important since it shows how a riskless farm enterprise, cash-rent land, can be combined with risky portfolios to decrease risk (Johnson; Collins and Barry; Turvey and Driver, 1987).

The separation theorem for farm operating decisions is that "the optimal strategy for combining risky enterprise options is independent of the ratio of the amount of land in risky enterprises to the amount of land owned" (Johnson, p. 615). Figure 1 is used to introduce this concept. The parabola represents efficient portfolios when only risky enterprises are available, and the line represents efficient portfolios when riskless cash rental land is available. The two EV frontiers are tangent. At tangency portfolio all of the land is used in the production of risky farm enterprises; no land is cash rented. All other points on the line represent portfolios which include rental land. For points below the tangency portfolio the farmer becomes a landlord by renting out part of the land base and planting the rest of the land to risky crops in the same proportions as the tangency portfolio. For points above the tangency portfolio the farmer becomes a tenant by renting land in and planting on it the risky crop mix in the same proportions as the tangency portfolio.

The separation theorem suggests that the selection of the crop mix does not depend upon the decision-maker's risk preferences, since it is constant along the expanded EV frontier. Instead, the amount of land rented in or out is the variable affected by risk preferences. Additionally, the availability of a riskless enterprise improves the risk-return possibilities over utilizing only risky assets. For a given level of expected income farmers can decrease risk by renting land in or out and growing a crop mix in proportion to the tangency portfolio on all the land they operate. The resulting operating decision dominates farm plans lying on the original EV frontier.

When cash renting is added, equation (7) can be rewritten in terms of n risky activities and 1 riskless activity as:

$$\hat{X}^* = \hat{Q}^{-1} \; [\hat{C} \; \hat{e}] \; \hat{A}^{-1} \begin{bmatrix} P_p \\ L \end{bmatrix} \tag{16}$$

The hats (^) denote the fact that a riskless enterprise is included as part of the farm operation. The variance-covariance matrix $\hat{Q}$ can be represented as:

$$\hat{Q} = \begin{bmatrix} Q & \phi \\ \phi' & \epsilon \end{bmatrix} \tag{17}$$

where Q is the n x n matrix of variances and covariances, ϕ is a n x 1 null vector and ϵ is a very small number close to zero. The use of ϵ implies that the riskless enterprise has a variance of ϵ. However, with ϵ equal to zero, $\hat{Q}$ is singular and cannot be inverted. Setting ϵ equal to a small number, such as 1 or .001, allows $\hat{Q}$ to be inverted. The primary requirement for choosing ϵ is that its contribution to the curvature of the linear EV frontier be negligible. Thus, the separation theorem can be illustrated using the previous equations.[3] The inverse of $\hat{Q}$ is:

$$\hat{Q}^{-1} = \begin{bmatrix} Q^{-1} & \phi \\ \phi' & \epsilon^{-1} \end{bmatrix} . \tag{18}$$

---

[3]The approach described somewhat ad hoc but has the advantage of utilizing the previous set of equations.

The fundamental matrix and the efficient set constants are computed by substituting $\hat{Q}^{-1}$ for Q^{-1} in (5). The previous formulas for the portfolios, variance, and the global minimum variance portfolio can then be used.

To illustrate how the separation theorem works, EV portfolios were computed with and without a riskless enterprise. They were computed over a range of expected net revenue from \$6,500 to \$12,800 using equations (7), (9), and (16). Results are shown in Table 2. The rental value of land was assumed to be \$65/acre. Both EV frontiers are plotted in figure 1. The curved EV frontier represents optimal portfolios considering only the risky assets.

The inclusion of cash-rented land as a riskless farm enterprise causes the efficient frontier to become linear (when graphed in expected income and standard deviation space) and dominates the EV frontier of risky activities. The minimum variance portfolio occurs when the 100 acres of land is cash rented out, giving net revenue of \$6,500 with a zero standard deviation (Figure 1, point b). The EV frontier allowing the farmer to rent land in or out will become tangent to the EV frontier of risky activities. The tangency portfolio in Figure 1 represents a crop mix of 15.9 acres of corn, 61.2 acres of soybeans, and 22.9 acres of wheat. The expected net return is \$11,012 and the standard deviation is \$6,275. Every portfolio along the line has the same crop mix (same proportions). For example, at all levels of expected income, the risky enterprises with renting land in Table 2 are found to be in constant proportions of 15.9%, 61.2% and 22.9% for corn, soybeans and wheat, respectively. This crop mix is determined by the expected returns and variance of risky activities and the risk free cash rental rate; it does not depend upon the degree of risk aversion.

The improvement in the risk-return opportunities using the leverage of cash rented land over risky crops alone is clear from figure 1. For any level of expected income, other than the tangency portfolio, farmers can decrease risk by utilizing cash renting land. For example, in order to obtain expected returns of $11,600, a portfolio comprised of 33.5 acres of corn, 57.5 acres of soybeans and 9.0 acres of wheat can be formed. This portfolio has a standard deviation of $7,171. Alternatively, the farmer can become a tenant, renting in 13 acres of land and forming a portfolio comprised of 17.9, 69.2 and 25.9 acres of corn, soybeans, and wheat respectively. The standard deviation of this portfolio is $7,093 which, for the same level of expected net revenues, is a substantial decrease in risk. For portfolios below the tangency portfolio the farmer becomes a landlord; renting land out and reducing risk.

Conclusions

The intent of this paper was to provide a pedagogic review of the efficient set mathematics and the separation theorem as it applies to farm operating decisions. The efficient set formulas make two important contributions to the teaching of intermediate risk management concepts to university students. First, the efficient set mathematics adds substantial rigor to the EV concepts found in undergraduate agricultural finance textbooks. Second, the efficient set formulas provide a method for students to generate EV frontiers for homework and classroom examples without using quadratic programming software.

The importance of the separation theorem as it applies to cash-rented land in agriculture was developed and illustrated. It is interesting to note that the economics of combining cash-rented land with risky farm portfolios has received only limited attention in the agriculture economics literature. The contents of this paper can be used to broaden the understanding of this important concept.

Figure 1: Efficient Set Frontiers

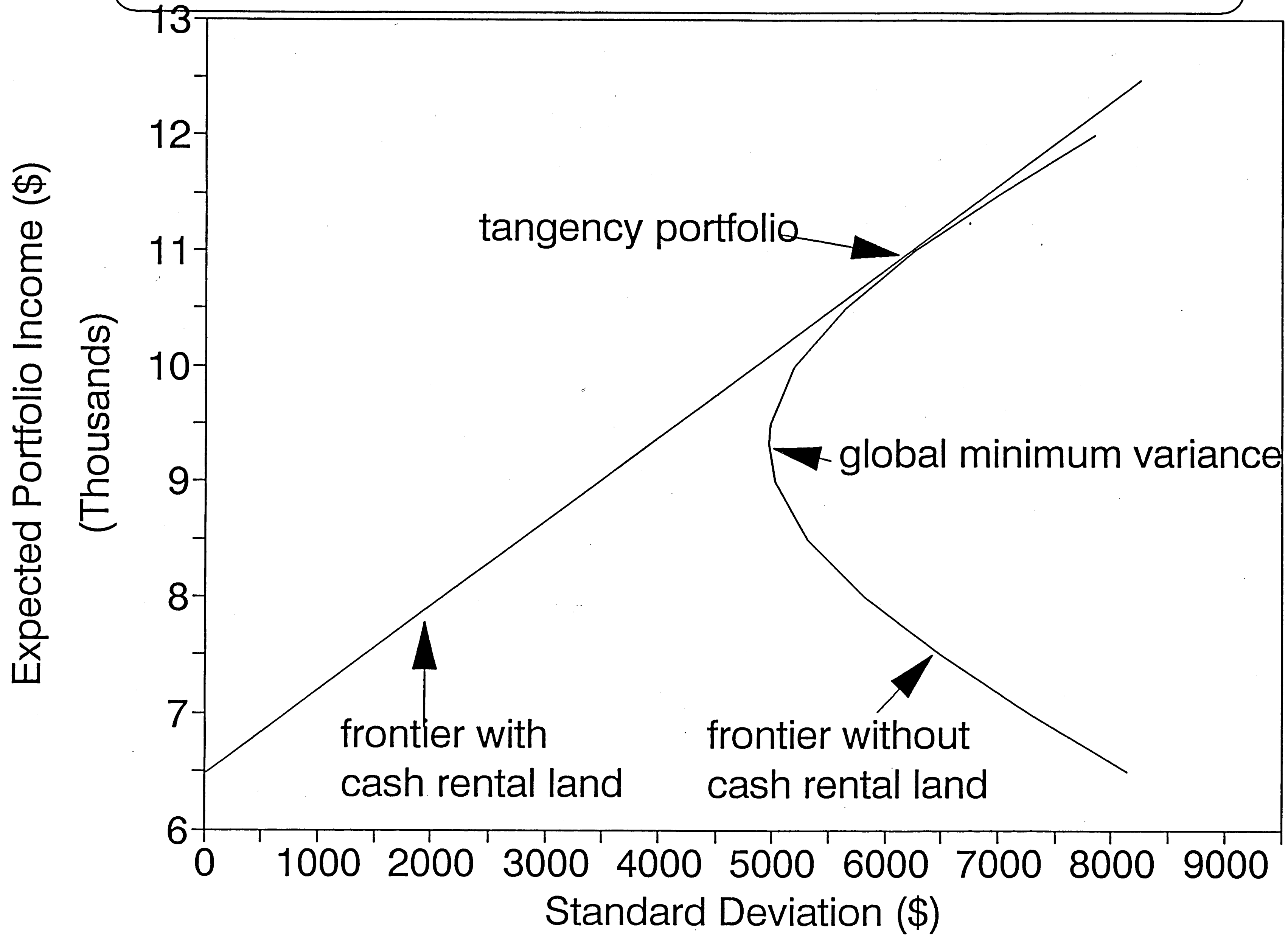

Table 1. Variance-Covariance Matrix, Inverse Variance-Covariance Matrix, and Net Revenues for Real Corn, Soybean and Wheat, Gross Revenues (1970-1986)

	Corn	Soybean	Wheat
	Variance-Covariance Matrix (Q)		
Corn	8,825.00	5,485.76	2,524.44
Soybeans	5,485.76	5,319.73	874.74
Wheat	2,524.44	874.74	4,326.11
	Inverse Variance-Covariance Matrix (Q^{-1})		
Corn	.000383908	-.000381076	-.000152806
Soybeans	-.000381076	.000563106	.000108511
Wheat	-.000152806	.000108511	.000298381
	Net Revenue Data		
Expected Gross Revenues ($/acre)	326.16	248.59	196.08
Variable Production Costs ($/acre)	200.00	134.00	109.00
Expected Net Revenues ($/acre)	126.16	114.59	87.08

Table 2. Efficient Portfolios With and Without Cash Rented Land.

Expected Net Revenues ($)	Standard Deviation of Net Revenues ($)	Corn (acres)	Soybean (acres)	Wheat (acres)	Cash Rented Land (acres)
		Portfolios Including Only Risky Enterprises			
6,500	8144.1	-119.6	89.6	130.0	-
8,000	5823	-74.6	80.2	94.4	-
9,331[b]	4971.2	-34.6	71.8	62.8	-
10,800	5992.3	9.5	62.6	27.9	-
11,012[c]	6274.8	15.9	61.2	22.9	-
12,800	9336.2	69.5	50.0	-19.5	-
		Portfolios Including Cash Rented Land			
6,500[b]	0	0	0	0	100
8,000	2086.1	5.3	20.3	7.6	66.8
9,331	3937.6	10.0	38.4	14.4	37.3
10,800	5980.	15.1	58.3	21.8	4.7
11,012[c]	6274.8	15.9	61.2	22.9	0.0
12,800	8761.4	22.2	85.5	32.0	-39.6

[a] Positive values indicate cash renting land out to others and negative values indicate renting land from others.

[b] Global minimum variance portfolio.

[c] Tangency portfolio.

Appendix 1

Substituting X* (equation (5)) into (8) gives

$$\text{(8a)} \qquad \sigma_p^2 = \left[Q^{-1} [C\ e]\ A^{-1} \begin{bmatrix} K_p \\ L \end{bmatrix}\right]' \quad Q \left[Q^{-1} [C\ e]\ A^{-1} \begin{bmatrix} K_p \\ L \end{bmatrix}\right],$$

$$= \left[Q^{-1} [C\ e]\ A^{-1} \begin{bmatrix} K_p \\ L \end{bmatrix}\right]' \quad \left[[C\ e]\ A^{-1} \begin{bmatrix} K_p \\ L \end{bmatrix}\right],$$

$$= [K_p \quad L]\ A^{-1} \quad [C\ e]'\ Q^{-1} \quad [C\ e]\ A^{-1} \begin{bmatrix} K_p \\ L \end{bmatrix},$$

$$= [K_p \quad L]\ A^{-1} \quad A \quad A^{-1} \begin{bmatrix} K_p \\ L \end{bmatrix},$$

$$\text{(8b)} \qquad \sigma_p^2 = [K_p \quad L]\ A^{-1} \begin{bmatrix} K_p \\ L \end{bmatrix},$$

Defining (8b) in terms of K_p, L and the efficient set constants gives:

$$\text{(8c)} \qquad \sigma_p^2 = [K_p \quad L] \left[\frac{1}{ac-b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}\right] \begin{bmatrix} K_p \\ L \end{bmatrix},$$

where $ac-b^2$ is the determinant and

$$\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$

is the transposed co-factor matrix of A. Expanding (8c) gives (9).

Appendix 2

Substituting (12) into (7) gives

$$X_o = Q^{-1} [C\ e] \begin{bmatrix} \dfrac{1}{ac-b^2} \end{bmatrix} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \begin{bmatrix} Lb/c \\ L \end{bmatrix} ,$$

$$= \frac{Q^{-1}}{ac-b^2} [(cC - be) \quad (-bC + ae)] \begin{bmatrix} Lb/c \\ L \end{bmatrix} ,$$

$$= \frac{Q^{-1}}{ac-b^2} [Lbc - \frac{Lb^2 e}{c} - Lbc + aLe] ,$$

$$= \frac{Q^{-1}}{ac-b^2} \begin{bmatrix} \dfrac{Le[ca - b^2]}{c} \end{bmatrix} .$$

Cancelling $ac-b^2$ gives (15).

REFERENCES

Adams, R.M., D.J. Menkhaus and B.A. Woolery, (1980), "Alternative Parameter Specification in E-V Analysis: Implications for Farm Level Decision Making", Western Journal and Agricultural Economics 5:13-20.

Barry, P.J., J. A. Hopkin and C.B. Baker. Financial Management in Agriculture, Interstate Printers and Publishers, 1979.

Black, F. "Capital Market Equilibrium with Restricted Borrowing," Journal of Business 45(1972):444-455.

Collins, R.A. and P.J. Barry, (1986), "Risk Analysis with Single-Index Portfolio Models: An Application to Farm Planning," American Journal of Agricultural Economics 35:152-161.

Fama, E.F. Foundations of Finance. New York: Basic Books, 1976. Johnson, S.R. "A Re-Examination of the Farm Diversification Problem," Journal of Farm Economics 49(1967):610-621.

Gemesaw, C.M., A.M. Tambe, R.M. Nayga, and U.C. Toensmeyer, (1988), "The Single Index Market Model in Agriculture", Northeastern Journal of Agricultural and Resource Economics 56:497-508.

Lin, W., G.W. Dean and C.V. Moore, (1974), "An Empirical Test of Utility vs. Profit Maximization in Agricultural Production", American Journal of Agricultural Economics 56:497-508.

Markowitz, H. Portfolio Selection: Efficient Diversification of Investments. New York: John Wiley and Sons, 1959.

Murtagh, B.A. and M.A. Saunders. MINOS 5.0 User's Guide, Technical Report. SOL 83-20, Systems Optimization Laboratory, Department of Operations Research, Stanford University, December,1983.

Ontario Ministry of Agriculture and Food. Agricultural Statistics for Ontario. Publication 20, various issues.

Ontario Ministry of Agriculture and Food. Farm Business Management Handbook. August, 1986.

Penson, J.G. and D.A. Lins. Agricultural Finance: An Introduction to Micro and Macro Concepts. Prentice Hall, NJ, 1980.

Pratt, J.W. "Risk Aversion in the Small and in the Large," Econometrica 32(1964):122.136.

Preckel, P.V., A.M. Featherstone and T.G. Baker. "Interpreting Dual Variables for Optimization with Nonmonetary Objectives," American Journal of Agricultural Economics, 69(1987):849-851.

Robison, L.J. and P.J. Barry. The Competitive Firm's Response to Risk, MacMillan Publishing Co., NY, 1987.

Roll, R. "A Critique of the Asset Pricing Theory's Tests: Part 1; On Past and Potential Testability of the Theory," Journal of Financial Economics 4(1977):129-176.

Roll, R. "Orthogonal Portfolios," Journal of Financial and Quantitative Analysis 15(1980):1005-1023.

Sharpe, W.F. (1963), "A Simplified Model for Portfolio Analysis", Management Science 2:277-293.

Turvey, C.G. and H.C. Driver. "Economic Analysis and Properties of the Risk Aversion Coefficient in Constrained Mathematical Optimization," Canadian Journal of Agricultural Economics 34(1986):125-137.

Turvey, C.G. and H.C. Driver. "Systematic and Nonsystematic Risks in Agriculture," Canadian Journal of Agricultural Economics, 35(1987):387-401.

Turvey, C.G., H.C. Driver and T.G. Baker, (1988), "Systematic and Nonsystematic Risk in Farm Portfolio Selection", American Journal of Agricultural Economics 70:831-836.