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# STAFF PAPER



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Interpretation and Transformations of Scale for  
the Pratt-Arrow Absolute Risk Aversion Coefficient:  
Implications for Stochastic Dominance

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INTERPRETATIONS AND TRANSFORMATIONS OF SCALE FOR THE PRATT-ARROW  
ABSOLUTE RISK AVERSION COEFFICIENT: IMPLICATIONS FOR STOCHASTIC  
DOMINANCE

Abstract

The Pratt-Arrow measure of absolute risk aversion, as defined by  $r(x) = u''(x)/u'(x)$ , is well known to be "invariant to linear transformations." However, this invariance property applies with respect to transformations of  $u$  and not with respect to arbitrary rescalings of  $x$ . The effects of this misunderstanding has led to ambiguity as to what actually constitutes behavior that is "slightly risk averse," "very risk averse," etc. The use of the coefficient in "per acre" analyses is particularly addressed.

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DOMINANCE

The Pratt-Arrow absolute risk aversion coefficient, defined as  $r(x) = -u''(x)/u'(x)$ , has been used in many analyses which order alternative action choices under conditions of uncertainty. Problems arise, particularly in applications of the Stochastic Dominance with Respect to a Function (SDWRF) when Pratt-Arrow coefficients elicited in one study are used as secondary data in other studies with different outcome ranges. It is well known that the Pratt-Arrow measure is invariant to linear transformations. However, this invariance property applies only to transformations of  $u$  and not to arbitrary rescalings of  $x$ , the outcome measure. Confusion about rescaling can lead to faulty classifications of preferences into such fuzzy groups as "slightly risk averse," "moderately risk averse," etc. Furthermore, inaccurate rankings of action choices can be produced with SDWRF when inappropriate rescaling of the outcome variable has been made.

INTERPRETATION IN TERMS OF CHANGES IN MARGINAL UTILITY

The Pratt-Arrow measure of absolute risk aversion can be defined in several equivalent ways:

$$r(x) = - \frac{u''(x)}{u'(x)}$$

$$= - \frac{(du'/dx)}{u'}$$

$$= - \frac{d}{dx} \ln u'$$

$$= - \frac{(du'/u')}{dx}$$

The last of the above expressions suggests that the coefficient can be interpreted as the percent change in marginal utility per unit of outcome space. Therefore,  $r$  has associated with it a unit - the reciprocal of the unit with which the outcome space is measured. For instance, suppose with outcomes measured in dollars,  $r$  is elicited as .0001. Such a value is actually .0001/\$ and should properly be specified with its unit intact. It indicates that near the outcome level at which the elicitation was made, the decision maker's marginal utility is dropping at the rate of .01% per dollar. Similarly,  $r(x) = -.00005/\$$  implies that around the outcome value  $x$ , marginal utility is rising at the rate of .005% per dollar.

Similarly, if  $r$  is known only to lie in the interval (.00004/\$, .00006/\$), then marginal utility is falling at a rate between .004% and .006% per dollar.

#### CONVERSION TO MARGINAL UTILITIES

To get a feel for plausible values of  $r$ , it is illuminating to convert quoted values of  $r$  from the literature into marginal utilities

at various outcome levels. For constant absolute risk aversion,  $u'(x) = \exp(-rx)$ , where  $u'(0) = 1$  (see Table 1). The values in the table, therefore, represent the worth of the next dollar, given that the first dollar is worth 1 unit.

What is surprising is the rapid falloff in marginal utility for what has been considered "moderately risk averse" behavior, such as .0002/\$. Even at "allegedly risk neutral",  $r = .00005/\$$ , the value of a dollar at \$50,000 is just one-seventh the value of a dollar at \$10,000.

One of the hazards of working with number as tiny as these is to underestimate the distinction between them. A pair of decision makers exhibiting seemingly close values of  $r$  such as .0002/\$ and .0003/\$, respectively, would disagree on the value of the 10,001st dollar by a factor of three and on the value of the 50,001st dollar by a factor of 160. Furthermore, elicitation procedures have often lumped "reasonable" values of  $r$  i.e. all (those below  $r = .0001$ ) in the same interval and must be construed as less than homogenous group at some values of  $x$ . At some values of  $x$ , the preferences may be similar, but at  $x = 50,000$  the weight at  $r(x) = .00005$  is eight times as large as the weight at  $r(x) = .0001$ . Such a wide interval at these larger values of  $x$  may be very weak in its discriminatory abilities and may result in a large Type II error.

Table 1  
Marginal Utility as a Function of Income  
and Constant Absolute Risk Aversion

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Marginal Utility (relative to  $u'(\$0) = 1$ )\*

	\$50	\$250	\$10,000	\$50,000
$r$ (/ \$)				
-.0001	1.005	1.03	2.71	148
-.00001	1.0005	1.002	1.11	1.65
0.0	1.0	1.0	1.0	1.0
.00001	.99995	.998	.90	.61
.00005	.998	.99	.61	.08
.0001	.995	.98	.37	.01
.0002	.99	.95	.14	.00005
.0003	.985	.93	.05	$3 \times 10^{-7}$
.0004	.98	.90	.018	$2 \times 10^{-9}$
.001	.95	.78	.00005	$2 \times 10^{-22}$
.01	.61	.08	--	--
.1	.007	$1 \times 10^{-11}$	--	--

Values are obtained from  $u'(x) = \exp(-rx)$

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\*(In using the Table with an interval representation of  $r$ , the bounds on the marginal utility are precisely the marginal utilities of the bounds.)



## CHANGE OF OUTCOME SCALES

The need for the explicit specification of the unit of  $r$  might arise when elicited values are used outside the context of the original study. If a risk aversion coefficient elicited over an outcome space measured in one unit is later applied over outcomes measured in another unit, it must be converted by the appropriate factor. Such a conversion is often trivial, as the following two examples illustrate.

## Example 1: Change of Currencies;

Suppose that  $r = .0001/\$$  is to be used as an upper bound of risk aversion in modeling decision making among Australian farmers. To become a meaningful risk measure in Australian currency,  $r$  must be converted by the appropriate exchange rate. At 2 Australian pounds per dollar,

$$r(x) = .0001/\$ * \frac{\$1}{2 \text{ pounds}} = .00005/\text{pound}.$$

## Example 2: Normalization of Outcomes;

Suppose that values of  $x$  have been normalized to  $[0,1]$  through a division by \$100,000 (perhaps in an attempt to simplify computation). The outcome scale now represents hundreds of thousands of dollars rather than dollars. In this case, a previously elicited value of  $r(x) = .0001/\$$  would be multiplied by 100,000 to become meaningful over the new normalized scale:

$$r(x) = .0001/R * \frac{\$100,000}{1 \text{ hundred thousand dollars}} = \$10/\text{hundred thousand dollars}$$

## CHANGE IN THE SPATIAL OR TEMPORAL DIMENSION OF THE OUTCOMES

In the two examples given above, the underlying attitudes towards risk were not affected by the transformations of scale; the risk measures were merely expressed in different units. The situation may be more complex when the change of scale is brought about by a change in the temporal or spatial dimension of the outcomes. The following examples illustrate the potential problems:

## Example 3: Per Acre Analyses;

Suppose that  $r$  is elicited as  $.0001/\$$  as a measure of aversion to annual income risk and is to be applied to returns expressed on a per acre basis. It is common to neglect the distinction between "per acre" risk and total farm income risk, yet it is easy to show that the two values are not the same. A  $.01\%$  falloff in marginal utility per farm income dollar is very different from a  $.01\%$  falloff in marginal utility per acre income dollar; the latter is a far more risk neutral description. The necessity for a distinction arises because there are two different outcome scales are involved: total returns and per acre returns. If the crop under study represented the entire income of the farmer and the farm size were known, the conversion factor between the two scales would be the number of acres. Then for a 100-acre farm,  $r(x) = .0001/\$$  where  $x$  represents annual income dollars would be

equivalent to  $r(w) = .01/\$$  where  $w$  is in acre income dollars.

However, if other crops contribute to the farm income, the farmer might exhibit differing attitudes towards risk for each crop.

Example 4: 10-Year Horizon;

Suppose that  $r(x) = .0001/\$$  is now to be applied to returns expressed on a 10-year net present value basis. In analogy with the previous example, risk per annual income dollar must be distinguished from risk per 10-year NPV dollar. The  $r$  over the new 10-year NPV scale would be obtained by dividing the old  $r$  by the 10-year NPV of a dollar. Once again, we are confronted with a problem. Will the marginal utility curve retain its shape over varying time horizons? Or, put in another way, will utility for wealth have the same properties as utility for annual income? Sinn suggests that with the passage of time, the degree of relative risk aversion will move towards unity (Sinn). Empirical evidence is lacking to make more than a tentative conclusion at this point, but concern must be expressed.

#### COMPARISON TO THE RELATIVE RISK AVERSION COEFFICIENT

The relative risk coefficient  $r^*$  is defined as  $r^* = r \cdot x$ , where  $x$  is an element of the outcome space. While  $r$  measures the percent change of marginal utility per unit change of the outcome space,  $r^*$  measures the same marginal utility change per percent change of the

outcome space. The relative risk aversion coefficient is therefore the elasticity of the marginal utility function and is unitless. However, it is unitless only when the spatial and temporal dimensions of the underlying marginal utility function are consistent with those of  $x$  itself. Therefore, while the relative risk aversion coefficient is not subject to the scaling problems of types presented in Examples 1 and 2, it is still susceptible to the problem of the marginal utility and returns being expressed in incommensurable units.

#### DIRECT ELICITATION PROCEDURES

A way around the scaling problems inherent in Examples 3 and 4 would be to elicit directly the aversion to per acre or 10-year NPV risk. Such a procedure would stray from the typical "after tax net farm income" questioning commonly used in the past and focus directly on preferences for per acre (or 10-year) returns before taxes and unrelated fixed expenses. Some interesting empirical questions could potentially be answered concomitantly. How do attitudes towards risk change (if at all) as the time horizon is varied (but the time origin remains fixed)? Do we become less prone to risk taking as wealth rather than short term income is at stake? Are our attitudes toward risk identical for each crop in a multicrop farm? Or can we even correctly measure risk for a single crop without the knowledge of the other income sources? Little work has been carried out to answer the above questions.

### CONVERTING VALUES OF R

In the absence of direct elicitations, the conversion from an  $r$  over one spatial-temporal scale to another should be viewed only as an approximation. The appropriate scaling factor to use would be that which brings these returns in one scale to the approximate level of the other. Therefore, if per acre returns are to be used with a "per farm" value of  $r$ , the factor that brings the per acre returns up to the level of the per farm returns (perhaps the farm size) would provide the desired approximation.

### INACCURATE RANKINGS FROM SDWRF

Inaccurate rankings of alternative action choices can be generated with SDWRF when scaling problems are uncorrected. By using Pratt-Arrow coefficients elicited at after-tax, whole farm income levels to evaluate action choices described in terms of per acre net returns, an efficient set is identified for a class of decision makers which is considerably less risk averse than the one intended. The probability of large Type I errors can be expected to surface.

An example of such inaccurate rankings is presented in Table II. Action choices for cotton irrigation strategies in Arkansas were ranked, first without an appropriate rescaling and then with a rescaling which converted the risk aversion measures from whole farm to per acre levels. The strategies identify a threshold tensionmeter

value which triggers an irrigation threshold. Probability distributions were generated with COTCROP-A, a computer simulation model. Further details on the strategies and the simulations can be found in Cochran, et al. It is important to note that in all four risk intervals, the efficient sets vary.

Table 2  
Comparison of Strategy Rankings With and Without  
Appropriate Output Scaling.

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Risk Interval Elicited at After Tax Whole Farm Income Levels	Without	With
-.0008 to -.0001	1) DYNCP2 2) TENSCP50 3) DYNCP3 4) TENSCP55 5) TENSCP45	1) TENSCP50 2) DYNCP2 3) TENSCP60 4) TENSCP55 5) DYNCP3
-.001 to .0001	1) DYNCP2 2) TENSCP50 3) DYNCP3 4) TENSCP55 5) TENSCP45	1) TENSCP50* 1) TENSCP55* 1) TENSCP60* 1) TENSCP65* 1) DYNCP2* 1) DYNCP3*
.0001 to .0004	1) DYNCP2 2) TENSCP50* 2) DYNCP3* 3) TENSCP45 4) TENSCP45	1) TENSCP65 2) TENSCP60 3) TENSCP55 4) DYNCP3 5) DYNCP2
.0004 to .001	1) DYNCP2 2) TENSCP50* 2) DYNCP3* 3) TENSCP55 4) TENSCP45	1) TENSCP65 2) TENSCP60 3) TENSCP55 4) DYNCP3 5) DYNCP2

\*denotes strategies appearing in the same efficient set

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(All returns were expressed in terms of per acre net revenue while risk aversion coefficients are associated with after tax net farm income. Scaling was performed by multiplying the per acre returns by the number of acres in an average size farm.)

## SUMMARY

In summary, the units of  $r$  and  $x$  must always be reciprocal to one another. If  $x$  is expressed in another unit,  $r$  can be converted to the reciprocal unit of  $x$  by the appropriate conversion factor.

Unfortunately, risk attitudes may change over varying temporal or spatial scales, and in such cases, conversion can be viewed only as approximations.

In general, it appears desirable to explicitly state the unit over the space with which  $r$  has been estimated. This is necessary because the value represents the percent change in marginal utility per outcome unit. Where an estimated value of  $r$  represents anything other than an aversion to annual income risk (e.g. per acre, per month, ten-year present value, etc.), and has not been converted, this should also be explicitly stated. When a value of  $r$  from one study is to be applied in another study, care must be taken that an implicit transformation of the outcome scale has not occurred. That is, per acre returns should not be used directly with an  $r$  that represents an aversion to annual income risk.



## APPENDIX

Proof of Theorem 1:

If  $w = x/c$ , then by the chain rule,

$$\frac{du}{dw} = \frac{du}{dx} \frac{dx}{dw} = c \frac{du}{dx} \quad \text{and}$$

$$\frac{d^2u}{dw^2} = c \frac{d^2u}{dx^2} \frac{dx}{dw} = c^2 \frac{d^2u}{dx^2} .$$

Therefore,

$$r(w) = \frac{-c^2 d^2u/dx^2}{c du/dx} = c r(x)$$

Proof of Theorem 2:

Since  $dv = dx$ ,  $dv/dx = du/dx$  and  $d^2v/dx^2 = d^2u/dx^2$ , therefore,  
 $r(v) = r(x)$ .

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