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1. Introduction

Country elevators play a vital role in the grain marketing system. Most of the grain in marketing channels originates in country elevators, and prices set by elevators are a critical link between world grain markets and the decisions of individual producers. Curiously, the price-setting behavior of country elevators has received little attention in the literature. A few studies have examined issues of spatial equilibrium (Lytle and Hill) and market efficiency (Farris), and there is some published evidence (Thompson and Dziura) on the determinants of marketing margins for a cross section of elevators. However, there have been no attempts to analyze elevator price-setting in terms of an explicit, dynamic optimization problem.

The manager of a grain elevator makes difficult marketing decisions. His objective is to maximize returns from grain marketing over some time horizon. In each period, he must decide what prices to offer producers (for both spot and forward delivery) based on the size of his current grain inventories and purchase plans. He must also decide how much grain to sell (spot and forward), taking prices in terminal markets as given. Returns from marketing will depend on the prices and quantities of grain bought and sold, and on costs associated with grain handling. If unit handling costs are positively related to the volumes shipped or delivered, the manager will have an incentive to reduce fluctuations in elevator throughput. The manager may also wish to keep the elevator's own grain inventories near specified levels. This will influence marketing decisions, since purchases and sales alter current and prospective inventory positions.

The problem is complicated by uncertainty about producer behavior and prices in terminal markets. The manager decides how much grain he wishes to purchase and chooses bid prices accordingly. The response of producers to elevator bid prices can be forecast by the manager, based on past experience and current market conditions, but the forecasts are subject to error. Hence, the amounts of grain purchased by the elevator and resulting inventory positions are uncertain. In addition, the manager will base his marketing decisions on expected price changes in terminal markets. Prices in future periods are not known, but may be forecast.

This paper presents a model that can be used to derive rules for optimal marketing decisions. The optimization problem is characterized as the maximization of a quadratic objective function over a finite time horizon subject to a set of linear difference equations. The objective function in this case represents net returns from grain marketing, and the constraints include linear forecasting equations and inventory accounting identities. The linear-quadratic structure is computationally tractable and yields a set of linear decision rules for variables under the control of the elevator manager.

In the next section, we briefly review some important features of linear-quadratic control problems. We then develop the elevator's marketing problem within this framework. The paper concludes with some comments on empirical application of the model.

2. Linear-Quadratic Control and Certainty Equivalence

The marketing model presented in this paper is based on a special formulation of the linear-quadratic control problem, due to Chow. Let y_t be a (px1) vector of variables, and let x_t be a (qx1) subvector of y_t. The variables in x_t are subject to the control of a decision-maker. The vector y_t is assumed to evolve according to:

$$y_t = A\, y_{t-1} + C\, x_t + b_t + u_t \qquad (1)$$

where A and C are matrices of coefficients, b_t is a vector of constants, and u_t is a vector of random errors. This equation is quite general, since y_t may be defined to include lagged, as well as current, endogenous and control variables; (1) could thus represent the reduced form of some higher-order autoregressive system. The vector b_t may include the effects of exogenous variables not subject to control.

The problem is to minimize a criterion of the form:

$$Z = E_o \sum_{t=1}^{T} (y_t - a_t)' K_t (y_t - a_t) \qquad (2)$$

subject to (1). E_o is the expectations operator, conditional on information available in period 0; a_t is a vector of known constants; and K_t is a known, symmetric, positive semidefinite matrix. (For maximization problems, K_t must be negative semidefinite.) The vector a_t may represent a preferred time path for variables in the criterion. When K_t is diagonal (as it often is in macro economic applications), the objective is to minimize squared deviations of selected variables from their target paths.

The solution to the problem is a linear feedback rule, specifying optimal values of the control variables x_t in terms of the known "state" values, y_{t-1}. Under certain conditions, the derivation of the feedback rule is relatively simple. Suppose A and C are known matrices (not subject to uncertainty). If the errors u_t have zero mean, known covariance, and are serially uncorrelated, the optimization problem can be solved as though it were deterministic, with errors evaluated at their means. This is the property of "certainty equivalence" originally introduced by Simon. The feedback rule is of the form

$$x_t = G_t \, y_{t-1} + g_t \qquad (3)$$

where G_t is a (qxp) matrix and g_t is a (qx1) vector. The coefficients are given by

$$G_t = -(C' H_t C)^{-1}(C' H_t A)$$

$$g_t = -(C' H_t C)^{-1} C' (H_t b_t - h_t)$$

where H_t and h_t are defined by the Ricatti equations

$$H_{t-1} = K_{t-1} + A' H_t (A + C\ G_t)$$

$$h_{t-1} = K_{t-1}\ a_{t-1} + (A + C\ G_t)' (h_t - H_t\ b_t)$$

with terminal conditions $H_T = K_T$ and $h_T = K_T\ a_T$. The feedback rule (3) is calculated recursively, starting in the terminal period and working backward to the initial period. If the planning horizon is sufficiently long and if A and C satisfy the convergence conditions described by Chow (pp. 170-72), the same feedback coefficients can be applied in successive periods. Optimal control settings would still change from period to period, however, because of the random shocks u_t that become embodied in the state vector.

This modeling approach has a number of advantages. It can handle large problems, and generates feedback rules that are simple to apply. As new information becomes available, the state equations can be reestimated and the decision rules revised. However, the solution procedure does not anticipate any updating of the state equations; matrices A and C are assumed to be known. Other, less restrictive approaches to the linear-quadratic problem (discussed by Chow and others) address learning behavior explicitly, but are difficult to solve.

3. The Elevator's Problem

Within the linear-quadratic control framework, the elevator manager's control variables include prices paid to farmers for spot and forward delivery, and contracted sales (spot and forward) to buyers in terminal markets. The manager's objective is to maximize the difference between grain sale receipts and purchase costs, less quadratic costs associated with grain handling, and with deviations of inventories from some target path. The dynamics of the model are determined by forecasting equations and a set of accounting identities.

Underlying the model are certain assumptions about elevator management. First, it is assumed that the manager exercises some discretion in the prices he offers to producers and is not simply deducting a fixed margin from prices received in terminal markets. Put another way, marketing margins are variables to some degree. Second, the elevator is willing to carry inventories in its own account--i.e., grain is not necessarily sold the same day it is purchased. It is recognized that much of the price risk associated with holding inventories can be eliminated through hedging; the manager may, nevertheless, wish to keep the elevator's own inventories near specified levels.

For simplicity, the model presented here does not capture income from storage rental, and no attempt is made to account for inventories stored but not owned by the elevator. The manager's targets for elevator-owned inventories may, however, reflect available storage capacity, which is determined by unowned inventories. It is also assumed that the elevator buys and sells only one type of grain. These simplifying assumptions can be relaxed in a more complete formulation of the model. A further simplification, imposed by the modeling framework, is to require all price changes to occur at discrete time intervals.

The planning horizon is of length T. For concreteness, the time intervals can be interpreted as weeks. Forward contracts, for sale and purchase, may be made up to m weeks in advance of delivery. Let $Q_{t,j}$ denote a quantity of grain purchased by the elevator, contracted in week t for delivery j weeks hence; and $S_{t,j}$ denote a grain sale contracted in t for delivery j weeks hence. Spot purchases and sales are denoted $Q_{t,o}$ and $S_{t,o}$. Let Q_t and S_t denote (m+1) vectors:

$$Q_t = (Q_{t,o}\ Q_{t,1}\ \ldots\ Q_{t,m})'$$
$$S_t = (S_{t,o}\ S_{t,1}\ \ldots\ S_{t,m})'$$

Two price vectors are associated with these transactions. P^q_t is the vector of prices bid by the elevator; and P^s_t is the vector of prices bid by buyers in terminal markets:

$$P^q_t = (P^q_{t,o}\ P^q_{t,1}\ \ldots\ P^q_{t,m})'$$
$$P^s_t = (P^s_{t,o}\ P^s_{t,1}\ \ldots\ P^s_{t,m})'$$

The vectors S_t and P^q_t are control variables for the elevator's marketing problem; Q_t will be specified as a linear function of P^q_t and other variables. Sale prices, P^s_t, are net of transport costs and are exogenous to the elevator.

Gross marketing returns (GMR) in week t are given by:

$$GMR_t = P^{s'}_t \Delta S_t - P^{q'}_t \Delta Q_t$$

where Δ is a diagonal (m+1) matrix of discount factors:

$$\Delta = \begin{bmatrix} 1 & & & & & 0 \\ & \delta^1 & & & & \\ & & \delta^2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \delta^m \end{bmatrix} ; \quad \text{with } (0<\delta<1).$$

This formulation is possible only in the absence of defaults, since discounted returns on forward transactions are treated as current income. It eliminates the need to retain lagged price and quantity variables in the model.

At the end of each week t, the elevator's net ownership position is described by a vector of inventories:

$$N_t = (N_{t,0}\ N_{t,1}\ \ldots\ N_{t,m})'$$

The first element, $N_{t,0}$ is the elevator's spot position at the end of week t, $N_{t,1}$ is the one-week forward position (purchase commitments less sales commitments), $N_{t,2}$ is the two-week forward position, and so on. Inventories increase by the amount of purchase commitments (spot or forward) entered in each period, and decrease by the amount of sales commitments, as follows:

$$
\begin{aligned}
N_{t,o} &= N_{t-1,o} + N_{t-1,1} + Q_{t,o} - S_{t,o} \\
N_{t,1} &= N_{t-1,2} + Q_{t,1} - S_{t,1} \\
&\vdots \\
N_{t,m-1} &= N_{t-1,m} + Q_{t,m-1} - S_{t,m-1} \\
N_{t,m} &= Q_{t,m} - S_{t,m}
\end{aligned}
\tag{4}
$$

with initial condition N_o given. In matrix notation,

$$
N_t = D\, N_{t-1} + Q_t - S_t \quad \text{where } D = \underset{(m+1)x(m+1)}{\begin{bmatrix} 1 & 1 & 0 & \ldots & & 0 \\ 0 & 0 & 1 & & & \vdots \\ \cdot & & & \cdot & & \\ \cdot & & & & 1 & 0 \\ \cdot & & & & 0 & 1 \\ 0 & & \ldots & & 0 & 0 \end{bmatrix}}
$$

Target levels for inventories are denoted N_t^* (an m+1 vector); these may reflect space availability or prudential interests, and are predetermined by the elevator manager.

In order to incorporate handling costs in the model, it is necessary to keep track of cumulative sale and purchase volumes by date of shipment or delivery. Let $V_{t,j}^S$ denote prospective shipments from the elevator, j periods hence. Shipments in the current period are denoted $V_{t,o}^S$. Shipment volumes evolve according to:

$$
\begin{aligned}
V^s_{t,0} &= V^s_{t-1,1} + S_{t,0} \\
V^s_{t,1} &= V^s_{t-1,2} + S_{t,1} \\
&\vdots \\
V_{t,m-1} &= V_{t-1,m} + S_{t,m-1} \\
V_{t,m} &= S_{t,m}
\end{aligned}
\tag{5}
$$

or in matrix notation,

$$
V^s_t = F\,V^s_{t-1} + S_t \qquad \text{where } F = \underset{(m+1)\times(m+1)}{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \cdot & & & \cdot & \\ \cdot & & & 1 & 0 \\ \cdot & & & 0 & 1 \\ 0 & \dots & & 0 & 0 \end{bmatrix}}
$$

Similarly, let V^q_t denote a vector of current and prospective purchase volumes, cumulated by delivery date:

$$
V^q_t = F\,V^q_{t-1} + Q_t \tag{6}
$$

Actually, the handling costs incurred each week depend on current shipments, $V^s_{t,0}$, and current deliveries, $V^q_{t,0}$. These can be approximated by a quadratic function:

$$
\begin{aligned}
HC_t = a\,[V^s_{t,c}]^2 + b\,[V^s_{t,0}V^q_{t,0}] + c\,[V^q_{t,0}]^2 \\
+ d\,V^s_{t,0} + e\,V^q_{t,c} + f
\end{aligned}
$$

where a through f are estimated coefficients. Note that forward purchases and sales can only affect handling costs in future periods. For modeling purposes, it will be useful to have a discounted measure of handling costs--one which closely parallels the definition of gross marketing returns but does not reflect the timing of actual cash flows. The discounted measure is given by:

$$
\begin{aligned}
DHC_t = \; & [2S_t{}' \Pi V_t^s - S_t{}' \Pi S_t] \\
& + [S_t{}' \theta V_t^q + Q_t{}' \theta V_t^s - S_t{}' \theta Q_t] \\
& + [2Q_t{}' \Omega V_t^q - Q_t{}' \Omega Q_t] + 2\lambda' S_t + 2\omega' Q_t
\end{aligned}
$$

where $\Pi = a\ \Delta$, $\theta = b\ \Delta$, $\Omega = c\ \Delta$, and λ and ω are vectors of appropriately discounted linear cost coefficients. In this formulation, all current decisions have immediate cost effects.

In addition to handling costs, quadratic costs attach to any deviations from desired net inventory levels. These are represented by $(N_t - N_t^*)' W (N_t - N_t^*)$, where W is a diagonal (m+1) matrix. The first diagonal element of W may reflect the cost of failing to meet a current sales commitment; other diagonal elements may reflect subjective preferences for future net inventory positions. If the manager wishes to balance forward purchase and sale commitments, the corresponding targets for net inventories will be zero.

Forecasting equations for Q_t and P_t^s complete the description of system dynamics. It is assumed that Q_t can be forecast with the following reduced-form equation:

$$
\begin{aligned}
Q_t = \; & \alpha_0 P_t^q + \alpha_1 P_{t-1}^q + \phi_0 P_t^s \\
& + \phi_1 P_{t-1}^s + \beta Q_{t-1} + b_t^q + \varepsilon_t
\end{aligned}
\tag{7}
$$

where α_o, α_1, ϕ_o, ϕ_1, and β are square (m+1) matrices of coefficients, b_t^q is a vector of constants, and ε_t is a vector of error terms. The inclusion of P_t^s and P_{t-1}^s is justified if the elevator's market share is related to changes in its marketing margin. For simplicity, P_t^s is assumed to follow an autoregressive process:

$$P_{t+1}^s = \gamma P_t^s + b_t^p + \nu_t \tag{8}$$

where γ is a matrix of coefficients, b_t^p is a vector of constants, and ν_t is a vector of errors. Note that the dependent variables in (8) have a different time subscript than the error terms. This reflects an informational assumption: that errors in the price forecasts are revealed before any decisions are made. That is, the elevator manager will fix his control variables P_t^q and S_t after he knows the prices bid in terminal markets. Finally, it is assumed that the errors in (7) and (8) have zero mean, known covariance, and are serially independent. This makes it possible to invoke certainty equivalence.

The optimization problem can now be stated. It is to maximize (over choice of controls, P_t^q and S_t) the discounted sum of marketing profits over the planning horizon

$$E_o \sum_{t=1}^{T} \delta^t \, [GMR_t - DHC_t - (N_t - N_t^*)' W (N_t - N_t^*)]$$

subject to equations (4) through (8), which encompass the system dynamics. (A matrix representation is given in the appendix.) The recursive solution procedure outlined in the previous section applies to this problem, with $K_t = \delta^t K$, provided that certain definiteness conditions are satisfied. The feedback rules express optimal levels of the choice variables--spot and forward bid prices, and spot and forward grain sales--as linear functions of the known state variables.

4. Requirements for Empirical Application

Before the model presented here can be applied empirically, it will need to be modified in several respects. First, the treatment of contract periods is unrealistic. Elevators change their bid prices daily, and forward price bids typically apply to a contract month. Ideally, the model should generate rules for daily pricing and sales decisions based on daily spot inventories, monthly forward inventories, and other state variables. The timing of deliveries on forward contracts is subject to uncertainty, but may be predicted to some extent. These features will complicate the dynamics of inventory change. In addition, it may be necessary to incorporate storage rental in the model--particularly if (as seems likely) producers who incur storage costs respond differently to elevator bid prices.

The forecasting equations for Q_t and P_t^s may also require modification. As specified in (7) and (8), the forecasts are based on a small set of explanatory variables. These include price and quantity variables that are recorded by elevators as part of normal business operations. Other explanatory variables could be introduced. For example, futures prices might contain relevant information for predicting prices in terminal markets. However, given the recursive nature of the solution procedure, including such variables would require specifying additional forecasting relationships--e.g., it would be necessary to know how futures prices evolve over time.

Predicting the response of producers to elevator bid prices may prove especially difficult. Supply response will depend to some extent on a seasonal pattern of grain sales outside the control of any individual elevator. Elevators compete for market share through their pricing policies. If the elevators in a marketing region face the same terminal market bids, marketing margins should determine their respective market shares. This suggests that careful consideration be given to assumptions about competitive behavior in future periods.

The assumption that forecast errors are serially independent is not limiting. If the forecast errors are correlated but follow a finite-order ARMA process, the state equations (1) can be augmented in a way that will satisfy the statistical conditions for certainty equivalence (see Pagan).

Various coefficients in the objective function also have to be estimated. The coefficients associated with handling costs may be estimated on the basis of historical cost data. Other coefficients--in particular, those reflecting subjective preferences over future net inventory positions--have to be elicited from the elevator manager, along with the values of any target variables. In practice, it may be necessary to modify objective function coefficients and target values and derive the corresponding decision rules until the decision maker is satisfied that his preferences are well represented.

Despite these problems, the framework presented here can be the basis for a model that is simple, yet realistic enough to be applied in a practical setting. Country elevators will be under greater pressure to earn satisfactory marketing returns in the next few years as CCC stocks are liquidated and storage income is reduced. In this environment, support systems for marketing decisions will be increasingly important.

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APPENDIX

The elevator's control problem can be formulated in a way similar to equations (1) and (2). Following Chow's notation, define the vectors y_t, x_t and a_t as follows:

$$
\begin{aligned}
y_t' &= (N_t' \quad Q_t' \quad S_t' \quad P_t^{q\prime} \quad P_t^{s\prime} \quad P_{t+1}^{s\prime} \quad V_t^{q\prime} \quad V_t^{s\prime} \quad 1) \\
x_t' &= (S_t' \quad P_t^{q\prime}) \\
a_t' &= (N_t^{*\prime} \quad 0 \ldots \qquad \ldots \qquad \ldots 0)
\end{aligned}
$$

Note that y_t includes a scalar one. This will allow the criterion to include linear cost terms. The optimization problem is to maximize (over choice of control variables, x_t)

$$
E \sum_{t=1}^{T} \delta^t
\begin{bmatrix} N_t - N_t^* \\ Q_t \\ S_t \\ P_t^q \\ P_t^s \\ P_{t+1}^s \\ V_t^q \\ V_t^s \\ \hline 1 \end{bmatrix}'
\left[\begin{array}{cccccccc|c}
-W & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Omega & \theta & -\Delta/2 & 0 & 0 & -\Omega & -\theta/2 & -\omega \\
0 & \theta & \Pi & 0 & \Delta/2 & 0 & -\theta/2 & -\Pi & -\lambda \\
0 & -\Delta/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Delta/2 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\
0 & -\Omega & -\theta/2 & 0 & 0 & 0 & 0 & 0 & \\
0 & -\theta/2 & -\Pi & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -\omega' & -\lambda' & 0 & & \ldots & & 0 & 0
\end{array}\right]
\begin{bmatrix} N_t - N_t^* \\ Q_t \\ S_t \\ P_t^q \\ P_t^s \\ P_{t+1}^s \\ V_t^q \\ V_t^s \\ \hline 1 \end{bmatrix}
$$

$$[8(m+1) + 1] \times [8(m+1) + 1]$$

subject to the system dynamics:

$$
\underset{[8(m+1)+1] \times 1}{\left[\begin{array}{c} N_t \\ Q_t \\ S_t \\ P^q_t \\ P^s_t \\ P^s_{t+1} \\ V^q_t \\ V^s_t \\ \hline 1 \end{array}\right]}
=
\underset{[8(m+1)+1] \times [8(m+1)+1]}{\left[\begin{array}{cccccccc|c}
D & \beta & 0 & \alpha_1 & \phi_1 & \phi_0 & 0 & 0 & 0 \\
0 & \beta & 0 & \alpha_1 & \phi_1 & \phi_0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \gamma & 0 & 0 & \\
0 & \beta & 0 & \alpha_1 & \phi_1 & \phi_0 & F & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & F & 0 \\
\hline
0 & & & \cdots & & & & 0 & 1
\end{array}\right]}
\left[\begin{array}{c} N_{t-1} \\ Q_{t-1} \\ S_{t-1} \\ P^q_{t-1} \\ P^s_{t-1} \\ P^s_t \\ V^q_{t-1} \\ V^s_{t-1} \\ \hline 1 \end{array}\right]
$$

$$
+ \underset{[8(m+1)+1] \times [2(m+1)]}{\left[\begin{array}{cc} -I & \alpha_0 \\ 0 & \alpha_0 \\ I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \\ 0 & \alpha_0 \\ I & 0 \\ \hline 0 & \ldots 0 \end{array}\right]}
\underset{2(m+1) \times 1}{\left[\begin{array}{c} S_t \\ P^q_t \end{array}\right]}
+ \underset{[8(m+1)+1] \times 1}{\left[\begin{array}{c} b^q_t \\ b^q_t \\ 0 \\ 0 \\ 0 \\ b^p_t \\ b^q_t \\ 0 \\ \hline 0 \end{array}\right]}
+ \left[\begin{array}{c} \varepsilon_t \\ \varepsilon_t \\ 0 \\ 0 \\ 0 \\ \nu_t \\ \varepsilon_t \\ 0 \\ \hline 0 \end{array}\right]
$$

If the number of forward contracts is not too large (say, $m \leq 9$), this problem is small enough to be solved on a personal computer, using the solution algorithm outlined in section 2.