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TWO-SECTOR MODEL WITH GENERALIZED DEMAND

by

Yair Mundlak & Ran Mosenson

Two-Sector Model with Generalized Demand[1/]

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1. Introduction

This paper deals with a two sector model with a general demand function which requires only that the two final products will be globally normal. In this sense, the paper generalizes the work of Jones [3] which constituted a generalization in its own right in the sense that the price elasticity was not restricted to a particular value. However, Jones implicitly assumed unitary income elasticities for the two products. As it becomes plainly clear from our analysis, such an assumption simplifies results considerably and in fact covers up some important possibilities embeded in the model. We do not attempt to generalize all known results about two sector models although the basic structure for such a generalization constitutes part of the analysis. The specific utilization made here of this structure is addressed toward the question of differential growth.

The differential growth examined in this paper is of the two sectors which constitute a closed economy. The ingredients that generate differential growth are non-unitary price and income elasticities of demand and different rates of technical change which are assumed here to be of the Hicks-neutral type. Differential growth may stem from other sources, for instance specific inputs or non-constant returns to scale [6]. We concentrate here only on demand and technical change in order to simplify the analysis and to isolate the contribution of these two sources which are of course of particular importance. It turns out that the set of possible outcomes is quite rich and complex even if other sources are ignored.

The study of differential growth should yield verifiable propositions with respect to industrial development. The case of agriculture serves as a classical illustration for a declining industry. The role of demand and technical change in such a development has been well known (See for instance [8]). The emphasis in this study is to trace their effects formally within a well defined analytic framework. This framework is general enough so that it can immediately be applied to other dichotomies, such as commodities vs. services which is of interest in studying mature economies.

2. The Model

Supply

The supply side of the model is the usual one: competitive with full employment, and production functions homogeneous of first degree. In addition we include Hicks neutral technical change:

$$(2.1)\quad Y_1 = \tau_1(t)\, F_1(K_1, L_1)$$

$$(2.2)\quad Y_2 = \tau_2(t)\, F_2(K_2, L_2)$$

where Y_j, L_j and K_j are output, employment and capital in sector j respectively; $j = 1, 2$. It is assumed that the production functions meet all the "derivative assumptions" specified explicitly by Uzawa [12, p. 108].

Demand

$$(2.3)\quad y_1 = D(p, y_2);\; y_j = \frac{Y_j}{L}$$

p is the price of the agricultural product in terms of the non-agricultural product. It is assumed that:

$$(2.4)\quad \frac{\partial D(p, y_2)}{\partial y_2} > 0, \quad \frac{\partial D(p, y_2)}{\partial p} \leq 0, \qquad D(p, 0) = 0$$

The demand function in (2.3) may seem somewhat strange at first sight. It is therefore desirable to elaborate on its relationship with the conventional demand system that can be written as:

$$(2.5)\quad \begin{aligned} y_1 &= H_1\,(y, p) \\ y_2 &= H_2\,(y, p) \end{aligned} \qquad \text{where } y = py_1 + y_2$$

The first question is under what conditions the existence of the system (2.5) secures existence of (2.3)? The condition is that for every p, H_2 will be a one to one function from y onto y_2. The one to one property of H_2 is equivalent to a requirement that y_2 is a normal good throughout the commodity space. The same assumption on y_1 is implied by (2.4). Consequently, the existence of (2.3) implies (2.3)' and vice versa, where:

$$(2.3)' \quad y_2 = \tilde{D}(\frac{1}{p}, y_1)$$

So we have Proposition 1 : The existence of the demand functions (2.5) together with the condition that y_1 and y_2 are globally normal goods imply the existence of the demand functions (2.3) and (2.3)' . The converse is also true as can be easily verified.

Using (2.3) rather than (2.5) simplifies the analysis. We shall use throughout the elasticities of (2.3) and it is of interest to state explicitly their relationships to usual demand elasticities obtained from (2.5). Let the elasticities obtained from (2.3) be: $\eta = \frac{\partial \ln y_1}{\partial \ln y_2}$ and $-\sigma_D = \frac{\partial \ln y_1}{\partial \ln p}$ and those obtained from (2.5) be:

$-E_{1p} = \frac{\partial \ln y_1}{\partial \ln p}$ $\quad E_{jy} = \frac{\partial \ln y_j}{\partial \ln y}$. Noting that $\Pi_1 E_{1y} + \Pi_2 E_{2y} = 1$ where $\Pi_j = \frac{p_j y_j}{y}$ we have:

$$(2.6) \quad \eta = \frac{\Pi_2 E_{1y}}{1 - E_{1y}\Pi_1} = \frac{E_{1y}}{E_{2y}} \qquad \text{or} \qquad E_{1y} = \frac{\eta}{\Pi_2 + \eta\Pi_1}$$

$$(2.7) \quad -\sigma_D = \frac{\Pi_1 E_{1y} - E_{1p}}{1 - E_{1y}\Pi_1} \qquad \text{or} \quad -E_{1p} = -\sigma_D + (\sigma_D - 1)\frac{\Pi_1 \eta}{\Pi_2 + \eta\Pi_1}$$

It then follows that $\eta \gtreqless 1 \Longleftrightarrow E_{1y} \gtreqless 1$, and $\sigma_D \lesseqgtr 1 \Longleftrightarrow E_{1p} \gtreqless 1$.

In some sense, our formulation of demand destroys the symmetry possesed by the system (2.5). However, the asymmetry of our system is only superficial since it is rather easy to shift results obtained under D in (2.3) to results in terms of $\tilde{D}$ in (2.3)'.This can be done by expressing the elasticities of $\tilde{D}$ in terms of those of (2.3): $\tilde{\eta} = \frac{1}{\eta}$ and $\tilde{\sigma}_D = \frac{\sigma_D}{\eta}$.

In a two-sector growth model the demand relation is given by the saving relation. Under the fixed saving relation we have: $y_2 = sy$ and $y_1 p = (1 - s)y$. Using the notation $d \ln \chi = \hat{\chi}$, we have $\hat{y}_2 = \hat{y} = \hat{p} + \hat{y}_1$. In our notation, this assumption implies $\eta = 1$, $-\sigma_D = 1$.As it is clear from the forthcoming discussion, many of the results are greately simplified under this assumption.

If we now allow s to change and set $\hat{s}(y) = \varepsilon\hat{y}$ where is the elasticity of s with respect to percapita income, we get :

$$\hat{y}_1 = -\hat{p} + \hat{y}_2 \frac{1 - \varepsilon\Pi_2/\Pi_1}{1 + \varepsilon} = -\hat{p} + \eta\hat{y}_2$$

and for $\varepsilon > 0$, $\eta < 1$ whereas $\sigma_D = 1$. Thus, our demand relation easily accomodates this generalization. Again, $\sigma_D = 1$ simplifies some of the results. These examples should make it clear that (2.3) is a rather general demand condition.

The specification of (2.3) and (2.4) serves the discussion up to section 11 on capital accumulation. At that point the specification has to be extended to include saving. It is assumed that agriculture produces only consumption goods whereas the output of the non-agricultural sector is used for both consumption and investment. The demand for investment goods is given by the saving relation: we assume a fixed saving ratio, so we have:

(2.8) Gross investment $= s(pY_1 + Y_2)$.

This constitutes only part of the demand for the output of sector 2. It is embedded in the general demand function, given by (2.3) above. Since capital goods are produced only in sector 2, it is implied that the output of sector 2 will not be smaller than $s(py_1 + y_2)$. So the function D must satisfy:2/

(2.9) $y_1 = D(y_2, p) \leq \frac{1-s}{s} \frac{y_2}{p} = h(y_2, p)$.

3. Summary of Main Findings

For expository purpose we identify sector 1 with agriculture. In such an identification agriculture is representing the sector with the "low" demand elasticities, and consequently the second sector is that with the "high" demand elasticities.

Short Run Equilibrium

(1) There exists a unique and stable competitive solution for the economy in the short run.

Response of the system to capital accumulation, technology constant:

(2) The Wage Rental Ratio is monotonically increasing with the capital labor ratio (k) . This result which is trivial in a one-sector model is not at all obvious in a two-sector model with general demand.

(3) The Price (of agricultural product) monotonically declines (increases) with k as long as agriculture is capital (labor) intensive.

(4) The Proportion of Agriculture in the Labor Force(ℓ_1) is likely to decline for sufficientely large elasticity of substitution in that sector (σ_1) and sufficientely small income elasticity (η). To illustrate some sufficient conditions for such a decline, it is sufficient that agriculture be labor intensive and $\frac{\sigma_1}{\sigma_2} > \eta \frac{\beta_1}{\beta_2}$ where β_j is the capital share in sector j. On the other hand if agriculture is capital intensive, it is sufficient that $\sigma_1 > \eta\sigma_2$ and $\sigma_1 > \sigma_D$ where $-\sigma_D$ is the price elasticity of demand.

(5) The Proportion of Agriculture in Capital Stock (ρ_1) : Conditions leading to decline in ℓ_1 are likely also to result in a decline in ρ_1 providing $\frac{\sigma_1}{\sigma_2}$ is not large. Specifically, $\sigma_2 \geq \sigma_1$ and a decline in ℓ_1 will result in a decline in ρ_1.

(6) Differential Growth in Output: Growth in agricultural output is likely to lag behind that of the non-agricultural sector. When agriculture is labor intensive, it is sufficient for such a lag that the income elasticity is not larger than 1. If agriculture is capital intensive, it is necessary that the income elasticity be smaller than $1 - \varepsilon$ where ε is a small positive number. The particular values for the income elasticity that result in such a lag depend on other parameters but is likely to be close (but smaller) to 1. The values for the various parameters in question observed in mature and moderately mature economies secure such a lag.

In the particular case that the income elasticity is 1, the sign of the differential growth depends on factor intensity alone. This is a familiar result. The conditions obtained in this paper (proposition 11) indicates the specifity of that result.

Response of the System to Technical Change, Resources Held Constant

(7) General: Apart from some very special conditions, such as unitary income and price elasticities, a technical change will shift the equilibrium position, including the allocation of the factors of production to the sectors. It turns out that this process can appropriately be described in two steps: Keeping the factors' distribution unchangedthe technical change will create excess supply of one of the products and excess demand to the other. In the second step equilibrium is restored by moving both factors of production toward the excess demand sector. The sign of the excess supply of the first stage depends crucially on the demand elasticities. When both price and income elasticities are smaller than 1, the excess supply is likely to be positive.

(8) The Proportions of Agriculture in the Labor Force and Capital Stock: Both proportions depend solely on the sign of the excess supply caused by technical change (7 above). A positive excess supply of the agricultural product leads to a decline in both proportions.

(9) Wage Rental Ratio: may increase or decrease, depending again on the sign of the excess supply and factor intensity. However, the combined effect of technical change and capital accumulation is such that any decline in the wage rental ratio is bounded and eventually it must increase with time if the capital labor ratio does so.

(10) Price. changes depend largely on the differential technical change. A larger technical change in agriculture leads to a decline in the price of the agricultural product and vice versa.

(11) Differential Growth in Output. A positive excess supply of agricultural product and technical change in agriculture not exceeding much that of the non-agricultural sector lead to a smaller rate of growth of the agricultural product.

Aggregate Production Function

The following results are by-products of the present system which have a more general theoretical interest outside the specific questions of differential growth.

(12) The value of total output in equilibrium is a homogeneous function of degree one in labor and capital. Namely there exists an aggregate production function which displays constant returns to scale. Consequently, in a variety of subjects, the two-sector model can be reduced to and analyzed as a one-sector-model.

It can be easily shown that the property of first degree homogeneity of the aggregate production function is shared by most other two-sector models. It stems from first degree homogeneity of the production system and the demand relation.

(13) When the production functions in both sectors are Cobb-Douglas and when the income and price elasticities are unitary, the aggregate production function, including the technical change component, will be identical, up to a scalar, to that of sector 2. As such, of course it is also a Cobb-Douglas function. The asymmetry with respect to the two sectors is a result of the choice of numeraire.

(14) The reduced aggregate "one-sector" model differs from the usual one sector model with a single production function. For instance, the competitive wage rate differs in general from the marginal product of labor evaluated from the aggregate production function. Furthermore, it is possible that the marginal productivity evaluated from the aggregate production function will be negative inspite of the fact that it is positive in both sectors. In as much as the model is valid it provides another example for a fallacy of composition in relating the marginal distribution theory to the aggregate production function. This may not be a big surprise since the aggregate production function reflects a system in equilibrium. It is not a pure physical production system. The demand function is embeded in the aggregate function. The aggregate "production function" is not invariant

to the specification of the demand. But note that this is always the case when we present the value product as a function of the available factors of production. In particular such kind of an aggregate is the only sensible interpretation which we can see to our sectorial production functions. We are assuming that marginal distribution theory is valid with respect to the sectorial production functions and get out with the result that it is invalid with respect to the over-all aggregate function. This result certainly acts as a boomerang against the validity of the same assumption with respect to the sectorial functions. The assumption stems only from careless analogy to simpler systems. The present paradox, which is shared by most macro models, is serious as much as it is predictable. It is only nice to see it in a form of an exact expression.

Capital Accumulation

(15) The model is converging to a steady state if the aggregate production function does not display technical change. This happens when (a) there is no technical change in both sectors, (b) when there is no technical change in the non-agricultural sector and the price elasticity is unitary as is the case when the demand consists only of the saving relation. A sufficient condition for the stability of a steady state is as in the usual case of fixed saving ratio, that the marginal productivity of labor with respect to the aggregate production function is positive. Global fulfillment of this condition is evaluated explicitly the, result is a generalization of the condition obtained by Jones [3].

(16) In the presence of technical change in the aggregate production function, there is in general, no exponential growth. Yet, somewhat weaker results are stated. The rate of capital accumulation is chasing the ratio of rate of technical change to the labor elasticity. In general, this ratio is not constant. Under the special case mentioned in (13) above, the ratio is constant and there is a convergance to a path of exponential growth with a constant rate of accumulation. This

rate is equal also to the rate at which the stationary capital labor ratio is progressing due to technical change.

(17) In general, the asymptotic properties of the accumulation process are dominated by the production function and the technical change of the sector which produces capital goods. If there is technical change in this sector, the capital labor ratio will be increasing indefinitely with time. The long term rate of growth is determined by the parameters of that function alone.

4. Short Run Equilibrium

Proposition 2: Under the assumptions made in Section 2 and given K, L and t, there exists a unique and stable equilibrium. It is stable in the sense that a too high price will result in a positive excess supply.

This is statement (1) of Section 3. ~~To prove~~ it we note that given K, L and t, there exists a transformation curve from which a supply schedule can be derived. This is presented in Figure 1 where p^s refers to the supply price at a particular point. The demand curve is drawn for a particular price, p^d, that appears in the demand function. Referring to Figure 1 we now proceed: (a) Existence: Suppose $p_A^d = p_A^s$ then A is an equilibrium point. Suppose $p_A^s > p_A^d$, draw a new demand curve $D(y_2, p_A^s)$. By (2.4) it will be to the right and below $D(y_2, p_A^d)$. Hence at the intersection we have $p_B^s < p_A^s = p_B^d$ and the inequality is reversed. Assuming continuity, (in fact we assume differentiability) equilibrium exists between A and B. (b) Uniqueness and stability: Suppose A is the equilibrium point. A rightward rotation of D is caused by higher demand prices but will generate lower supply prices. Hence, no other equilibrium exists. Also, the price differences will make it profitable to increase production of y_1, and beneficial to increase consumption of y_1 hence stability.

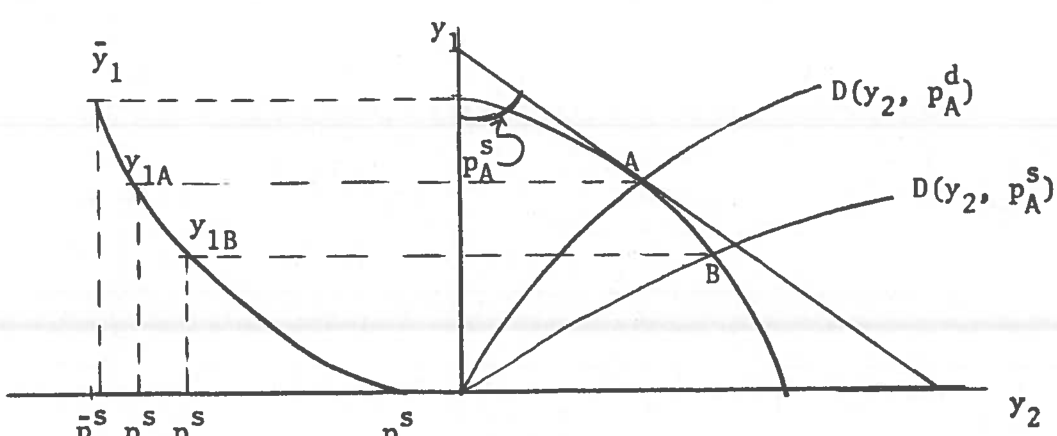


Figure 1

Analytically, the proof can be stated as follows:

The supply of y_1, is denoted by $y_1^s = y_1^s(p|\ k, t)$ with $\frac{\partial y_1^s}{\partial p} > 0$. The quantity demanded at a given p is $y_1^d\ [y_2^s(p), p]$. This function is declining in p:

$$\frac{dy_1^d}{dp} = \frac{\partial y_1^d}{\partial y_2^s(p)} \ \frac{dy_2^s}{dp} + \frac{\partial y_1^d}{\partial p} < 0$$

Let $\phi\ (p)$ = excess demand function = $y_1^d(p) - y_1^s(p)$.

So $\phi'(p) < 0$. Let $\bar{p} = \max\limits_{y_1^s} p(y_1^s\ |k, t)$ and $\underline{p} = \min\limits_{y_1^s} p(y_1^s\ |k, t)$ then

$$p \geq \bar{p} \Rightarrow y_2^s = 0 \implies y_1^d = 0 \implies \phi(p) < 0$$

$$p \leq \underline{p} \Rightarrow y_1^s = 0 \implies \phi(p) > 0 .$$

Thus, $\phi(p)$ changes sign and is always declining which completes the proof.

Holding technology constant, the transformation curve is uniquely determined by k. [3] In view of proposition 2, for any given technology and demand relation, p, y_1 and y_2 are uniquely determined by k. Similarly, $y = py_1 + y_2$ is uniquely determined by k. Consequently we have:

Proposition 3: Under our assumptions there exists an aggregate per capita production function $y = y(k, t)$. Hence total output $Y = Ly$ is determined by a production function $Y = F(K, L, t)$ which is homogeneous of degree 1 in K and L. This proves statement 12 in Section 3.

It should be noted that y is a locus of equilibria points and depends on the demand relation. Two economies which are identical in all pertinent respects except for the demand relation will have different aggregate production functions even though the production function in each of the sectors are the same in the two economies.

5. The Displacement of the System and the Behavior of the Wage Rental Ratio.

We turn now to evaluate the displacement of the system. We start with differentiating the system of equations describing the supply side. The resulting differentials are related to changes in the wage rental ratio (ω), resources and technology. After introducing the demand relation the wage rental ratio and consequentely the other differentials are related to changes in resources and technology alone. Such a two step presentation allows to trace the effects of working with our general demand relation. At the same time such a presentation requires some care when dealing with the case of equal factor intensities in the two sectors. All the differentials of the supply side alone hold only for unequal intensities ($i \neq 0$ in the notation below) whereas the final differentials which also take into account the demand hold also for equal factor intensities. This point is further discussed in Appendix 6.

Some of the following findings are well known and are presented briefly for further reference. The emphasis is on presentation in terms of parameters which are subject to intensive empirical work. Outline of the proofs is given in the appendix.

Let $\ell_j = \frac{L_j}{L}$, $L = L_1 + L_2$, $\rho_j = \frac{K_j}{K}$, $K = K_1 + K_2$, $\hat{x} = \frac{d \ln x}{dt}$, and $\sigma_j = \hat{k}_j/\hat{\omega}$ is the elasticity of substitution in sector j. We then have: 4/

(5.1) $\hat{\ell}_1 = \frac{\ell_2}{i}(-\hat{k} + \bar{\sigma}_\rho \hat{\omega})$ $\quad i \neq 0$

(5.2) $\hat{\ell}_2 = \frac{\ell_1}{i}(\hat{k} - \bar{\sigma}_\rho \hat{\omega})$ $\quad i \neq 0$

where $\bar{\sigma}_\rho = \rho_1\sigma_1 + \rho_2\sigma_2$

$i = \ell_1 - \rho_1$ is a measure of capital intensity; $i > 0$ implies that sector 1 is labor intensive.

(5.3) $\hat{\rho}_1 = \frac{\rho_2}{i}(-\hat{k} + \bar{\sigma}_\ell \hat{\omega})$ 5/ $\quad i \neq 0$

(5.4) $\hat{\rho}_2 = \frac{\rho_1}{i}(\hat{k} - \bar{\sigma}_\ell \hat{\omega})$ $\quad i \neq 0$

where $\bar{\sigma}_\ell = \ell_1\sigma_1 + \ell_2\sigma_2$

(5.5) $\hat{y}_1 = \frac{-\ell_2}{i} \hat{k} + E_1 \hat{\omega} + \gamma_1$ 6/ $\quad i \neq 0$

(5.6) $\hat{y}_2 = \frac{\ell_1}{i} \hat{k} + E_2 \hat{\omega} + \gamma_2 \quad i \neq 0$

where $\gamma_j = \frac{d \, \ell n}{dt} \tau_j(t)$

where the elasticities of outputs with respect to ω are given by:

$$E_1 = \frac{\ell_2}{i} \bar{\sigma}_\rho + \beta_1\sigma_1 \qquad i \neq 0$$

$$E_2 = \frac{-\ell_1}{i} \bar{\sigma}_\rho + \beta_2\sigma_2 \qquad i \neq 0$$

where β_j is the capital share in the j-th sector:

$$\beta_j = 1 - \alpha_j = \frac{rK_j}{p_jY_j}, \quad \alpha_j = \frac{wL_j}{p_jY_j} \qquad p_2 = 1$$

It can be shown that the signs of E_j are given by: 7/

(5.7) $iE_1 > 0$, $iE_2 < 0$.

The first component on the r.h.s. of (5.5) and (5.6) is a statement of the so called Rybczynski theorem, the second component indicates the movement along a contract curve whereas the last component indicates the effect of technical change.

(5.8) $\hat{p} = I\hat{\omega} - T$ 8/

where $I = \alpha_1 - \alpha_2$, $iI > 0$

$T = \gamma_1 - \gamma_2$.

The first component gives $I = \left.\frac{\hat{p}}{\hat{\omega}}\right|_t$ 9/ which is the well known result on factor price theory in international trade. The second component summarizes the effect of technology on p.

Combining (5.5), (5.6) and (5.8): 10/

(5.9) $\hat{y} = \Pi_1 I\hat{\omega} + \beta\hat{k} + \gamma_2$.

$\beta = \frac{rK}{Y}$ is the share of capital in total output and $\Pi_1 = \frac{py_1}{y}$ is the share of sector 1 in total output.

Holding k constant and $\gamma_2 = 0$, y increases with ω if the sector whose price is the numeraire is capital intensive. Second, for $\hat{\omega} = \gamma_2 = 0$, y increases with k. Finally, y is affected directly only by technical change in the sector which serves as numeraire.

The developments up to this point deal only with the supply side. We now close the system by bringing in the displacement in the demand relation (2.3):

(5.10) $\hat{y}_1 = \eta\hat{y}_2 - \sigma_D\hat{p}$,

Combining (5.5), (5.6), (5.8) and (5.10) we get:

(5.11) $\hat{\omega} = \frac{1}{\sigma} [\hat{k} - \frac{i}{\ell} E_0]$

where

(5.12) $\sigma = (E_1 + \sigma_D I - \eta E_2) \frac{i}{\ell}$ is a quasi overall elasticity of substitution

(5.13) $\ell = \ell_2 + \eta\ell_1$

(5.14) $E_0 = \gamma_1(1 - \sigma_D) - \gamma_2(\eta - \sigma_D)$

Since $iE_1 > 0$, $iI > 0$ and $-iE_2 > 0$, σ is always positive for $\sigma_D > 0$ and $\eta > 0$. For $E_0 = 0$ the wage rental ratio is increasing in k. A sufficient condition for $E_0 = 0$ is that there is no technical change in the two sectors. If there is technical change, it is possible to obtain $\hat{k} < \frac{i}{\ell} E_0$ so that k and ω will move in different directions. This however, can only hold locally. To see this we note that $\underline{\omega}(k(t)) = \min [.] \leq \omega(t) \leq \max [.] = \bar{\omega}(k(t))$ where $[.] = [\omega_1(k_1 = k(t))$, $\omega_2(k_2 = k(t))]$ and since $\omega_j'(k_j) > 0$, $\underline{\omega}(k(t))$ is monotonically increasing and the Uzawa derivative conditions make $\underline{\omega}(k) \to \infty$ as $k \to \infty$ and ω must eventually increase with k. So we have:

<u>Proposition 4:</u> (a) $\left.\frac{\hat{\omega}}{\hat{k}}\right|_t > 0$. (b) when k(t) is increasing indefinetely in t, ω is eventually increasing in t as well.

We refer to σ as a quasi overall elasticity of substitution since σ is not derivable from the aggregate production function as elasticity of substitution is usually derived[11/]. However, $1/\sigma$ does indicate the relative change in ω associated with such change in k and it is in that sense that the term is applied.

For further analysis, we write (5.12) explicitly:

(5.15) $\sigma = \bar{\sigma}_\rho + \frac{i}{\ell} D$ where $D = [\beta_1(\sigma_1 - \sigma_D) - \beta_2(\eta\sigma_2 - \sigma_D)]$

The term E_0 has a revealing economic meaning, it is equal to $\left.(\hat{y}_1^s - \hat{y}_1^d)\right|_{\text{allocation of resources}}$ where y_1^s and y_1^d are the supply and demand of y_1 respectively. Holding resources and their allocation constant, $\hat{y}_1^s = \gamma_1$, $\hat{y}_2^s = \gamma_2$ and $\hat{p} = \gamma_2 - \gamma_1$. Thus, $\hat{y}_1^d = \eta\hat{y}_2^s - \sigma_D\hat{p} =$ $= \eta\gamma_2 - \sigma_D\gamma_2 + \sigma_D\gamma_1$ and $\hat{y}_1^s - \hat{y}_1^d = \gamma_1(1 - \sigma_D) - \gamma_2(\eta - \sigma_D) = E_0$.

Consequently, y_1E_0 is the excess supply caused by technical change had the economy not reacted to it. Since the economy is always in equilibrium this excess supply is not observed. It is closed by price adjustment and according to (5.11) such adjustment leads to a change in ω at the rate $-\frac{i}{\sigma\ell}E_0$.

The measure of technologically caused excess supply, E_0, plays an important role in the analysis. The following proposition states some sufficient conditions for the sign of the excess supply.

<u>Proposition 5:</u> Given $\gamma_1, \gamma_2 > 0$ sufficient conditions for sign E_0 (sign excess supply) are:

(a) $E_0 > 0$.

(a.1) $\eta \leq \sigma_D < 1$; (a.2) $\eta < \sigma_D \leq 1$; (a.3) $\frac{\gamma_1}{\eta} > \gamma_2 > \gamma_1$; (a.4) $\sigma_D < 1$, $\eta < 1$ and $\gamma_1 \geq \gamma_2$; (a.5) $\sigma_D < 1$, $\eta \leq 1$ and $\gamma_1 > \gamma_2$.

(b) $E_0 = 0$

(b.1) $\sigma_D = \eta = 1$; (b.2) $\gamma_1 = \gamma_2$, $\eta = 1$.

(c) $E_0 < 0$

(c.1) $1 < \sigma_D \leq \eta$; (c.2) $1 \leq \sigma_D < \eta$; (c.3) $\frac{\gamma_1}{\eta} < \gamma_2 < \gamma_1$; (c.4) $\sigma_D > 1$, $\eta > 1$ and $\gamma_1 \geq \gamma_2$; (c.5) $\sigma_D > 1$, $\eta \geq 1$ and $\gamma_1 > \gamma_2$.

The proofs follow immediately from (5.14). It is interesting to note the dependence of the conditions on the demand elasticities. For instance, (a.1), (a.2), (b.1), (c.1) and (c.2) are invariant to the values of γ_1 and γ_2 and are stated in terms of η and σ_D alone. We cannot state a condition which will be invariant to σ_D and η . At the most, we can do without σ_D but not without η . We also note from (b.1) that under the conventional fixed saving growth model $E_0 = 0$; technical change does not create excess supply.

6. Rate of Change of Composition of the Labor Force

Combining (5.1) with (5.11):

$$(6.1) \quad \hat{\ell}_1 = - \frac{\ell_2}{\ell \sigma} (D\hat{k} + \bar{\sigma}_\rho E_0)$$

We take up the response to technical change and to capital accumulation seperately.

Response to capital accumulation

$$\left.\frac{\hat{\ell}_1}{\hat{k}}\right|_t = - \frac{\ell_2}{\ell} \frac{D}{\sigma}$$

Hence, sign $\left.\frac{\hat{\ell}_1}{\hat{k}}\right|_t$ = - sign D.

Referring to (5.15) we get

Proposition 6. (Effect of capital accumulation on the composition of the labor force)

(6.2) $\left.\hat{\ell}_1/\hat{k}\right|_t \gtreqless 0 \Longleftrightarrow \beta_1\sigma_1 + \beta_2\sigma_D \lesseqgtr \beta_2\eta\sigma_2 + \beta_1\sigma_D$

Some sufficient conditions immediately follow. Sufficient for $\left.\hat{\ell}_1/\hat{k}\right|_t < 0$:

(6.3) $\sigma_1 \geq \sigma_D \geq \eta\sigma_2$

(6.4) $\beta_2\sigma_1 \geq \beta_1\sigma_1 \geq \beta_2\eta\sigma_2$

(6.5) $\beta_1 > \beta_2$ and $\sigma_D \leq \sigma_1 \geq \eta\sigma_2$

(6.6) $\sigma_1 \geq \sigma_D + \frac{\beta_2}{\beta_1}\eta\sigma_2$

Sufficient conditions for $\left.\hat{\ell}_1/\hat{k}\right|_t > 0$ are given by (6.3) and (6.4) with the ordering reversed and by

(6.5)* $\beta_2 > \beta_1$ and $\sigma_1 \leq \eta\sigma_2 \geq \sigma_D$

(6.6)* $\eta\sigma_2 \geq \sigma_D + \frac{\beta_1}{\beta_2}\sigma_1$

A sufficient condition for $\left.\hat{\ell}_1/\hat{k}\right|_t = 0$

(6.7) $\sigma_1 = \sigma_D = \eta\sigma_2$

This condition is met under $\eta = 1$ and $\sigma_1 = \sigma_2 = \sigma_D$ of which the Cobb-Douglas fixed saving rate is a special case.

Basically (6.2), and the derived sufficient conditions state that for σ_1 sufficiently large as compared with $\eta\sigma_2$, ℓ_1 declines with k. The economic meaning is related to the effect of change in k on the system. Heuristically, it can be stated as follows: Suppose $\hat{k} > 0$, then the wage rental ratio increases. As a result, the larger is σ_1, the larger will be the increase in k_1 and consequentely the less will the increase in y_1 caused by the initial increase in k, depend on increasing employment in that sector. Similarly, the smaller is σ_2 the more will the associated increase in y_2 depend on increasing employment in sector 2. The actual increase in y_1 and y_2 is determined by the demand equation. The smaller is η the smaller will be the increase in y_1 and it is possible that the increase in k_1 will substitute capital for labor to the extent of decreasing ℓ_1. Similarly, for small η , there will be a relatively large increase in y_2 which will have to come about from increase in ℓ_2 in addition to whatever increase there has been in k_2. To all this, we add the price effect whose direction depends on the factor intensity. All these remarks can be supplemented by noting that:12/

$$\frac{\partial D}{\partial \sigma_1} > 0 \ , \ \frac{\partial D}{\partial \sigma_2} < 0, \ \frac{\partial D}{\partial \eta} < 0, \ \frac{\partial D}{\partial \sigma_D} = I \ .$$

It is clear that if η and σ_D are small as is the case of agriculture, D is likely to be positive and $\hat{\ell}_1 < 0$. The reverse would be true if η and σ_D were large as may be the case for services. Thus, the whole pattern of mobility of the labor force depends crucially on the demand elasticities.

<u>Response to technical change, k constant</u>

$$\hat{\ell}_1\Big|_k = -\frac{\ell_2\bar{\sigma}_\rho}{\ell\sigma} E_0 \ , \text{ hence}$$

<u>Proposition 7:</u> sign $\hat{\ell}_1\Big|_k = -$ sign E_0. Consequently, parts a, b and c of proposition 5 state sufficient conditions for decline, constancy and increase in ℓ_1 respectively.

If we identify sector 1 with agriculture, it is very likely by part a of proposition 5 that ℓ_1 will be declining. This is simply due to the fact that with low demand elasticities, technical change creates positive excess supply of the agricultural product. Furthermore, with $\sigma_D < 1$, the larger is γ_1 the larger is E_0. Again, had σ_D and η been larger than 1 the whole pattern of development would have been different and technical change in sector 1 would have drawn some more resources into it.

Another question of interest is what happens to the total employment in sector 1. Clearly, the answer depends on the rate of change of the labor force, since $\hat{L}_1\big|_k = \hat{\ell}_1\big|_k + \hat{L} = \hat{L} - \frac{\ell_2}{\ell}\ \frac{\bar{\sigma}_\rho}{\sigma} E_0$. When $\hat{L} = 0$, we have $\hat{\ell}_1 = \hat{L}_1$. In general, if σ_D and η and ℓ_1 are small, $(\ell_2\bar{\sigma}_\rho/\ell\sigma)$ is not far from 1, $\hat{\ell}_1$ is nearly $-\gamma_1$ so that for L_1 to decline we need roughly $\gamma_1 > \hat{L}$.

Finally, the change in ℓ_1 over time as a result of both capital accumulation and technical change is indicated in (6.1). The contribution of capital accumulation to outflow of labor from agriculture may be of substantial magnitude.

7. Rate of Change of Composition of Capital Utilization

Combining (5.3) and (5.11):[13/]

$$(7.1)\quad \hat{\rho}_1 = \frac{\rho_2}{\ell\sigma}\,[\ell(\sigma_1 - \sigma_2) - D]\hat{k} - \frac{\rho_2}{\ell\sigma}\ \bar{\sigma}_\ell\ E_0$$

or

$$(7.2)\quad \hat{\rho}_1 = \rho_2\ [\frac{(\sigma_1 - \sigma_2)}{\sigma} + \frac{1}{\ell_2}\ (\hat{\ell}_1/\hat{k}\big|_t\ )]\hat{k} + \frac{\rho_2\bar{\sigma}_\ell}{\ell_2\bar{\sigma}_\rho}\ (\hat{\ell}_1\big|_k\ ) .$$

Response to capital accumulation

Given $\sigma_1 \leq \sigma_2$, $\hat{\ell}_1/\hat{k}\big|_t < 0$ is sufficient for $\hat{\rho}_1/\hat{k}\big|_t < 0$ and $\sigma_1 \geq \sigma_2$, $\hat{\ell}_1/\hat{k}\big|_t > 0$ is sufficient for $\hat{\rho}_1/\hat{k}\big|_t > 0$. Consequentely if $\sigma_1 > \sigma_2$, ρ_1 is less likely to decrease with accumulation than ℓ_1. The reason is that a large value for σ_1 act in favor of substituting capital for labor in sector 1 as k increases. As a consequence ℓ_1 tends to decline but at the same time ρ_1 tends to increase. For ρ_1 to decrease, it is necessary that the main expansion in production due to capital accumulation will be in sector 2 and this requires a relatively low value for η. This statement gives a heuristic explanation for the asymmetry that exists between the behavior of the two factors. All this can be shown by a formal partial differentiation of $(.) = \ell(\sigma_1 - \sigma_2) - D$ with respect to the parameters in question: $\frac{\partial(.)}{\partial\sigma_1} = \ell - \beta_1$ which is likely to be positive whereas $\frac{-\partial D}{\partial\sigma_1}$ is negative. $\frac{\partial(.)}{\partial\sigma_2} = \eta\beta_2 - \ell$ is likely to be negative whereas $\frac{-\partial D}{\partial\sigma_2}$ is positive. $\frac{\partial(.)}{\partial\eta} = \ell_1(\sigma_1 - \sigma_2) + \beta_2\sigma_2$ is positive for $\sigma_1 > \sigma_2$ or for $\beta_2 > \ell_1$. The same sign holds for $\frac{-\partial D}{\partial\eta}$. $\frac{\partial(.)}{\partial\sigma_D} = \frac{-\partial(D)}{\partial\sigma_D} = -I$.

To obtain exact conditions on the sign of $\hat{\rho}_1/\hat{k}|_t$, we write explicitly $\ell(\sigma_1 - \sigma_2) - D$ and get an expression analogous to (6.2):

Proposition 8: (Effect of capital accumulation on the composition of capital utilization).

$$(7.3) \quad \hat{\rho}_1/\hat{k}|_t \gtreqless 0 \iff \beta_1\sigma_1 + \ell\sigma_2 + \beta_2\sigma_D \lesseqgtr \ell\sigma_1 + \beta_2\eta\sigma_2 + \beta_1\sigma_D$$

Some further sufficient conditions are given in the Appendix (A.7).

Response to technical change, k constant.

Proposition 9. (Effect of technical change on composition of capital accumulation)

Sign $\hat{\rho}_1|_k = -$ sign $E_0 =$ sign $\hat{\ell}_1|_k$.

This result was predictable and the remarks that follow Proposition 7 apply here as well.

8. Rate of Change of p.

Combining (5.8) and (5.11):

$$(8.1) \quad \hat{p} = \frac{I}{\sigma}\hat{k} - \frac{iI}{\sigma\ell}E_0 - T.$$

Proposition 10 (rate of change of p) : sign $\hat{p}/\hat{k}|_t =$ sign I.

It should be noted that the proposition deals with movements which satisfy the demand conditions and as such it is not a trivial repetition of a similar result in factor price theory.

In evaluating the effect of technical change on p, it should be noted that the term $\frac{iI}{\sigma\ell} E_0$ is likely to be small relative to T. Consequentely T is likely to be dominant in determining sign $\hat{p}\big|_k$. Ignoring $\frac{iI}{\sigma\ell} E_0$, we also state that for the price of agricultural product to decline in a process of growth it is necessary that agriculture will be either capital intensive or display a larger rate of technical change.

9. Rate of Change of Outputs.

The importance of the demand equation becomes immediately apparent when we focus our attention on rates of change of outputs in the two sectors. Rewriting (5.10) we have $\hat{y}_1 - \hat{y}_2 = (\eta - 1)\hat{y}_2 - \sigma_D\hat{p}$. Assuming $\hat{y}_2 > 0$, it is sufficient for $\hat{y}_1 < \hat{y}_2$ that $\eta < 1$ and $\hat{p} > 0$. Similarly, it is sufficient for $\hat{y}_1 > \hat{y}_2$ that $\eta > 1$ and $\hat{p} < 0$. This result is an obvious generalization of the result usually quoted in the absence of technical change for $\eta = 1$ [3] which, in view of proposition 10 depends on factor intensity alone. The general case is of much greater variety as we now proceed to show. Using (5.5), (5.6) and (5.11) we get:14/

$$(9.1) \quad \hat{y}_1 = \frac{1}{\ell\sigma}(\eta\bar{\sigma}_{\beta\ell} - \ell_2 I\sigma_D)\hat{k} + (\gamma_1 - \frac{iE_1}{\ell\sigma} E_0)$$

$$(9.2) \quad \hat{y}_2 = \frac{1}{\ell\sigma}(\bar{\sigma}_{\beta\ell} + \ell_1 I\sigma_D)\hat{k} + (\gamma_2 - \frac{iE_2}{\ell\sigma} E_0)$$

where $\bar{\sigma}_{\beta\ell} = \ell_1\beta_1\sigma_1 + \ell_2\beta_2\sigma_2 > 0$.

It then follows:

$$(9.3) \quad \hat{y}_1 - \hat{y}_2 = \frac{-1}{\ell\sigma}[(1 - \eta)\,\bar{\sigma}_{\beta\ell} + I\sigma_D]\hat{k} + [T - \frac{i(E_1 - E_2)}{\ell\sigma} E_0]$$

Effect of capital accumulation

Proposition 11 (Effect of capital accumulation on differential growth of outputs):

$$(9.4) \quad \frac{\hat{y}_1 - \hat{y}_2}{\hat{k}}\bigg|_t \gtreqless 0 \Longleftrightarrow \frac{1 - \eta}{\sigma_D}\,\bar{\sigma}_{\beta\ell} \lesseqgtr - I \qquad \sigma_D \neq 0$$

Given $1 \geq \eta$, $I > 0$ is sufficient for (9.4) to be negative. However, for "small" η and σ_D (9.4) is likely to be negative for all conceivable values of I. On the other hand, for $\eta>1$, $I > 0$ becomes necessary but not sufficient for (9.4) to be negative. The simplification which is usually obtained for the convenient but narrow assumption of $\eta = 1$ is now obvious.

Effect of technical change

Since $iE_1 > 0$, $iE_2 < 0$, sign $i(E_1 - E_2)E_0$ = sign E_0. Under the case of equal rates of technical change, $\gamma_1 = \gamma_2 = \gamma$, we have sign $(\hat{y}_1 - \hat{y}_2)\big|_k = -$ sign E_0. But under this assumption $E_0 = \gamma(1 - \eta)$. In view of proposition 11, it is likely that for $\gamma_1 = \gamma_2$ the effect of technical change will be in the same direction as that of capital accumulation: $\hat{y}_2 \gtrless \hat{y}_1$ for $1 \lessgtr \eta$.

10. Rate of Change of Output Composition.

The rate of change of the share of sector 1 in total output is $\hat{\Pi}_1 = \left(\dfrac{\widehat{py_1}}{y}\right) = \Pi_2(\hat{y}_1 - \hat{y}_2 + \hat{p})$, which in view of the demand relation can be written as:

$$\hat{\Pi}_1 = \Pi_2\,[(\eta - 1)\hat{y}_2 + (1 - \sigma_D)\hat{p}]$$

From which we get:

(a) $\sigma_D = \eta = 1 \implies \hat{\Pi}_1 = 0$

(b) $\sigma_D = 1 \implies \hat{\Pi}_1 = \Pi_2(\eta - 1)\hat{y}_2$

(c) $\eta = 1 \implies \hat{\Pi}_1 = \Pi_2(1 - \sigma_D)\hat{p}$

(d) $\eta < 1,\ \sigma_D < 1,\ \hat{y}_2 > 0,\ \hat{p} < 0 \implies \hat{\Pi}_1 < 0$

(e) $\eta > 1,\ \sigma_D > 1,\ \hat{y}_2 > 0,\ \hat{p} < 0 \implies \hat{\Pi}_1 > 0$.

The first three parts are obvious. They are stated explicitly in view of the common use of such assumptions. Part e is in a way the inverse of part d. For when $\sigma_D > 1$ and $\eta > 1$ sector 2 becomes the one which lags behind, and consequently Π_1 increases. Part d is the one which is more pertinent for our identification of sector 1 with agriculture.

In order to derive conditions on sign $\hat{\Pi}_1$ in terms of the parameters of the model we combine (8.1) and (9.3) to get:

$$(10.1)\quad \hat{\Pi}_1 = -\frac{\Pi_2}{\ell\sigma}[(1-\eta)\bar{\sigma}_{\beta\ell} + (\sigma_D - \ell)I]\hat{k} - \frac{i}{\ell\sigma}(E_1 - E_2 + I)E_0$$

Response to capital accumulation.

Proposition 12. (Effect of capital accumulation on rate of change of output composition.[15]

$$(10.2)\quad \hat{\Pi}_1/\hat{k}\Big|_t \gtreqless 0 \Longleftrightarrow (1-\eta)\bar{\sigma}_{\beta\ell} \lesseqgtr (\ell - \sigma_D)I.$$

To get sufficient conditions we let $\underline{\sigma}_2 = \dfrac{\ell - \sigma_D}{\ell - \eta}$ and $\underline{\sigma}_1 = \dfrac{\ell - \sigma_D}{\ell - I}$, then

(a) Assume $\eta < 1$. Sufficient conditions for $\hat{\Pi}_1\big|_t < 0$ are:

(a.1) Given $\ell \geq \sigma_D$:(a.1.1) $\sigma_2 \geq \underline{\sigma}_2$, or (a.1.2) $I < 0$.

(a.2) Given $\sigma_D \leq \ell$: (a.2.1) $\sigma_1 \geq \underline{\sigma}_1$, or (a.2.2) $I > 0$.

(b) Assume $\eta > 1$. Sufficient conditions for $\hat{\Pi}_1\big|_t > 0$ are:

(b.1) Given $\ell \geq \sigma_D$: (b.1.1) $\sigma_1 \geq \underline{\sigma}_1$, or (b.1.2) $I > 0$.

(b.2) $\sigma_D \geq \ell$: (b.2.1) $\sigma_2 \geq \underline{\sigma}_2$, or (b.2.2) $I < 0$.

(c) $\eta = 1$. Sign $\hat{\Pi}_1\big|_t$ = sign $(1 - \sigma_D)\, I$.

It is interesting to note the symmetry between the sufficient conditions in parts a and b. The same conditions that lead to a decline in Π_1 under $\eta < 1$, result in an increase of Π_1 under $\eta > 1$. Thus, conditions that lead to contraction under low values of demand elasticities lead to expansion under large such values.

Response to technical change

Referring to (10.1) and recalling that $iE_1 > 0$, $iE_2 > 0$ and $iI > 0$, we state:

Proposition 13: (Effect of technical change on rate of change of output composition)

$$\text{Sign}\left[-\frac{i}{\ell\sigma}(E_1 - E_2 + I)E_0\right] = -\text{ sign } E_0.$$

Recalling the discussion of E_0 in section 5 we note that conditions leading to excess supply in sector 1 due to technical change will result in a decline in Π.

11. Capital Accumulation.

Finally, we turn to analyze the pattern of capital accumulation in our economy. At this stage we broaden the demand relation by adding (2.8) and the implied constraint (2.9). In this discussion we utilize the aggregate production function, and assume for convenience that γ_j are constants so that $\tau_j(t) = e^{\gamma_j t}$.

Constant technology

Let λ be the rate of depreciation, $\hat{L}$ be a constant exogeneous rate of growth of the labor force, then

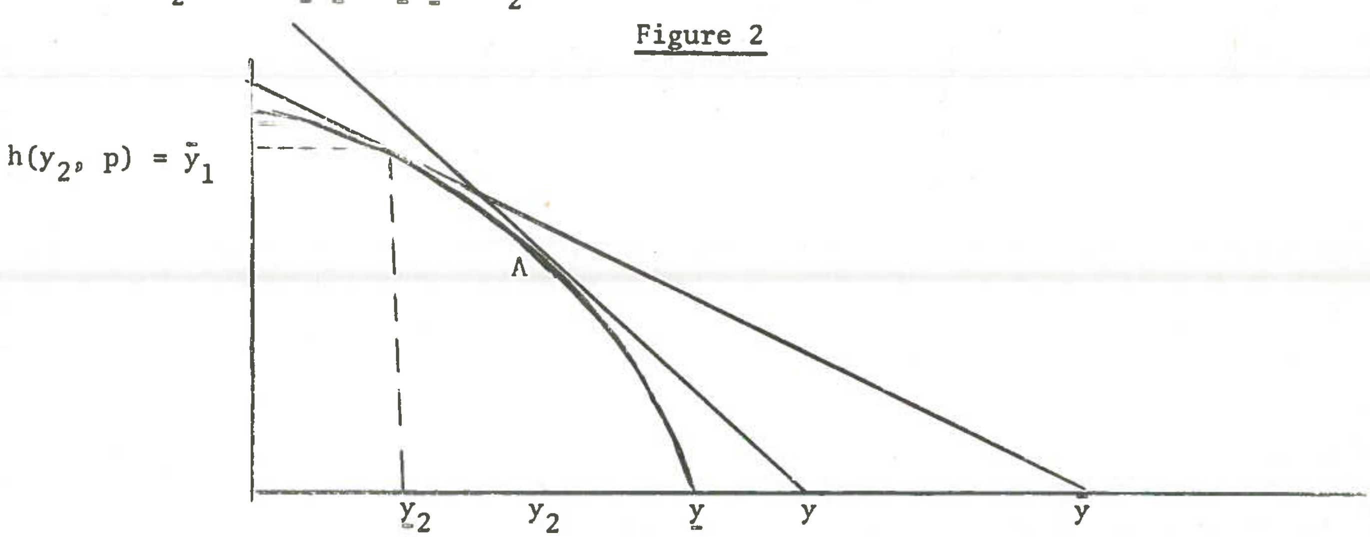
(11.1) $\hat{k} = \frac{s\,y(k)}{k} - n \qquad n = \lambda + \hat{L}$

Proposition 14: There exists a stable steady state.

Proof: we note that $y \geq \underline{y} = f_2(k)$ .This is illustrated in Figure 2.
So we have

(11.2) $sf_2(k) \equiv s\underline{y} \leq sy \leq \underline{y} \equiv f_2(k)$

Figure 2



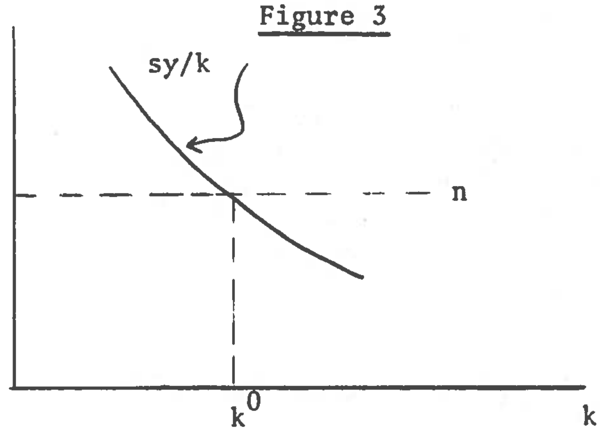
By the assumption on the production functions:

(11.3) $\lim\limits_{k\to\infty} \frac{f_2(k)}{k} = 0$, Hence $\lim\limits_{k\to\infty} \frac{sy}{k} = 0$

$\lim\limits_{k\to 0} \frac{f_2(k)}{k} = \infty$, Hence $\lim\limits_{k\to 0} \frac{sy}{k} = \infty$.

Combining the two cases we can draw Figure 3 which demonstrates existence of a stationary solution k^0, in whose neighborhood $\frac{sy}{k}$ is declining.

Figure 3



There is also a very singular possibility in which we cannot find a stable steady state in the strict sense. In this case, we find a steady state which is stable from one side and has a whole interval of steady states to its other side.

Proposition 15: A sufficient condition for the uniqueness of the steady state is that the marginal productivity of labor evaluated from the aggregate production function is positive everywhere. A local fulfillment of this condition is sufficient for the uniqueness of a steady state.

Proof: for uniqueness it is sufficient that $\frac{y}{k}$ will decline everywhere in k.

$y/k = Y/K$ is a capital productivity - its first derivative with respect to $1/k$ is the marginal productivity of labor in the same way as the marginal productivity of capital is the first derivative of the labor productivity with respect to k.

It is clear that once the existence of aggregate production function homogeneous of degree 1 is established the proof is simple. Nevertheless, the fulfillment of the condition is not as automatic as it may seem. The rate of change in y is obtained by combining (5.9) and (5.11):

$$(11.4) \quad \hat{y} = (\beta + \frac{\Pi_1 I}{\sigma})\hat{k} + \gamma_2 - \frac{iI}{\ell\sigma}\Pi_1 E_0$$

$$= B\hat{k} + \Gamma$$

Where B is the capital elasticity of the aggregate production function and Γ is the rate of technical change of that function $= \frac{\partial \ln y}{\partial t}(k, t)$

It then follows that

$$(11.5) \quad \left.\frac{\widehat{(y/k)}}{\hat{k}}\right|_t < 0 \iff \sigma > \frac{\Pi_1 I}{\alpha}$$

where $\alpha = 1 - \beta = \frac{WL}{y}$ is the labor share in total output.

Condition (11.5) is a generalization of that obtained by Jones in that it refers to a more general demand relation as is reflected in the definition of σ. This condition always holds when $I < 0$, that is, when the sector in which capital goods are produced is more labor intensive. It also holds in the extreme case when $\Pi_1 = 0$, which is the one sector model.

Technical change

Once we allow for technical change in the aggregate production function, a stationary capital - labor ratio will grow through time. With technical change we define a stationary $k^0(t)$ as a solution to (11.6).

$$(11.6) \quad sy(k^0, t) = n\, k^0$$

Differentiating (11.6) (Cf. Goldman 1, p. 34):

$$(11.7) \quad \hat{k}^0(t) = \frac{\partial \ln k^0(t)}{\partial t} = \frac{\dfrac{\partial \ln y(k^0(t), t)}{\partial t}}{1 - \dfrac{\partial \ln y}{\partial \ln k}(k^0(t), t)} = \frac{\Gamma\,(k^0(t), t)}{A(k^0(t), t)}$$

when $A = 1 - B$ is the labor elasticity in the aggregate production function.

In one sector model (11.7) can be written as:

$$(11.8) \quad \frac{d \ln k^0(t)}{dt} = \frac{\gamma}{\alpha\,(k^0(t), t)} = \frac{\gamma}{\alpha^0}$$

where γ is the technical change in the one sector model. In the special case of Cobb-Douglas production function, where α^0 is constant, the stationary capital labor ratio, $k^0(t)$ increases at the constant rate of $\frac{\gamma}{\alpha}$.

The similarity of (11.7) and (11.8) indicates the structural resemblance of the one and two sector models. However, (11.7) is a more complex expression. In particular (11.7) may be negative even though technical change in the two sectors is positive. This cannot happen in (11.8). More specifically, for (11.7) to be positive, Γ and A should have the same sign. It should be noted that a positive value for A is the sufficient condition for uniqueness of the stationary solution as was stated above. Given that A is positive, it still may be possible to have various combinations of the parameters in question that will result in a negative value for Γ and consequently, in a decline in the stationary capital labor ratio $k^0(t)$. However, as we shall see below, a decline in $k^0(t)$, if it occurs, is bounded and given $\gamma_2 > 0$, there will be a tendency for $k^0(t)$ to increase with t.

The rate of capital accumulation at an arbitrary point in time t is treated in <u>proposition 16</u>: (a) when $A > 0$, $\mathring{\hat{k}} \gtreqless 0 \Longleftrightarrow \dfrac{\Gamma\;(k(t), t)}{A\;(k(t), t)} \gtreqless \hat{k}$.

(b) when $A < 0$, $\mathring{\hat{k}} \gtreqless 0 \qquad \dfrac{\Gamma\;(k(t), t)}{A\;(k(t), t)} \lesseqgtr \hat{k}$; (c) when $A = 0$, $\mathring{\hat{k}} \gtreqless 0 \Longleftrightarrow \Gamma \gtreqless 0$.

Proof: Differentiating (11.1) :

(11.9) $\overset{\circ}{\hat{k}} = \frac{sy}{k}(\hat{y} - \hat{k})$

Combining (11.9) and (11.4) we get:

(11.10) $\overset{\circ}{\hat{k}} = \frac{sy}{k}(\Gamma \;\; - A\,\hat{k})$

which proves the proposition.

Note that $A > 0$ is the stability condition for stationary solution under constant technology ($\Gamma = 0$). Part b of the proposition indicates that $\hat{k}$ will run away from Γ/A. In view of (11.3) and proposition 14, A will eventually become positive.

In a very special case, the aggregate production function is identical up to a scalar to that of sector 2 - the sector producing capital goods. This is the case of two-sector growth model with fixed saving ratio and Cobb-Douglas production functions in the two sectors. In this case we have $\sigma_D = \eta = \sigma_1 = \sigma_2 = 1$ and hence $\sigma = 1$. From proposition 5, $E_0 = 0$. Since, by definition $\beta = \beta_1\Pi_1 + \beta_2\Pi_2$ we have for (11.4):

$$\hat{y} = \beta_2\hat{k} + \gamma_2 \;.$$

In this particular case there is a convergence to exponential growth with a constant rate of accumulation.

In the general case, the asymptotic properties of the accumulation process are dominanted by the production function and the technical change in Sector 2, the sector which produces capital goods. To see this we rewrite (11.2).

(11.11) $f_2(k, t) = \underline{y}(k, t) \geq sy(k, t) \geq s\underline{y}(k, t) = sf_2(k, t)$

Let there be two economies (A, B) whose production function is f_2 and their saving ratios are $s_A = 1$, $s_B = s$ (the same as in our economy). The rates of capital accumulation in these economies are given by:

(11.12) $\dot{k}^A = f_2(k^A, t) - n\, k^A$

(11.13) $\dot{k}^B = s\, f_2(k^B, t) - n\, k^B$.

If at any initial point in time t_0, we have

(11.14) $k^A \geq k \geq k^B$

where k is the overall capital labor ratio in our model, (11.14) will be maintained for any $t > t_0$. This is a result of combining (11.11), (11.12) and (11.13). (11.13) tells us that k^B will grow indefinitely through time and so will k. This results requires no more than $\gamma_2 > 0$. It tells us further that the long run rates of accumulation are dictated by the production function and technical change in the capital producing sector alone.

APPENDIX

(1) Displacement of Labor Allocation

The conditions of full employment are rewritten as:

(A.1.1) $K = K_1 + K_2$ $\qquad\qquad$ $L = L_1 + L_2$

Differentiating:

(A.1.2) $\hat{K} = \rho_1 \hat{K}_1 + \rho_2 \hat{K}_2$

(A.1.3) $\hat{L} = \ell_1 \hat{L}_1 + \ell_2 \hat{L}_2$

Subtracting (A.1.3) from (A.1.2), using $\hat{k}_j = \sigma_j \hat{\omega}$ and

(A.1.4) $\ell_1 \hat{\ell}_1 + \ell_2 \hat{\ell}_2 = 0$

We obtain (5.1) and (5.2)

(2) Displacement of Capital Allocation

(A.2.1) $\rho_1 = \dfrac{K_1}{K} = \dfrac{k_1 \ell_1}{k}$, hence

(A.2.2) $\hat{\rho}_1 = \hat{k}_1 + \hat{\ell}_1 - \hat{k}$

Using (5.1) :

(A.2.3) $\hat{\rho}_1 = \left(\sigma_1 + \dfrac{\ell_2}{i} \bar{\sigma}_\rho \right) \hat{\omega} - \left(1 + \dfrac{\ell_2}{i}\right) \hat{k}$ $\qquad\qquad$ $i \neq 0$

Some further simplification results in (5.3) and a similar development leads to (5.4).

(3) Displacement of Supply

Total differentiation of the production functions gives:

(A.3.1) $\hat{y}_1 = \beta_1 \hat{k}_1 + \hat{\ell}_1 + \gamma_1$

(A.3.2) $\hat{y}_2 = \beta_2 \hat{k}_2 + \hat{\ell}_2 + \gamma_2$

Using (5.1) and (5.2) and rearrangement of terms tead to (5.5) and (5.6).

(4) Displacement of Price

Since the production functions are homogeneous of first degree, we can write:

(A.4.1) $Y_1 = F_{1L}L_1 + F_{1K}K_1$

Differentiation of (A.4.1) gives

(A.4.2) $\hat{Y}_1 - \alpha_1\hat{L}_1 - (1 - \alpha_1)\hat{K}_1 = \alpha_1\hat{F}_{1L} + (1 - \alpha_1)\hat{F}_{1K} = \gamma_1$

rewriting

(A.4.3) $\alpha_1(\hat{F}_{1L} - \hat{F}_{1K}) + \hat{F}_{1K} - \gamma_1 = 0$

Similarly for Sector 2

(A.4.4) $\alpha_2(\hat{F}_{2L} - \hat{F}_{2K}) + \hat{F}_{2K} - \gamma_2 = 0$

Writing the competitive conditions:

(A.4.5) $w = pF_{1L} = F_{2L}$

(A.4.6) $r = pF_{1K} = F_{2K}$

Differentiation of (A.4.5) and (A.4.6) gives:

(A.4.7) $\hat{w} = \hat{p} + \hat{F}_{1L} = \hat{F}_{2L}$

(A.4.8) $\hat{r} = \hat{p} + \hat{F}_{1K} = \hat{F}_{2K}$

combining (A.4.3) - (A.4.8) results in (5.8).

(5) Displacement of Per Capita Product

(A.5.1) $\hat{y} = \Pi_1(\hat{py}_1) + \Pi_2\hat{y}_2$

Using (5.5), (5.6) and (5.8):

(A.5.2) $\hat{y} = [\pi_1(E_1 + I) + \pi_2E_2]\ \hat{\omega} + \frac{1}{i}[-\pi_1\ell_2 + \pi_2\ell_1]\hat{k} + \pi_1(\gamma_1 - \gamma_1 + \gamma_2) + \pi_2\gamma_2$

$= a\hat{\omega} + b\hat{k} + \gamma_2 \qquad\qquad i \neq 0$

First evaluate b : $(i \neq 0)$

$$b = \frac{1}{i} \; [\ell_1(1 - \pi_1) - (1 - \ell_1)\pi_1] = \frac{\ell_1 - \pi_1}{i}$$

but

$$\Pi_1 = \frac{pY_1}{Y} = \frac{1}{Y} [rK_1 + wL_1] = \beta\rho_1 + \alpha\ell_1$$

$$= \beta\rho_1 + (1 - \beta) \; \ell_1 = \ell_1 - i\beta$$

Hence

(A.5.3) $$b = \frac{1}{i} [\ell_1 - \ell_1 + i\beta \;] = \beta$$

To evaluate a, we use the definitions of E_j given in the text following (5.6):

$$a = \pi_1 \; [\frac{\ell_2}{i} \; \bar{\sigma}_\rho \; + \beta_1\sigma_1 \;] + \pi_2 \; [- \; \frac{\ell_1}{i} \; \bar{\sigma}_\rho \; + \beta_2\sigma_2 \;] + \pi_1 I$$

$$a = \frac{\bar{\sigma}_\rho}{i} \; (\pi_1\ell_2 - \pi_2\ell_1) + [\pi_1\beta_1\sigma_1 + \pi_2\beta_2\sigma_2] + \pi_1 I$$

$$a = a_1 + a_2 + \pi_1 I$$

But

$$a_1 = -b\bar{\sigma}_\rho \; = - \beta\bar{\sigma}_\rho$$

Write out a_2 as:

$$a_2 = \frac{pY_1}{Y} \frac{rK_1}{pY_1} \; \sigma_1 + \frac{Y_2}{Y} \frac{rK_2}{Y_2} \; \sigma_2 = \beta(\rho_1\sigma_1 + \rho_2\sigma_2) = \beta\bar{\sigma}_\rho \; = -a_1$$

Hence,

(A.5.4) $$a = \pi_1 I$$

combining (A.5.3) - (A.5.4)

$$\hat{y} = \pi_1 I\hat{\omega} \; + \beta\hat{k} + \gamma_2$$

which appears as (5.9) in the text.

(6) The Case of Equal Factor Intensities

Our algebra is full of divisions by i. Those partial differentials in which i appears in the denominator are meaningless in the case when $i = 0$, the case of equal factor intensities. It so happens that in all the final (= total) differentials i disappears from the denominator. These expressions are algebraically meaningful for $i = 0$, but the algebra leading to them is not clean from division by zero and thus meaningless.

Taking for granted the existence of the differentials at these points, we are going to prove that the expressions we have got are valid for $i = 0$ despite the flow in the algebra leading to them. One may guess that this is true (or may even prove it in a more general way than we do) from continuity considerations.

The present algebraic difficulty does not lack its economic meaning. Our procedure was first to differentiate the production system alone. In this differentiation, we have regarded the variables of the production system as functions of three arguments - k, w and t. But those one valued functions do not exist for values of k which correspond to equal factor intensities. The algebraic tool was very faithful in telling us that there is something wrong with our procedure.

Our ultimate total differentials correspond to the system as a whole, including the demand. The variables of this system are functions of only two variables, k and t. These one valued functions do exist (and are assumed differentiable) for every k (and t), including those (if any or however many) which give $i = 0$.

To show all these, it suffices to show the route of the development. We start with (A.3.1), (A.3.2) (5.8), (5.10) and (A.1.4) and obtain:

$$\text{(A.6.1)} \quad \hat{\ell}_1 = -\frac{\ell_2}{\ell}[D\hat{\omega} + E_0]$$

$$\text{(A.6.2)} \quad \hat{\ell}_2 = \frac{\ell_1}{\ell}[D\hat{\omega} + E_0] \ .$$

Similar to (A.1.4) we have $\rho_1\hat{\rho}_1 + \rho_2\hat{\rho}_2 = 0$, which together with (A.6.1), (A.6.2), (A.2.1) and (A.2.2) yields:

$$\text{(A.6.3)}\quad (\bar{\sigma}_\rho + \frac{i}{\ell} D)\hat{\omega} - \hat{k} + \frac{i}{\ell} E_0 = 0$$

which in view of (5.15) is the same as (5.11). Thus, under the special case of $i = 0$ we have

$$\text{(A.6.4)}\quad \hat{\omega}\Big|_{i=0} = \frac{1}{\bar{\sigma}_\rho} \hat{k}$$

Combining (A.6.4) and (A.6.1) :

$$\text{(A.6.5)}\quad \hat{\ell}_1\Big|_{i=0} = \frac{-\ell_2}{\ell} [\frac{D}{\bar{\sigma}_\rho} \hat{k} + E_0]$$

(A.6.5) is identical to (6.1) under the constraint $i = 0$ and could be obtained from it directely.

Using $\hat{k}_j = \sigma_j\hat{\omega}$, (A.6.4), (A.6.5) in (A.3.1) and noting that $i = 0 \implies \beta_1 = \beta_2 = \beta$ yields

$$\text{(A.6.6)}\quad \hat{y}_1 = \frac{\beta n}{\ell} \hat{k} + \gamma_1 - \frac{\ell_2}{\ell} E_0$$

A similar procedure yields:

$$\text{(A.6.7)}\quad \hat{y}_2 = \frac{\beta}{\ell} \hat{k} + \gamma_2 + \frac{\ell_1}{\ell} E_0$$

(A.6.6) and (A.6.7) could be obtained directely from (9.1) and (9.2) respectively by imposing $i = I = 0$ after first multiplying $iE_1 = \ell_2\bar{\sigma}_\rho$ and $iE_2 = -\ell_1\bar{\sigma}_\rho$ (see definitions of E_j in Section 5 in the text).

Finally to obtain $\hat{y}\Big|_{i=0}$, we use (A.5.1), (A.6.6), (A.6.7) and (5.8) and note that under $i = 0$

$pY_1 = rK_1 + WL_1 = \rho_1 rK + \ell_1 WL = \rho_1 Y = \ell_1 Y$ and therefore $\Pi_1 = \ell_1 = \rho_1$ and hence

$$\text{(A.6.8)}\quad \hat{y} = \beta\hat{k} + \gamma_2$$

Again (A.6.8) is obtainable from (11.4) under $i = I = 0$.

(7) Evaluation of $\hat{\rho}_1$ and Related Developments.

Combining (5.3) and (5.4) we get $(i \neq 0)$:

(A.7.1) $\hat{\rho}_1 = (\frac{\bar{\sigma}_\ell}{\sigma} - 1) \frac{\rho_2}{i} \hat{k} - \frac{\rho_2 \sigma_\ell}{\ell \sigma} E_0$.

Note that $\bar{\sigma}_\ell - \bar{\sigma}_\rho = i(\sigma_1 - \sigma_2)$

Hence, using (5.15)

$$\frac{\rho_2}{i} (\frac{\bar{\sigma}_\ell}{\sigma} - 1) = \frac{\rho_2}{i\sigma} [\bar{\sigma}_\ell - \bar{\sigma}_\rho - \frac{i}{\ell} D] = \frac{\rho_2}{\sigma\ell} [\ell(\sigma_1 - \sigma_2) - D]$$

as given in (7.1).

Writing D explicitly,

(A.7.2) $\ell(\sigma_1 - \sigma_2) - D = \sigma_1(\ell - \beta_1) - \sigma_2(\ell - \eta\beta_2) - \sigma_D(\beta_2 - \beta_1)$.

This is used in (7.3).

We now state some sufficient conditions related to Proposition 8.

Sufficient conditions for $\hat{\rho}_1/\hat{k}\Big|_t < 0$:

(A.7.3) $\beta_2 > \beta_1 \geq \ell \geq \eta\beta_2$

(A.7.4) $\sigma_2 \geq \sigma_1 \geq \sigma_D \geq \eta\sigma_2$

(A.7.5) $\beta_2 > \beta_1 \geq \eta\beta_2 \leq \ell$ and $\sigma_2 \geq \sigma_1$

(A.7.6) $\sigma_1 = \sigma_2 > \sigma_D$ and $\frac{\beta_1}{\beta_2} > \frac{\eta\sigma_1 - \sigma_D}{\sigma_1 - \sigma_D}$

This is fulfilled regardless of β_j if $\eta\sigma_1 < \sigma_D < \sigma_1 = \sigma_2$

Note that (A.7.3) - (A.7.5) $\Longrightarrow \eta < 1$.

Sufficient conditions for $\hat{\rho}_1/\hat{k}\Big|_t > 0$:

(A.7.3) and (A.7.4) in reverse orderings.

(A.7.5)* $\eta\beta_2 \geq \beta_1 > \beta_2$; $\ell > \beta_1$ and $\sigma_1 \geq \sigma_2$

(A.7.6)* $\sigma_1 = \sigma_2$, $\eta\sigma_1 > \sigma_D$ and $\dfrac{\beta_2}{\beta_1} > \dfrac{\sigma_1 - \sigma_D}{\eta\sigma_1 - \sigma_D}$

The first three conditions imply $\eta > 1$.

Finally, $\sigma_1 = \sigma_2 = \eta = \sigma_D \Longrightarrow \hat{\rho}_1/\hat{k}\Big|_t = 0$.

Next we note how (A.7.2) is simplified by assuming $\eta = 1$.

(A.7.2) (under $\eta = 1$) $= \alpha_1\sigma_1 + \beta_1\sigma_D - (\alpha_2\sigma_2 + \beta_2\sigma_D)$

Hence $\hat{\rho}_1/\hat{k}\Big|_t \gtreqqless 0 \Longleftrightarrow \alpha_1\sigma_1 + \beta_1\sigma_D \gtreqqless \alpha_2\sigma_2 + \beta_2\sigma_D$.

The discussion can also be applied to evaluate:

(A.7.7) $\hat{k}_1 - \hat{k} = \hat{\rho}_1 - \hat{\ell}_1$

$$= \sigma_1\hat{\omega} - \hat{k} = (\frac{\sigma_1}{\sigma} - 1)\,\hat{k} - \frac{i}{\ell}\,\frac{\sigma_1}{\sigma}\,E_0$$

$$= \frac{1}{\ell\sigma}\{\,[(\sigma_1 - \sigma_2)\,\rho_2\ell - iD]\hat{k} - i\sigma_1 E_0\,\}$$

We donot specify sufficient conditions except for noting that given $\sigma_1 = \sigma_2$, sign $\dfrac{\hat{k}_1 - \hat{k}}{\hat{k}} = -$ sign $i\,(D\,\hat{k} + \sigma_1 E_0)$ and for $D > 0$, $E_0 > 0$, and agriculture being capital intensive we have $\hat{k}_1 > \hat{k}$ or equivalently $\hat{\rho}_1 > \hat{\ell}_1$, with the opposite for agriculture labor intensive.

(8) Rates of Change of Outputs

(a) Combining (5.5) and (5.11) :

(A.8.1) $\hat{y}_1 = (\dfrac{-\ell_2}{i} + \dfrac{E_1}{\sigma})\,\hat{k} + \gamma_1 - \dfrac{iE_1}{\ell\sigma}\,E_0$

Using the definition of E_1 .

$$(.) \quad = \frac{E_1}{\sigma} - \frac{\ell_2}{i} = \frac{1}{i\sigma}(\ell_2\sigma_\rho + i\beta_1\sigma_1 - \sigma\ell_2)$$

Using (5.15) and simplifying

$$(.) \quad = \frac{1}{\sigma}(\beta_1\sigma_1 - \frac{\ell_2}{\ell} D)$$

Writing D explicitly and simplifying we get expression (9.1) in the text. A similar development follows for y_2 to yield (9.2).

b. The condition that appears in (9.4) can be also written as:

$$(A.8.2) \quad \left.\frac{\hat{y}_1 - \hat{y}_2}{\hat{k}}\right|_t \gtreqless 0 \iff \ell_2\beta_2\sigma_2 + \ell_1\beta_1\sigma_1 + \beta_2\sigma_D \lesseqgtr \eta(\ell_2\beta_2\sigma_2 + \ell_1\beta_1\sigma_1) + \beta_1\sigma_D .$$

c. The relationship between the rates of output changes and those of factor utilization can be given as follows:

$$(A.8.3) \quad \hat{y}_1 - \hat{y}_2 = \beta_1\hat{\rho}_1 - \beta_2\hat{\rho}_2 + \alpha_1\hat{\ell}_1 - \alpha_2\hat{\ell}_2 - I\hat{k} + T$$

Applying (A.1.4) and a similar expression for $\hat{\rho}_1$ and simplifying:

$$(A.8.4) \quad \hat{y}_1 - \hat{y}_2 = [\frac{\bar{\rho}}{\rho_2}\hat{\rho}_1 + \frac{\bar{\ell}}{\ell_2}\ell_1] - I\hat{k} + T$$

where $\bar{\rho}_\beta = \beta_1\rho_1 + \beta_2\rho_2$ and $\bar{\ell}_\alpha = \alpha_1\ell_1 + \alpha_2\ell_2$

Given $\left.\hat{\rho}_1\right|_t < 0$, $\left.\hat{\ell}_1\right|_t < 0$ and $I > 0$ are sufficient for $\left.\frac{\hat{y}_1 - \hat{y}_2}{\hat{k}}\right|_t < 0$. This is a similar, but not identical, to (9.4) given $I > 0$ and $\eta < 1$.

(9) Supplements to Proposition 12

Referring to (10.1) in the text we write $\left(\left.\frac{\hat{\Pi}_1}{\hat{k}}\right|_t\right)\frac{\ell\sigma}{\Pi_2} = \varepsilon_1$.

Writing out $\bar{\sigma}_{\beta\ell}$ in (10.1) explicitly and rearranging terms:

$$(A.9.1) \quad \varepsilon_1 = \beta_1[(\ell - 1)\sigma_1 - (\ell - \sigma_D)] - \beta_2[(\ell - \eta)\sigma_2 - (\ell - \sigma_D)]$$

$$= \beta_1F_1 - \beta_2F_2$$

note that

$$(A.9.2)\quad \frac{F_1}{F_2} = \frac{1 - \frac{1 - \ell}{\sigma_D - \ell}\sigma_1}{1 + \frac{\ell - \eta}{\sigma_D - \ell}\sigma_2}$$

when $\eta = 1 \Longrightarrow \ell = 1$ and $\varepsilon_1 = (1 - \sigma_D)$.

Thus, $\eta = 1$ is sufficient to the make sign ε_1 independent of the elasticities of substitution σ_1 and σ_2. It is clear that $\eta = 1$ simplifies matters considerably. Another simplification is obtained when $\sigma_1 = \sigma_2 = 1$ in which case

$\varepsilon_1 = \beta_1(\sigma_D - 1) - \beta_2(\sigma_D - \eta)$ which yields:

$$\eta \leq \sigma_D < 1 \Longrightarrow \varepsilon_1 < 0, \quad 1 \leq \sigma_D < \eta \Longrightarrow \varepsilon_1 > 0 .$$

Thus Cobb-Douglass production functions and low demand elasticities are sufficient for Π_1 to decline as a result of capital accumulation. We now turn to obtain those conditions in proposition 12 which are stated in terms of lower bounds on the elasticities of substitution. Those stated in terms of I immediately follow from (9.2).

(a) $\eta < 1 \Longrightarrow \eta < \ell < 1 \Longrightarrow F_1 < F_2$.

(a.1) $\ell \geq \sigma_D \Longrightarrow F_1 < 0$. If we have $F_2 \geq 0$ then $\varepsilon_1 < 0$. This is obtained under

$$F_2 \geq 0 \Longleftrightarrow \sigma_2 \geq \underline{\sigma}_2 = \frac{\ell - \sigma_D}{\ell - \eta} \lesseqgtr 1 \Longleftrightarrow \sigma_D \gtreqless \eta.$$

(a.2) $\sigma_D > \ell \Longrightarrow F_2 > 0$. Then $F_1 \leq 0 \Longrightarrow \varepsilon_1 < 0$.

$$F_1 \leq 0 \Longrightarrow \sigma_1 \geq \underline{\sigma}_1 = \frac{\sigma_D - \ell}{1 - \ell} \lesseqgtr 1 \Longleftrightarrow \sigma_D \gtreqless 1 .$$

(b) $\eta > 1 \Longrightarrow \eta > \ell > 1 \Longrightarrow F_1 > F_2$.

(b.1) $\ell \geq \sigma_D \Longrightarrow F_2 < 0$. Then $F_1 \geq 0 \Longrightarrow \varepsilon_1 > 0$.

$F_1 \geq 0 \Longleftrightarrow \sigma_1 \geq \underline{\sigma}_1$

(b.2) $\sigma_D > \ell \Longrightarrow F_1 > 0$. Then $F_2 \leq 0 \Longrightarrow \varepsilon_1 > 0$

$F_2 \leq 0 \Longleftrightarrow \sigma_2 > \underline{\sigma}_2$.

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FOOTNOTES

1/ This research has been financed in part by a Ford Foundation Faculty Research Fellowship at the University of Chicago and by a grant made by the United States Department of Agriculture under P.L. 480.

2/ This adds some global restrictions on the size of the elasticities which cannot be uniformally high or low for every $y_2 > 0 \quad p > 0$. These restrictions are not so stringent if we are interested in local properties or in the growth starting from an initial finite point.

3/ Formally, we should say holding t constant. However, in general, t stands here for technology and we therefore use t and technology interchangeably wherever ambiguity does not result.

4/ See A.1 - Appendix Section 1.

5/ See A.2.

6/ See A.3.

7/ To show that $iE_1 > 0$ write $iE_1 = \ell_2\bar{\sigma}_\rho + i\beta_1\sigma_1$.
if $i > 0$ then we have a sum of positive terms. So suppose $i < 0$, multiply out and expand terms to get:
$iE_1 = -i\alpha_1\sigma_1 + \ell_1\ell_2\sigma_1 + \ell_2\rho_2\sigma_2$ which is again a sum of positive terms. A similar procedure leads to $iE_2 < 0$.

8/ See A.4.

9/ By $\hat{p}/\hat{\omega}\Big|_t$ we mean $\frac{d \ \ell n \ p}{dt} \Big/ \frac{d \ \ell n \ \omega}{dt}\Big|$ technology constant.
This notation is used throughout.

10/ See A.5.

11/ Define $\tilde{\sigma} = \dfrac{d \ \ell n \ k}{d\ell n(\frac{\partial Y}{\partial L} / \frac{\partial Y}{\partial K})}$
but we show later that in general $\frac{\partial Y}{\partial L} / \frac{\partial Y}{\partial K} \neq \omega = \frac{\partial Y_j}{\partial L_j} / \frac{\partial Y_j}{\partial K_j}$.

12/ The derivatives here mean a formal differentiation of the rational function D.

13/ See A.7.

14/ See A.8 for details.

15/ See A.9 for details.

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