Vertical Price Transmission in the US Pork Industry: Evidence from Copula Models

Christos Emmanouilides and Panos Fousekis
Aristotle University of Thessaloniki, Department of Economics, Thessaloniki, Greece
(cemman@econ.auth.gr & fousekis@econ.auth.gr)

Abstract
This paper investigates vertical price transmission in the US pork industry using the statistical tool of copulas and monthly data from 1970 to 2012. The empirical results indicate that the degree and the structure of price dependence differs across markets and time periods. In the first half of the sample, there was a relatively high degree of co-movement with symmetric tail dependence for the pair of markets farm-wholesale and asymmetric for the pair wholesale-retail. In the second half of the sample, tail dependence disappeared for both markets pairs and the association between price changes at the wholesale and the retail became very weak.

Keywords: Price Dependence, Copula Families, US Pork Industry
JEL Classification: Q13, C01

1. Introduction

The analysis of vertical price linkages has been an important topic in agricultural and food economics for a long period of time. Price inertia and incomplete pass-through along a food supply chain are indications of market inefficiency, and as such cause the concern of economists and policy makers. In well functioning (integrated) markets price shocks in any market level are transmitted to other market levels; primary producers benefit from price increases at the wholesale and the retail levels and final consumers benefit from cost reductions upstream.

Empirical investigation of vertical price transmission has been conducted with a variety of quantitative tools, ranging from simple correlation analysis and regression models to recently developed econometric approaches such as the linear Error Correction Model (ECM), the asymmetric/non-linear ECM, the Threshold Vector ECM, the Smooth Transition Cointegration Model, and the Markov-Switching ECM. Most of the researchers have focused their attention to potential asymmetries in the speed of price transmission (e.g. Goodwin and Holt, 1999; Goodwin and Harper, 2000; Ben-Kaabia and Gil, 2007). Fewer studies allowed for asymmetries in both the speed and the magnitude of transmission (e.g. Lass, 2005; and Gervais, 2011). The findings appear to depend on the methods employed, the time period considered, and the type of data used; nevertheless, the majority of earlier empirical works has obtained some evidence of asymmetric price transmission regarding either its speed or magnitude or even both its
speed and magnitude. Typically, price increases at the primary (farm) level have been found to be transmitted to the wholesale and the retail levels faster and/or more fully than price decreases. Such patterns of price transmission are consistent with the presence of market power.

In this context, the objective of the present work is to investigate vertical price transmission in the US pork industry using monthly prices at three market levels (farm, wholesale, and retail) over the period 1970:1 to 2012:12 and copulas, a statistical tool that offers an alternative and a very flexible way to analyze price dependencies /co-movements, particularly during extreme market events (upturns and downturns). Copulas have been extensively applied in engineering, insurance, risk management, and finance since the late 1990s, but only recently have found their way into applied and agricultural economics. As noted by Reboredo (2011), the rationale behind using copulas to analyze price linkages is that in well functioning markets, prices at different levels or at different locations move together, i.e. they boom and they crash together. An empirical finding that prices are linked with different intensities during extreme market upswings and downswings would be an indication of price transmission asymmetry.

A pork supply chain connects the primary, the pig meat processing, and the pig meat distribution sectors. Meat supply chains are complex and heterogeneous including a large variety of commodities, firms, and markets. The high concentration levels, especially at the meat processing and the meat distribution, are often a cause of concern for public bodies and for anti-trust authorities. Asymmetry in price transmission from one level of a chain to another implies that the distribution of benefits among the participants (stakeholders) of that chain is not fair; for example, final consumers do not benefit from cost reductions upstream or primary producers do not gain from demand rises at the retail level. In this respect, meat supply chains, in general, and the pork supply chain, in particular present interesting cases for an empirical analysis of price transmission.

Thus far, it appears that there have been only two published copula-based empirical works on price transmission. Reboredo (2011) studied four regional crude oil markets. He found symmetric upper and lower tail dependence (i.e. co-movement during extreme market upswings and downswings) between crude oil benchmark price returns, something which supports the hypothesis that crude oil markets constitute one great pool. Serra and Gil (2012) investigated co-movement between biodiesel, diesel, and crude oil prices in Spain. Their results indicated symmetric tail dependence for the crude oil-diesel pair but only lower tail co-movement for the crude oil-biodiesel pair.

The structure of the present work is as follows: Section 2 presents the analytical framework, that is modeling price co-movements via copulas and rank-based dependence measures. Section 3 presents the data, the empirical models, and the empirical results. Section 4 offers conclusions and suggestions for future research.

2. Analytical Framework

2.1 The Copula Approach to Modeling Dependence

The use of copulas to represent flexible dependence structures has been based on Sklar’s (1959) theorem according to which a multivariate distribution of a vector of random variables is completely specified by the individual marginal densities and a
joining function known as \textit{copula}. In the simple bivariate case, let the joint cumulative density (cdf) function of a pair of continuous random variables \((X_1, X_2)\) be \(F(x_1, x_2)\) and the marginal cdfs be \(F_1(x_1)\) and \(F_2(x_2)\), respectively. Then, Sklar’s theorem suggests that

\[
F(x_1, x_2) = C[F_1(x_1), F_2(x_2)]
\]

where \(C\) is the copula function. Provided that the marginal distributions are continuous, \(C, F_1,\) and \(F_2\) are uniquely determined by \(F(x_1, x_2)\). Conversely, a valid joint cdf for \((X_1, X_2)\) arises from relation (1) when \(C, F_1,\) and \(F_2\) are chosen from given parametric families of distributions.

The copula is a bivariate cdf with uniform marginal distributions, \(C:[0,1]^2 \rightarrow [0,1]\), and it can be obtained from (1) as

\[
C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2))
\]

where \(F_i^{-1} (i = 1, 2)\) are marginal quantile functions and \(u_i\) are quantiles on \(U[0,1]\). The joint density function (pdf) of the pair of random variables may be written as

\[
f(x_1, x_2) = c(F_1(x_1), F_2(x_2))f_1(x_1)f_2(x_2)
\]

where \(c\) is the joint pdf associated with \(C\). A joint pdf contains information for both the marginal behavior of each variable and the dependence between them. In the term \(c(F_1(x_1), F_2(x_2))\), each random variable is fed into its own cdf. In this way, all information contained in the marginal distributions is swept away and what is left is the pure joint information between \(X_1\) and \(X_2\). Thus, the copula captures the information missing from the individual marginal pdfs to complete the joint pdf (Meucci, 2011).

Using copulas to model and to analyze co-movement between random variables offers a number of important advantages: (a) Just as marginal distributions provide an exhaustive description of the behavior of two random variables when considered separately, copulas completely and uniquely characterize the dependence structure between them. (b) Copulas model co-movement independently of the marginal distributions. (c) Copulas examine general functional dependence between \(X_1\) and \(X_2\) and provide information about its intensity. In contrast, standard non rank-based measures such as Pearson’s correlation coefficient assess only linear dependence, which is only a special case of functional dependence. (d) Since copulas are based on the ranks of \(X_1\) and \(X_2\), they are invariant to continuous and monotonically increasing transformations of them.

2.1 Bivariate Copula Families and Dependence Structures

In the relevant literature there is a large number of bivariate parametric copula functions that are used to study a multitude of possible dependence structures. Next, the paper discusses only functions that are often employed in finance, risk management, and economics (e.g. Embrechts, et al., 2002; Patton, 2006; Reboredo, 2011; Serra and Gil, 2012; Czado et al., 2012). These functions we also use in the present study.

Two important members in the family of the so called \textit{elliptical} copulas are the
Gaussian and the \( t \)-copula. The Gaussian contains a single dependence parameter, \( \rho \) (the linear correlation coefficient corresponding to the bivariate normal distribution). The \( t \)-copula contains two parameters, namely, the linear correlation coefficient and the degrees of freedom (denoted as \( \nu \)). When \( \nu \geq 30 \) the \( t \)-copula becomes a Gaussian one. The Archimedean copulas are developed from a number of different generator functions. From the Archimedean copulas employed in this work, the Clayton, the Gumbel and the Frank copulas contain a single dependence parameter (denoted as \( \theta \)) while the Gumbel-Clayton and the Joe-Clayton involve two dependence parameters (denoted as \( \theta_1 \) and \( \theta_2 \)).

Given that comparison of dependence parameters across copula families is (in most cases) without much meaning, the Kendall’s \( \tau \) is typically employed to measure functional dependence. It is calculated as

\[
\tau = 4 \int \int C(u_1, u_2) dC(u_1, u_2) - 1
\]

and it provides information on co-movement across the entire joint distribution function (both at the center, as well as at the tails of it). Often times, however, information concerning dependence at the tails (at the lowest and the highest ranks) is extremely useful for economists, managers, and policy makers. Tail (extreme) co-movement is measured by the upper, \( \lambda_u \), and the lower, \( \lambda_l \), dependence coefficients defined as

\[
\lambda_u = \lim_{u \to 1} \text{prob}(U_1 > u | U_2 > u) = \lim_{u \to 1} \frac{1 - 2u + C(u, u)}{1 - u} \in [0,1]
\]

and

\[
\lambda_l = \lim_{u \to 0} \text{prob}(U_1 < u | U_2 < u) = \lim_{u \to 0} \frac{C(u, u)}{u} \in [0,1]
\]

\( \lambda_u \) measures the probability that \( X_1 \) is above a high quantile, given that \( X_2 \) is also above that high quantile, while \( \lambda_l \) measures the probability that \( X_1 \) is below a low quantile given that \( X_2 \) is also below that low quantile. In other words, the two measures of tail dependence provide information about the likelihood for the two random variables to boom and to crash together, respectively. Note that since \( \lambda_u \) and \( \lambda_l \) in (5) and (6) are expressed via copula, certain properties of copulas (e.g. invariance to monotonically increasing transformations of the underlying random variables) apply to tail coefficients as well.

The Gaussian copula is symmetric and exhibits zero tail dependence. The \( t \)-copula exhibits symmetric non-zero tail dependence (joint booms and crashes have the same probability of occurrence). From the Archimedean copulas, the Clayton copula exhibits only left co-movement (lower tail dependence); the Gumbel exhibits only right co-movement (upper tail dependence); the Frank copula exhibits zero tail dependence; the Gumbel-Clayton and Joe-Clayton copulas allow for (potentially asymmetric) both right and left co-movement. The Appendix presents the way the parameters of different copula families are related to Kendall’s \( \tau \) and to tail dependence coefficients.
3. The Data, the Empirical Models, and the Empirical Results

3.1. The Data and the Empirical Models

As mentioned in the Introduction, the data for the empirical analysis are monthly prices at the farm, the wholesale, and at the retail levels of the pork industry in the US. They have been obtained from the Economic Research Service of the United States Department of Agriculture (ERS-USDA). Let \( p_n^a \) and \( p_{n+1}^a \) be the pig meat prices at two consecutive market levels (the farm and the wholesale or the wholesale and the retail one) at time \( n \). Let also that these two prices be linked by a smooth function \( \Psi \), that is,

\[
    p_{n+1}^a = \Psi_a(p_n^a) \tag{7}
\]

From (7), after a simple manipulation, it follows

\[
    \frac{dp_{n+1}^a}{p_{n+1}} = \Psi_a(p_n^a) \left( \frac{dp_n^a}{p_n} \right) \frac{dp_n^a}{p_{n-1}} \tag{8}
\]

linking the rates of change in the prices at the two market levels in a potentially non-linear way which depends on the market conditions (reflected in prices). In the present study we investigate the degree and the structure of dependence between \( dp_{n+1}^a / p_{n+1} \) and \( dp_n^a / p_n \) using copulas.

For the investigation we employ the semi-parametric approach. This involves the non-parametric estimation of margins from their empirical counterparts in a first stage and the parametric estimation of a copula model in a second stage with Maximum Likelihood. The relevant estimator is called a Canonical Maximum Likelihood or a Pseudo Maximum Likelihood one. Its consistency and asymptotic normality (with i.i.d. data) has been established by Genest et al. (1995). However, the i.i.d. assumption often does not hold for time series data. This does not affect consistency but, for inferential purposes, the asymptotic distributions of the parameters of interest, such as Kendall’s \( \tau \) and the tail dependence coefficients, should be approximated using bootstrap or jackknife methods (e.g. Choros et al., 2010; Fermanian and Scaillet, 2004).

Subsequently we provide a brief presentation of the semi-parametric approach. Let the pairs of ranks \((R_{1i}, R_{2i}), \ldots, (R_{Ni}, R_{2Ni})\) where \( R_{1i} \) is the rank of observation \( X_{1i} \) in the random vector \( X_1 \) and \( R_{2i} \) is the rank of observation \( X_{2i} \) in the random vector \( X_2 \). Normalizing the ranks by a factor of \( 1/(N+1) \) one obtains the domain \((0,1)^2\) of the so called empirical copula

\[
    C_{\theta}(u_1, u_2) = \frac{1}{N} \sum_{i=1}^{N} I\left( \frac{R_{1i}}{N+1} \leq u_1, \frac{R_{2i}}{N+1} \leq u_2 \right) \tag{9}
\]

with \( I(.) \) being an indicator function. The Canonical Maximum Likelihood involves maximizing the rank-based, log-likelihood function of the form

\[
    l(\theta) = \sum_{i=1}^{N} \ln \{ C_{\theta}(\frac{R_{1i}}{N+1}, \frac{R_{2i}}{N+1}) \} \tag{10}
\]
where \( c_{\theta} \) stands for the density function of a bivariate copula with parameter vector \( \theta \) (e.g. Genest, et al., 1995; Genest and Favre, 2007). The selection among the seven alternative copulas, presented above, has been carried out using the AIC and the BIC information criteria.

Over the last 40 years, there have been substantial structural changes in the US pork industry expressed in high concentration and integration at the different market levels. Much of these changes have occurred since the late 1980s (Sirrolli, 2006) and they may have affected the price dependence patterns. To allow for that possibility, the sample has been split in, roughly, two equal parts (1970:1 to 1990:12 and 1991:1 to 2012:12) and copulas have been fitted to each sub-sample.1

Figures 1 and 2 present scatterplots of the normalized ranks for the rates of price change at the different market levels. The overwhelming majority of pairs in Figure 1 (left panel) lies along and close to the positive diagonal suggesting a positive association between the rates of price change at the farm and the wholesale level for the US pork sector over the period 1970-90. The association between the rates of price change also appears to be positive for the wholesale and the retail market (Figure 1, right panel). However, the dispersion of pairs about the positive diagonal is considerably higher compared that in the left panel; this is true not only close to the medians but at the extreme quantiles as well. Figure 2 (left panel) also suggests a positive association between the rates of price change at the farm and the wholesale level over the period 1991 to 2012. In the right panel of Figure 2, however, one can hardly discern a positive association between the rates of price change at the wholesale and the retail level for the same period.

Figure 1: Scatterplots of Normalized Ranks. Rates of Price Changes (1970-1990).
Left Panel (farm-wholesale level), Right Panel (wholesale-retail level)

1 Conducting a formal test of structural change in co-movement is well beyond the scope of the present work. The examination of the periods prior and after 1991 separately relies on an “educated guess” drawing from our knowledge of the developments in the US pork industry. It has turned out that the empirical results offer support to our choice since not only the copula parameters but the copula families as well are found to have changed from the earlier to the latest time period.
3.2 The Empirical Results

All estimations and tests have been carried out using the CDVine package in R (Schepmeier and Brechmann, 2012). For the first half of the sample (1970-1990), both the AIC and the BIC criteria have selected the \( t \)-copula for the pair farm-wholesale and the Joe-Clayton copula for the pair wholesale-retail. For the second half of the sample (1991-2012), both the AIC and the BIC criteria have selected the Gaussian copula for the pair farm-wholesale and the Frank copula for the pair wholesale-retail. To assess the adequacy of the selected copula families we have use the rank-based versions of the Kolmogorov-Smirnov (KS) and the Cramer-von Mises (CvM) tests (Genest et al., 2009). Table 1 presents the results of the goodness-of-fit tests. The p-values in all cases are above the typical levels of statistical significance, thus providing a strong indication that the selected copula families fit the actual data very well.

### Table 1. The Results of the Rank-Based CvM and KS Tests+

<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>farm-wholesale</td>
<td>wholesale-retail</td>
</tr>
<tr>
<td></td>
<td>( t )-copula</td>
<td>Joe-Clayton Copula</td>
</tr>
<tr>
<td>Test</td>
<td>Statistic</td>
<td>p-value</td>
</tr>
<tr>
<td>CvM</td>
<td>0.053</td>
<td>0.426</td>
</tr>
<tr>
<td>KS</td>
<td>0.629</td>
<td>0.386</td>
</tr>
</tbody>
</table>

* Based on 500 bootstrap samples

Table 2 presents the parameter estimates of the copula models. Starting with the pair farm-wholesale and for the period 1970-1990, the \( t \)-copula points to fat tails of the joint distribution of the respective rates of price change and to symmetric tail dependence; the degrees of freedom \( (v = 4.91) \) are well below 30 suggesting a very strong departure from normality. Kendall’s tau is above 0.7 indicating that the number of concordant pairs of observations exceeds by far the number of discordant ones. The coefficients of
tail dependence (co-movement at the extremes) are statistically significant at any reasonable level and give a probability of 0.55 that the two price changes are together at the upper or at the lower joint tails. All these taken together imply that the farm and the wholesale markets for pork were very well integrated in the first half of the sample. Turning now to the pair wholesale-retail, the Joe-Clayton copula points to a certain degree of asymmetry in the tails of the joint distribution; the coefficient of right co-movement exceeds that of left co-movement by almost 10 percentage points. Kendall’s tau is only 0.39 (considerably lower compared to that for the pair farm-wholesale over the same period).

Table 2. Parameter Estimates of the Copula Models +

<table>
<thead>
<tr>
<th>Period 1970-1990 (farm-wholesale)</th>
<th>Copula Model</th>
<th>Copula Parameters (s)</th>
<th>Kendall’s τ</th>
<th>Λ_L</th>
<th>Λ_U</th>
</tr>
</thead>
<tbody>
<tr>
<td>t-Copula</td>
<td>ρ=0.894</td>
<td>0.707 (0.017)</td>
<td>0.551</td>
<td>0.551</td>
<td></td>
</tr>
<tr>
<td></td>
<td>v=4.912</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.023)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Period 1970-1990 (wholesale-retail)</td>
<td>Joe-Clayton</td>
<td>θ_1=1.598 (0.131)</td>
<td>0.394</td>
<td>0.457</td>
<td>0.362</td>
</tr>
<tr>
<td></td>
<td>θ_2=0.679 (0.141)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Period 1990-2012 (farm-wholesale)</td>
<td>Gaussian</td>
<td>ρ=0.891 (0.009)</td>
<td>0.7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(0.014)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Period 1990-2012 (wholesale-retail)</td>
<td>Frank</td>
<td>θ=1.319 (0.376)</td>
<td>0.144</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(0.039)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

+ standard errors for Kendall’s τ and for the tail dependence coefficients have been obtained using the jackknife method (Efron, 1979)

In the second half of the sample, the Gaussian copula for the pair farm-wholesale indicates that close enough to the tails of the joint distribution of price changes, extreme events occur independently in each margin (i.e., a price boom at the wholesale level is not associated with a price boom at the farm level). Kendall’s tau, however, is quite high (0.7) suggesting a strong propensity of co-movement in parts of the joint distribution other than its tails. Finally, the Frank copula for the pair wholesale-retail points also to zero dependence at the extremes. However, Kendall’s tau in this case is very low (only 0.14) implying a rather weak dependence over the entire joint distribution.

For a proper interpretation of the results from copula models when these are applied to assess price transmission (market integration), some information about the causal market is required. Causal is called the market in which the price of a commodity is established or equivalently the market from which price shocks stem from. Earlier empirical works on meat industries in the US have provided strong evidence that causality is typically uni-directional from the upstream to the downstream market levels (e.g. Goodwin and Holt, 1999; Goodwin and Piggott, 2001). Explanations offered for that causality pattern include the tendency of supply shocks in the meat industries to occur...
more frequently than demand shocks and the application of fixed mark-up pricing from the sellers.

In the light of the above information, the finding of a t-copula for the pair farm-wholesale over 1970-1990 implies that extreme positive and extreme negative changes in farm prices were equally likely to be passed to the wholesale level; the finding of an asymmetric Joe-Clayton copula for the pair wholesale-retail over the same period implies that positive price shocks (booms) at the wholesale level were transmitted with higher intensity to the farm level compared to negative price shocks (crashes). In other words, final consumers were more likely to feel price increases than price decreases occurring at the wholesale level. The finding of a Gaussian copula for the pair farm-wholesale over the period 1991 to 2012 together with a relatively high Kendall’s tau suggests that price increases and decreases at the farm level where generally transmitted to the wholesale level, expect from the very extreme ones. The finding of a Frank copula for the pair wholesale-retail together with a very low Kendall’s tau offers evidence of poor integration between these two market levels over the same period (changes in the retail prices of pig meat could be attributed to changes in wholesale prices only to a very limited extend).

4. Conclusions

Price transmission asymmetry and incomplete pass-through along supply chains or across spatial agricultural and food markets have long captured the attention of researchers and policy makers because of their implications for economic efficiency. Against this background the objective of the present work has been to investigate the degree and the structure of price co-movement in the US pork industry. This has been pursued using copulas, a flexible statistical approach to analyze price dependence/co-movement, especially during extreme market upswings and downswings.

Our empirical results indicate:

(a) The appropriate copula models for the US pork industry differ across time periods and across markets. The Gaussian copula, which assumes normality of the joint distribution of price changes, turned out to be appropriate for only one out of the four market pairs examined.

(b) In the first half of the sample, there was a relative high degree of dependence both across the entire joint distributions of price changes as well as at their respective extremes. For the pair farm-wholesale, extreme co-movement was symmetric while for the pair wholesale retail it was asymmetric (prices booms upstream were transmitted downstream with greater intensity compared to prices crashes).

(c) In the second half of the sample the structure of dependence for the pair farm-wholesale changed from a consistent with a t-copula to a consistent with a Gaussian copula. The degree of dependence, however, remained the same. For the pair wholesale-retail there was a change in the structure of dependence from a consistent with a Joe-Clayton copula to a consistent with a Frank copula and a change (considerable decrease) in the degree of dependence. It turned out that changes in the prices of pig meat at the retail level had little to do with those at the wholesale level.
It is certainly not always very wise for a researcher to draw policy recommendations based on the results of a single (or of a few empirical) study(-ies). The findings, however, of asymmetric price transmission and low degrees of price dependence (especially in the second half of the sample for the pair wholesale-retail) raise concerns about the efficiency and the distribution of benefits among the stakeholders of the pork supply chain in the US. Therefore, further research on this interesting topic is necessary. Also, the present work (as all earlier ones on price transmission and market integration) has been based on bivariate copulas. Future works may consider the use of multivariate ones which are more appropriate for analyzing price dependence across many product markets and product forms.

References


### APPENDIX

**A.1. Copula Parameters, Kendall’s τ, and Tail Dependence**

<table>
<thead>
<tr>
<th>Copulas</th>
<th>Parameters</th>
<th>Kendall’s τ</th>
<th>Tail Dependence (lower, upper)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>$\rho \in (-1, 1)$</td>
<td>$\frac{2}{\pi} \arcsin(\rho)$</td>
<td>$(0,0)$</td>
</tr>
<tr>
<td>t-Copula</td>
<td>$\rho \in (-1, 1)$, $\nu &gt; 2$</td>
<td>$\frac{2}{\pi} \arcsin(\rho)$</td>
<td>$(2t_{\nu+1}(-\sqrt{\nu+1} \sqrt{\frac{1-\rho}{1+\rho}}), 2t_{\nu-1}(-\sqrt{\nu+1} \sqrt{\frac{1-\rho}{1+\rho}}))$</td>
</tr>
<tr>
<td>Clayton</td>
<td>$\theta &gt; 0$</td>
<td>$\frac{\theta}{\theta+2}$</td>
<td>$\left(\frac{1}{2\pi}, 0\right)$</td>
</tr>
<tr>
<td>Gumbel</td>
<td>$\theta \geq 1$</td>
<td>$1 - \frac{\theta}{\theta}$</td>
<td>$(0, 2 - \frac{1}{\theta})$</td>
</tr>
<tr>
<td>Frank*</td>
<td>$\theta \in R \setminus {0}$</td>
<td>$1 - \frac{4}{\theta} + 4 \frac{D(\theta)}{\theta}$*</td>
<td>$(0,0)$</td>
</tr>
<tr>
<td>Gumbel-Clayton</td>
<td>$\theta_1 &gt; 0, \theta_2 \geq 1$</td>
<td>$1 - \frac{2}{\theta_1(\theta_1 + 2)}$</td>
<td>$\left(2^{\frac{1}{\theta_1}}, 2 - 2^{\frac{1}{\theta_1}}\right)$</td>
</tr>
<tr>
<td>Joe-Clayton</td>
<td>$\theta_1 \geq 1, \theta_2 &gt; 0$</td>
<td>$\int_0^{\theta_1} \frac{c^\theta}{\exp(x) - 1} dx$ where $c = \frac{1}{(1-t)^{\theta_2 - \theta_1}}$</td>
<td>$\left(2^{\frac{1}{\theta_1}}, 2 - 2^{\frac{1}{\theta_1}}\right)$</td>
</tr>
</tbody>
</table>

* $D(\theta) = \int_0^{\theta_1} \frac{c^\theta}{\exp(x) - 1} dx$ is the Debye function.

** Brechmann and Schepsmeier (2013).