Measuring technical efficiency of dairy farms with imprecise data: a fuzzy data envelopment analysis approach

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This article integrates fuzzy set theory in the data envelopment analysis (DEA) framework to compute technical efficiency scores when input and output data are imprecise. The underlying assumption in conventional DEA is that input and output data are measured with precision. However, production agriculture takes place in an uncertain environment, and, in some situations, input and output data may be imprecise. We present an approach of measuring efficiency when data are known to lie within specified intervals and empirically illustrate this approach using a group of 29 dairy producers in Pennsylvania. Compared to the conventional DEA scores that are point estimates, the computed fuzzy efficiency scores are interval bound allowing the decision maker to trace the performance of a decision-making unit at different possibility levels.

Key words: data envelopment analysis, fuzzy set theory, membership function, technical efficiency, α-cut level.

1. Introduction

The data envelopment analysis (DEA) approach has been extensively applied in agriculture to measure the productive efficiency of production entities. Charnes *et al.* (1978) developed the DEA methodology for measuring relative efficiencies within a group of decision-making units (DMUs) which utilise several inputs to produce a set of outputs. DEA constructs a nonparametric frontier over data points so that all observations lie on or below the frontier. A competing method for computing technical efficiency scores is the stochastic frontier approach (SFA) developed by Aigner *et al.* (1977) and Meeusen and van den Broeck (1977).

DEA approach has been favoured over the SFA for several seasons. First, it requires no assumption about the distribution of the underlying data and deviation from the estimated frontier is interpreted purely as inefficiency. Second, it does not require specification of a functional form for the frontier just as economic theory does not imply a particular functional form.

However, DEA requires detailed data about inputs and outputs. It is based on the assumption that all input and output data are crisp, that is, all the

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observations are considered as feasible with probability one, meaning no noise or measurement error is assumed (Simar 2007; Henderson and Zelenyuk 2007). Empirical analysis using DEA has increasingly used individual or household level data derived from survey responses that are not perfectly reliable (Bound *et al*. 2001). The use of survey in microdata has raised concerns about measurement error as some variables are difficult to measure with reasonable accuracy. For example, input and output data are usually collected by asking respondents to recall the details of events occurring during past agricultural seasons prior to the interview. This can introduce recall bias (under or over reporting) in survey data. The dominance of uncertainty in agricultural production has seen the flourish of studies of production under risk in agricultural economics (Just and Pope 2001). Factors used in production agriculture, such as labour, are sometimes difficult to measure in a precise manner. Input measures are often based on accounting data even though the definition of accounting costs differs from that of economic costs by excluding the opportunity cost (Kuosmanen *et al*. 2007). Producer data may also be available only in linguistic form such as 'high yield', 'low yield', 'labour intensive' or 'capital intensive'. The conventional DEA[1] approach is very sensitive to data measurement errors and changes in data, including outliers and missing data, can change the efficient frontier significantly. The DEA model does not account for statistical noise.

A number of techniques to account for the deterministic nature have been suggested in the literature, such as the techniques for detecting possible outliers (Cazals *et al*. 2002) and the stochastic programming approach (Cooper *et al*. 1998). Notably, Simar and Wilson (1998, 2000a) introduced bootstrapping into the DEA framework to allow for consistent estimation of the production frontier, corresponding efficiency scores, as well as standard errors and confidence intervals. However, as observed by Kuosmanen *et al*. (2007), the statistical properties and hypothesis tests suggested by Simar and Wilson (2000a,b) focus exclusively on the effect of the sampling of firms from the production possibilities set, and hence, the bootstrap approach does not allow for data errors of any kind. Therefore, there is need for a model that can adequately represent the stochastic nature of production data at a microlevel.

This paper introduces fuzzy DEA, an approach advanced in the field of industrial engineering, to measure technical efficiency where data are imprecise. A group of 29 dairy producers in Pennsylvania are used to illustrate how to empirically compute fuzzy technical efficiency scores. The approach incorporates fuzzy set theory in the DEA mathematical programming technique to compute technical efficiency indices under natural uncertainty inherent in the production processes. Unlike the conventional

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[1] Here, we refer to the Banker, Charnes and Cooper (BCC) model that assumes variable return to scale (Banker *et al*. 1984). The concept presented can equally be extended to the Charnes, Copper and Rhodes (CCR) model that assumes constant return to scale (Charnes *et al*. 1978).

DEA model, with a fuzzy DEA model, the decision maker can consider different degrees of measurement errors (possibility levels) when estimating technical efficiency. Expert judgment expressed in linguistic variables can also be incorporated into the fuzzy DEA models (Guo and Tanaka 2001).

Fuzzy DEA models are rare in the economics or agricultural economics literature. A search for 'fuzzy DEA' in the AGRICOLA, AgEcon Search and EconLit databases returned no items. The only recent application of fuzzy DEA in agriculture is by Hadi-Vencheh and Matin (2011) who compute efficiency scores for wheat provinces in Iran. Other applications of fuzzy set theory in agricultural economics include van Kooten *et al.* (2001) who proposed a fuzzy contingent valuation approach to measure uncertain preferences for nonmarket goods. Duval and Featherstone (2002) compared compromise programming and fuzzy programming to a traditional mean–variance approach, and Krcmar and Van Kooten (2008) developed a compromise-fuzzy programming framework to analyse trade-offs of economic development prospects of forest dependent aboriginal communities. The contribution of this paper is to allow an agricultural economist to expediently reach an introductory understanding of fuzzy data envelopment analysis.

Analysis of technical efficiency using fuzzy DEA models is very useful to the decision maker and presents several advantages. First, uncertainty in measurement can be incorporated in DEA model at different degrees. Second, linguistic variables can be incorporated into the DEA model, for example expert judgment and environmental variables. Third, fuzzy DEA can be used to deal with missing data, and fourth, the decision maker can trace how the efficiency scores vary at different levels of uncertainty.

In what follows, the conventional DEA model is presented followed by the basic concepts of fuzzy set theory and how those concepts are integrated into the DEA framework. Then, a literature review of numerical and empirical fuzzy DEA models is presented. The dataset is discussed next followed by an application of the fuzzy DEA model to that data and discussion of the results. Then, the article concludes.

2. Methodology

2.1. Conventional DEA model

Data envelopment analysis (DEA) is a nonparametric methodology for measuring efficiency within a group of decision-making units (DMUs) that utilise several inputs to produce a set of outputs. DEA models provide efficiency scores that assess the performance of different DMUs either in terms of the use of several inputs (input orientation) or the production of certain outputs (output orientation). The input-oriented DEA scores vary in [0, 1], the unity value indicating the technically efficient units (Leon *et al.* 2003). The assumption underlying DEA is that all data assume specific numerical values.

Consider N decision-making units, DMU_j, where $j = 1 \ldots N$. Each DMU consumes input levels x_{ij}, $i = 1 \ldots M$, to produce outputs levels y_{rj}, $r = 1 \ldots S$. Suppose that $x_j = [x_{1j} \ldots, x_{Mj}]^T$ and $y_j = [y_{1j} \ldots, y_{Sj}]^T$ are the vectors of input and output values for DMU_j, where $x_j \geq 0$ and $y_j \geq 0$. The relative efficiency score of the DMU_o, $o \in \{1, \ldots, N\}$, is obtained from the following input-oriented DEA model that aims at reducing the input amounts by as much as possible while keeping at least the present output levels:

$$MinZ = \theta \text{ subject to} : \theta x_{io} \geq \sum_{j=1}^{N} \lambda_j x_{ij}, \forall i, ; y_{ro} \leq \sum_{j=1}^{N} \lambda_j y_{rj}, \forall r, ; \lambda_j \geq 0 \quad (1)$$

where λ indicates the intensity levels which make the activity of each DMU expand or contract to construct a piecewise linear technology (Färe *et al.* 1994). The DMU_o is technically efficient if and only if $\theta = 1$, otherwise the DMU_o is inefficient. There is an extensive literature on classical DEA models and Cooper *et al.* (2007) provides a comprehensive review of some of the accomplishments and future prospects of DEA. A major drawback of the DEA model is that the computed relative efficiency scores are very sensitive to noise in data. Any outlier or missing value in the data may cause the efficiency measure of most DMUs to change drastically (Kao and Liu 2000a,b). This makes an approach that is able to deal with inexact numbers, numbers in range or vague measures desirable. Fuzzy set theory can be incorporated in the DEA framework to deal with imprecise data in both the objective function and constraints.

2.2. Fuzzy set theory

Optimisation techniques often used in economics are 'crisp' in that a clear distinction is made in a two-valued way between feasible and infeasible, and between optimal and nonoptimal solutions (Zimmerman 1994). The techniques do not allow for gradual transition between these categories, a limitation often referred to as the problem of artificial precision in formalised systems (Geyer-Schulz 1997). Bellman and Zadeh (1970) were the first to suggest modelling goals and/or constraints in optimisation problems as fuzzy sets to account for uncertainty and fuzziness of the decision-making environment.

Fuzzy set theory is a generalisation of classical set theory in that the domain of the characteristics function is extended from the discrete set $\{0, 1\}$ to the closed real interval $[0, 1]$. Zadeh (1965) defined a fuzzy set as a class of objects with continuum grades of membership. Suppose X is a space of objects, and x is a generic element of X. A fuzzy set, $\tilde{A}$, in X can be defined as the set of ordered pairs:

$$\tilde{A} = \{(x, u_A(x)) | x \in X\}, \quad (2)$$

where $u_A(x)$: $X \rightarrow M$ is the membership function and M is the membership space that varies in the interval $[0, 1]$. The closer the value of $u_A(x)$ is to one,

the greater the membership degree of X to $\tilde{A}$. However, if $M = \{0, 1\}$, the set A is nonfuzzy[2] (Triantis and Girod 1998). A fuzzy set $\tilde{A}$ can be defined precisely by associating with each object x a number between 0 and 1, which represents its grade of membership in A. Thus, $u_A(x) = 1$ if x is totally in A, $u_A(x) = 0$ if x is not in A, and $0 < u_A(x) < 1$ if x is partly in A.

Fuzzy set theory[3] is based on several topological concepts that are beyond the scope of this paper. The interested readers are referred to Kaufmann and Gupta (1991) and Zimmerman (1994) for an introduction to fuzzy sets theory and fuzzy mathematical models. However, terms like *fuzzy sets*, *membership functions* and *fuzzy numbers* will be used several times, but no real knowledge of the theory of fuzzy sets is required. Basic concepts relevant to understand this paper are defined:

1 A set in conventional set theory, A, such as a set of large dairy farms (x) that produce at least 1000 L of milk per day is represented as $A = \{x | milk(x) \geq 1000\}$. A universal set, U, is the set from which all elements are drawn, for example, all dairy farms. The conventional set is defined such that the elements in a universe are divided into two groups: members (those that do belong to it) and nonmembers (those that do not belong).

2 A fuzzy set, drawn from U, allows its elements to belong to A at various degrees, with '1' implying a full belongingness and '0' implying no belongingness. For example, from $U = \{x_1 = 500, x_2 = 900, x_3 = 1200\}$, we can have a crisp set $A = \{x_3 = 1200\}$ and fuzzy set $\tilde{A} = \{x_1 = (500, 0.5) x_2 = (900, 0.9), x_3 = (1200, 1)\}$. The values 0.5, 0.9 and 1 are membership functions, $u_A(x)$, and represent the grade of membership of x_1, x_2 and x_3 to the set $A = \{x | milk(x) \geq 1000\}$. The term 'large dairy farms' here is vague and vary depending with the perception of an individual. Therefore, farms x_1 and x_2 can be considered large farms too but with degrees of membership 0.5 and 0.9.

3 A fuzzy number is a quantity whose value is imprecise, rather than exact as is the case with single-valued numbers. Generally, a fuzzy number is a fuzzy subset of a real number, R, which is both *normal* and *convex* where normal implies that the maximum value of the fuzzy set in R is 1. It has a peak or plateau with membership grade 1, over which the members of the universe are completely in the set. The membership function is increasing towards the peak and decreasing away from it. Fuzzy numbers can be represented as linear, triangular, trapezoidal or Gaussian. In practice, triangular fuzzy numbers are used commonly used because they can easily be specified by the decision maker.

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[2] This rule outs degree of belongingness. It implies that x belong to the set 100% (1) or is not a member of the set (0).

[3] Fuzzy set theory focusses on how to deal with imprecision or inexactness analytically. The imprecision here is nonstatistical or nonprobabilistic (Levine 1997).

4　A triangular fuzzy number, $\tilde{A}$, is a number with piecewise linear membership functions $u_{\tilde{A}}(x)$ defined by:

$$u_{\tilde{A}}(x) = \begin{cases} 0, x < \pi^l \\ \frac{x - \pi^l}{\pi^m - \pi^l}, \pi^l \le x \le \pi^m, \\ \frac{\pi^m - x}{\pi^u - \pi^m}, \pi^m \le x \le \pi^u, \\ 0, x > \pi^u \end{cases} \tag{3}$$

5　This can be denoted as a triplet (π^m, π^l, π^u) where π^m, π^l, π^u are the centre, left spread and right spread of the number. Figure 1 illustrates an example of a triangular fuzzy number. Letting $\tilde{A}$ and $\tilde{B}$ to be two triangular fuzzy numbers denoted by (a_l, a_m, a_u) and (b_l, b_m, b_u), it follows that $\tilde{A} \le \tilde{B}$ if and only if $a_l \le b_l$, $a_m \le b_m$. and $a_u \le b_u$.

6　The α-cut level of a fuzzy set is a crisp subset of X that contains all the elements of X whose membership grades are greater than or equal to the specified value of α. This is denoted by $\tilde{A}_\alpha = \{(x, u_{\tilde{A}}(x)) | u_{\tilde{A}}(x) \ge \alpha, x \in X\}$. Each α-cut level of a fuzzy number is a closed interval which can be represented as $[L(\alpha), U(\alpha)]$, where $L(\alpha)$ is the lower bound and $U(\alpha)$ is the upper bound at a defined α-cut level, α. A family of α-cut levels determines a fuzzy number.

7　Therefore, using fuzzy mathematics, the interval of confidence at a given α-cut level, where L is lower bound and U is upper bound, can be characterised as:

$$\forall \alpha \in [0:1], A_\alpha = \left[L = \alpha(\pi^m - \pi^l) + \pi^l, U = \pi^u - \alpha(\pi^u - \pi^m) \right]$$

To illustrate concepts (3–6), assume that we want to define a medium dairy farm as one producing an average of 1000 kg of milk a day. Suppose we set the lower and upper bounds to 500 and 1500 kg. Letting x represent the average kilos of milk produced per day, the linear membership function in concept (4) can be represented as:
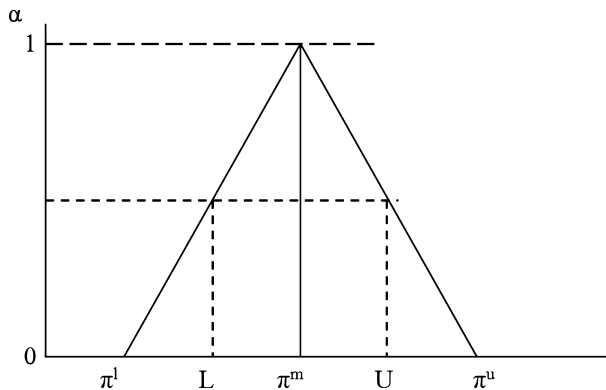

Figure 1 A theoretical triangular fuzzy number. Note: L = $\pi^l + \alpha(\pi^m - \pi^l)$ and U = $\pi^u - \alpha \times (\pi^u - \pi^m)$.

$$u_{\tilde{A}}(x) = \begin{cases} 0, x < 500 \\ \frac{x-500}{1000-500}, 500 \le x \le 1000, \\ \frac{1500-x}{1500-100}, 1000 \le x \le 1500, \\ 0, x > 1500 \end{cases}$$

Therefore, if $x < 500$ or $x > 1500$, the degree of membership is zero. If x is 700 or 1300, the degree of membership is 0.4, and if x is 900 or 1000, the degree of membership is 0.8. This implies that at $u_A(x) = 0.4$, the lower and upper bounds of x are $[L(\alpha), U(\alpha)] = [700, 1300]$.

Likewise, at $u_A(x) = 0.8$, we have $[L(\alpha), U(\alpha)] = [900, 1100]$. The family of the $u_A(x)$ levels determines the fuzzy number that defines a medium dairy farm. Concept 6 provides a simple way of determining the lower and upper bounds at any given α-cut level. For instance, at $u_A(x) = 0.8$, we can compute the lower and upper bounds as follows:

$$\forall \alpha \in [0:1], A_\alpha = \left[L = \alpha(\pi^m - \pi^l) + \pi^l, U = \pi^u - \alpha(\pi^u - \pi^m) \right]$$
$$\Rightarrow [0.8(1000 - 500) + 500, 1500 - 0.8(1500 - 1000)] = [900, 1100]$$

Therefore, it follows that the fuzzy number defined by α-cut level = 0.8 is a subset of the fuzzy number defined by α-cut level = 0.4.

2.2.1. Fuzzy DEA with triangular membership functions

Consider the conventional DEA model in equation 1 with the exception that the inputs and outputs are fuzzy where, '~', indicates fuzziness.[4] Suppose the input and output are triangular fuzzy numbers represented by $\tilde{x}_{ij} = (x_{ij}^l, x_{ij}^m, x_{ij}^u)$ and $\tilde{y}_{rj} = (y_{rj}^l, y_{rj}^m, y_{rj}^u)$. Kao and Liu (2000a) developed a method to find the membership function of the efficiency scores when the observations are fuzzy numbers based on the idea of the α-cut level and Zadeh's extension principle.[5] The main idea is to transform the levels of inputs and outputs such that the data lie within bounded intervals, that is $\tilde{x}_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $\tilde{y}_{rj} \in [y_{rj}^L, y_{rj}^U]$ where L and U represent the lower and upper bounds, respectively. Therefore, equation 1 can be reformulated, taking into consideration the fuzzy data, as:

$$MinZ = \tilde{\theta} s.t : \tilde{\theta}\tilde{x}_{io} \ge \sum_{j=1}^{N} \lambda_j \tilde{x}_{ij}, \forall i,; \tilde{y}_{ro} \le \sum_{j=1}^{N} \lambda_j \tilde{y}_{rj}, \forall r,; \sum \lambda_j = 1, \lambda_j \ge 0 \quad (4)$$

The above model can be expanded to indicate the centre, lower and upper bound values as follows:

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[4] The presented fuzzy model provides measures of weak efficiency because inefficiency represented by nonzero slacks is omitted when measuring technical efficiency. However, the concepts presented can be integrated in a slack-based DEA model to compute strong efficiency measures that allow for nonzero slacks.
[5] The extension principle states that the classical results of Boolean logic are recovered from fuzzy logic operations when all fuzzy membership grades are restricted to the classical set {0, 1}.

$$MinZ = \tilde{\theta}s.t:$$

$$(\tilde{\theta}x_{io}^m, \tilde{\theta}x_{io}^l, \tilde{\theta}x_{io}^u) \geq \left(\sum_{j=1}^{N} \lambda_j x_{ij}^m, \sum_{j=1}^{N} \lambda_j x_{ij}^l, \sum_{j=1}^{N} \lambda_j x_{ij}^u\right) \forall i,$$

$$(y_{ro}^m, y_{ro}^l, y_{ro}^u) \leq \left(\sum_{j=1}^{N} \lambda_j y_{rj}^m, \sum_{j=1}^{N} \lambda_j y_{rj}^l, \sum_{j=1}^{N} \lambda_j y_{rj}^u\right) \forall r,$$ (5)

$$\sum \lambda_j = 1, \lambda_j \geq 0$$

This model is fuzzy and the usual linear programming method cannot solve it without being defuzzified.[6] The α-cut level and extension principle are used to defuzzify the model by transforming the fuzzy triangular numbers to 'crisp' intervals that are solvable as a series of conventional DEA models as follows:

$$MinZ = \theta \text{ subject to}:$$

$$[\theta(\alpha x_{io}^m + (1-\alpha)x_{io}^l), \theta(\alpha x_{io}^m + (1-\alpha)x_{io}^u)] \geq$$

$$\left[\sum_{j=1}^{n} \lambda_j(\alpha x_{ij}^m + (1-\alpha)x_{ij}^l), \sum_{j=1}^{n} \lambda_j(\alpha x_{ij}^m + (1-\alpha)x_{ij}^u)\right] \forall i,$$

$$[\theta(\alpha y_{ro}^m + (1-\alpha)y_{ro}^l), \theta(\alpha y_{ro}^m + (1-\alpha)y_{ro}^u)] \leq$$ (6)

$$\left[\sum_{j=1}^{n} \lambda_j(\alpha y_{rj}^m + (1-\alpha)y_{rj}^l), \sum_{j=1}^{n} \lambda_j(\alpha y_{rj}^m + (1-\alpha)y_{rj}^u)\right] \forall i,$$

$$\sum \lambda_j = 1, \lambda_j \geq 0$$

The model is solved by means of comparing the left-hand side (LHS) and right-hand side (RHS) of each equality/inequality constraint. The main advantage of the α-cut level approach used in this paper is that it provides flexibility for the analyst to set their own acceptable possibility levels for decision-making in evaluating and comparing DMUs. Zadeh (1978) suggested that fuzzy sets could be used as a basis for the theory of possibility similar to the way that measurement theory provides the basis for the theory of probability. The fuzzy variable is associated with a possibility distribution in the same manner that a random variable is associated with a probability distribution. Therefore, the computed fuzzy efficiency scores are viewed as a fuzzy variable in the range [0, 1].

The main advantage of the FDEA is the ability to deal with imprecision in data (i.e. incompleteness of information). Rather than representing uncertain information by approximation, flexible data structures such as fuzzy numbers can be used. The FDEA approach makes it possible to converts fuzzy data into interval data that can be integrated into the DEA framework and

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[6] Equation (5) is a possibility problem and would require possibility programming to solve.

analysed using the linear programming model. The main disadvantage of this approach is that it requires the analyst to have prior and accurate knowledge of the units being analysed and their environment in order to detect data imprecision and formulate interval data using fuzzy set theory.

3. Literature review

Sengupta (1992) was the first to propose a mathematical programming approach where fuzziness was incorporated into DEA by allowing the objective function and the constraints to be fuzzy. The stochastic DEA model was to be solved using chance-constrained programming and required the analyst to supply information on expected values of variables, the variance–covariance matrices of all variables, and the probability levels at which the feasibility constraints are to be satisfied. This method was difficult to implement due to those data requirements.

Triantis and Girod (1998) suggested a mathematical programming approach that transforms fuzzy inputs and outputs into crisp data using membership function values. Efficiency scores would then be computed for different membership functions and averaged. Hougaard (1999) suggested an approach that allows the decision maker to include other sources of information such as expert opinion in technical efficiencies computation. Kao and Liu (2000a) suggested the use of α-cut level sets to transform fuzzy data into interval data so that the fuzzy model becomes a family of conventional crisp DEA models. This approach was much similar to Guo and Tanaka (2001) who proposed a fuzzy CCR model in which fuzzy constraints, including fuzzy equalities and fuzzy inequalities, were all converted to crisp constraints by predefining different possibility levels.

Lertworasirikul *et al.* (2003) proposed a possibility approach in which fuzzy constraints are treated as fuzzy events, and a fuzzy DEA model is transformed into a possibility DEA model by using possibility measures on fuzzy events. Saati *et al.* (2002) adopted the α-cut level approach, defined the fuzzy CCR model as a possibility-programming problem and transformed it into an interval programming problem. This model could be solved as a crisp linear programming model and produce crisp efficiency score for each DMU and for each given α-cut level. All the above authors used numerical examples to illustrate the application of the proposed fuzzy DEA approach.

4. Data

Fuzzy DEA is applied to compute the technical efficiency scores of 29 dairy farms in Pennsylvania using the α-cut level approach. The dairy producers use three inputs (land, labour and cows) to produce two outputs (milk and butterfat). The original data had 34 decision-making units, but five units were dropped after testing for outliers using the Wilson (1993, 2010) approach. The data are obtained from Strokes *et al.* (2007) who used the conventional

DEA to computed technical efficiencies, assuming that either the data are precise or the relationship between inputs and outputs is deterministic. However, the authors hint that the data may not be precise, '*Due to the structure of the dataset it was not possible to determine whether all resources such as land or labor were utilized by the dairy operations* (pp. 2558)'.

To illustrate the application of fuzzy DEA, uncertainty is introduced in the data by representing the inputs and outputs as symmetrical triangular fuzzy numbers with a fuzzy interval. The input and output data can be represented as pairs consisting of centres and spreads as $\tilde{x}_{ij} = (x_{ij}^m, \varepsilon_{ij})$ and $\tilde{y}_{rj} = (y_{rj}^m, \beta_{rj})$, respectively.[7] A representation of the input/output relationship is simply:

$$\tilde{Y}(\text{milk, butterfat}) = \tilde{X}(\text{land, labor, cows}), \qquad (7)$$

where $\tilde{Y}$ and $\tilde{X}$ are matrices of the fuzzy outputs and inputs. The data are listed in Table 1. The spread for each variable is generated as a random number using the random number generator in *R*. For the purpose of this study, we assume that the spread for labour is a random number between 0.1 and 0.5. The spread for cows is between 1 and 5 and that of land is between 1 and 15. The spread of milk is between 0 and 191 and for butterfat is between 0 and 10. In practice, the spread can be determined by eliciting expert knowledge on the accuracy of collected data.[8] The decision maker can construct the lower bound, centre and upper bound of each variable based on the following criteria: (i) the most pessimistic value (lower bound) which has a very low likelihood of belonging to the set of available values; (ii) the most possible value (centre) that definitely belongs to the set of available values; and (iii) the most optimistic value (upper bound) that has a very high likelihood of belonging to the set of available values. The membership function is applied to those triplet values to generate a triangular possibility distribution for each variable; the function will vary from zero and one and represent the degree to which a specific variable (input or output) is close to the most optimistic value.

Empirical studies on data accuracy can also be used to determine reasonable spreads. For example, Berry *et al.* (2005) used a control dataset consisting of 58,210 cows in Ireland to investigate the effect of reduced milking frequencies on the accuracy of computing 305-day milk and fat yields and to determine the accuracy of predicting milk and fat yields from alternative recording schemes. The mean error in estimating 305-day yield from records that are updated every 8 weeks was 6.8 kg (standard deviation of 191 kg) for milk yield and 0.3 kg (standard deviation of 10 kg) for fat yield. Those standard deviation values are used to determine the spread of

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[7] Symmetric fuzzy numbers means that the upper and lower spreads are equal, that is $x_{ij}^l = x_{ij}^u = \varepsilon_{ij}$
[8] It is not possible to suggest rules on how the spread should be chosen because this will vary from case to case. This is where the role of expert knowledge of the system being analysed is very important in determining degree of inaccuracy in data.

Table 1 Inputs and outputs used in the fuzzy DEA analysis models

DMU	Labour (FTE)	Cows	Land (ha)	Milk production (kg)	Butterfat production (kg)
Farm1	2.66	70	98	734,300	26,040
Farm2	3.06	67	97	585,312	22,579
Farm3	3.59	72	38	595,224	22,968
Farm4	1	60	48	600,600	23,520
Farm5	2.8	180	166	1,605,240	59,400
Farm6	2	112	66	1,114,736	40,208
Farm7	1.6	40	109	297,840	12,080
Farm8	2.28	55	105	514,910	18,535
Farm9	4.71	118	121	1,063,888	40,946
Farm10	1.8	55	19	498,685	17,435
Farm11	2	58	57	499,090	19,662
Farm12	2	87	63	795,876	29,232
Farm13	1.8	40	36	272,080	10,480
Farm14	2	53	136	446,949	15,794
Farm15	4.18	249	257	1,827,411	73,206
Farm16	1.6	43	40	366,790	13,029
Farm17	1.38	55	101	373,725	14,080
Farm18	1.6	36	85	175,320	6,588
Farm19	1.9	44	60	326,744	13,068
Farm20	1.51	54	81	450,900	17,010
Farm21	1	98	121	921,788	35,770
Farm22	1.65	36	89	257,976	9,612
Farm23	1.67	54	147	237,114	8,370
Farm24	3.2	110	127	1,097,910	38,390
Farm25	1	64	51	732,032	25,920
Farm26	3.72	110	42	989,450	38,720
Farm27	1.93	81	80	907,281	33,210
Farm28	2.17	56	74	392,840	14,952
Farm29	2	71	61	474,919	18,034
Farm30	1	30	45	183,150	7,350
Farm31	2	82	52	441,078	16,564
Farm32	2	73	113	572,612	20,294
Farm33	3	143	126	1,293,435	50,479
Farm34	1.15	62	86	534,502	19,964
Mean	2	77	88	652,403	24,514
SD	1	45	47	396,973	15,283
Minimum	1	30	19	175,320	6,588
Maximum	5	249	257	1,827,411	73,206

Source: Strokes *et al.* (2007).

milk and fat yield in our illustration. Figure A1 presents the average butterfat expressed as triangular fuzzy number.

We follow a three-stage approach to compute the technical efficiency scores. In the first stage, the inputs and outputs are expressed in terms of symmetrical triangular fuzzy numbers and membership functions at six different α-cut levels ranging from 0 to 1. Prespecified intervals of 0.2 are used. In the second stage, the classical DEA model is re-formulated as a series of DEA models in terms of the membership functions for each of the fuzzy input and output variables following equation (6). The adopted model is presented in the appendix. In the third stage, fuzzy technical efficiency scores are computed

for different membership functions to track how the relative efficiency scores of each farm varies at different possibility levels. The Benchmarking package in R is used to solve the different linear programming problems.

5. Empirical results

Each farm was evaluated at different α-cut level from zero to one at both the lower and upper bounds. The lower bound and upper bound input reducing technical efficiency scores $(\theta_{\alpha i})$ are presented in Tables 2 and 3. The input and output data were assumed to be imprecise, and, therefore, the computed efficiency scores are fuzzy too. In general, with some few exceptions, the lower bound technical efficiency scores $(E_{ji})^L_{\alpha i}$ decrease as the membership function shifts the input and output data from the most precise measurement (α = 1) to the most imprecise measurement (α = 0). The upper bound scores $(E_{ji})^U_{\alpha i}$ increase as α decrease from 1 to 0. The closer α approaches 1 the greater the level of possibility and the lower the degree of uncertainty is. The fuzzy efficiency scores lie in a range, and the different α-cut levels indicate those intervals and the uncertainty level associated with precision in data. Specifically, α = 0 has the widest interval. On the other hand, the value of α = 1 is the most likely value of efficiency score.

Using the α-cut level approach, the range of a farm's efficiency score at different possibility levels is derived. For example, the efficiency scores for Farm 1 at α-cut level = 1 is 0.918. This deterministic case assumes precision in measurement. At α-cut level = 0.8, the efficiency score range is [0.861, 0.927]. This indicates that it is possible that the efficiency score of Farm 1 will fall between 0.861 and 0.927 at the possibility level of 0.8. The range of the efficiency score at the extremes (α = 0) is [0.872, 0.962]. This implies that the efficiency score of Farm 1, relative to other farms, will never exceed 0.962 or fall below 0.872. Results of the other farms at different possibility levels can be interpreted in similar manner. As the degree of uncertainty in data measurement increases, what we observe is that in general, the technical efficiency scores will tend to be underestimated in the lower bound and overestimated in the upper bound. However, the Fuzzy DEA is also able to discriminate farms that will be affected differently. For example, the technical efficiency of Farm 15 is overestimated as the degree of uncertainty increases in the lower bound and underestimated in the upper bound. The technical efficiency of Farm 16 remains stable in the lower bound but unstable in the upper bound and that of Farm 9 is stable in the upper bound but unstable in the lower bound.

Figure 2 illustrates the membership function of the average triangular fuzzy efficiency scores for the 29 farms. Figure 3 plots the best practice frontiers for the upper bound (dashed lines) and lower bound (dotted lines) membership functions of inputs and outputs at α = 0.[9] This represents the extreme range that the frontiers defining the relative technical efficiency scores

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[9] Simple summation is used to construct the aggregate outputs and inputs.

of each farm are expected to shift due to imprecision in data. The shift of the frontier at $0 < \alpha < 1$ would fall within this range and would keep on narrowing as α approached 1.

The results from the fuzzy DEA model provide more information to the decision maker compared with the point estimates from the conventional DEA model. The analyst can observe the variation of the technical efficiency profile of each farm from the impossible value when α-cut level = 0 to the risk-free value when α-cut level = 1. For example, only nine farms (i.e. Farms 4, 5, 8, 18, 21, 22, 23, 26 and 28) have technical efficiency scores that define the frontier at all α-cut levels. The analyst can also identify those farms that will be affected differently due to imprecision in data.

The computed fuzzy efficiency scores need to be ranked in order to determine how each farm performs relative to the other farms in an uncertain environment. The ranking of the fuzzy efficiency scores can be compared with

Table 2 Lower bound technical efficiency scores at varying α-cut levels

DMU	θ_1	$\theta_{0.8}$	$\theta_{0.6}$	$\theta_{0.4}$	$\theta_{0.2}$	θ_0	Average
				Lower bound membership function value $(E_j)^L_{\alpha i}$			
Farm1	0.918	0.909	0.900	0.891	0.882	0.872	0.895
Farm2	0.864	0.861	0.857	0.854	0.849	0.844	0.855
Farm3	0.892	0.878	0.863	0.848	0.833	0.817	0.855
Farm4	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm5	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm6	0.967	0.961	0.956	0.950	0.939	0.927	0.950
Farm7	0.919	0.912	0.905	0.898	0.887	0.875	0.899
Farm8	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm9	0.906	0.900	0.895	0.889	0.883	0.876	0.892
Farm10	0.856	0.852	0.848	0.844	0.840	0.836	0.846
Farm11	0.982	0.982	0.982	0.983	0.983	0.984	0.983
Farm12	0.874	0.878	0.881	0.882	0.879	0.877	0.879
Farm13	0.994	0.990	0.985	0.980	0.974	0.968	0.982
Farm14	0.769	0.766	0.763	0.760	0.755	0.749	0.761
Farm15	0.833	0.845	0.857	0.869	0.882	0.896	0.864
Farm16	0.920	0.921	0.922	0.922	0.922	0.920	0.921
Farm17	0.883	0.886	0.889	0.892	0.892	0.889	0.888
Farm18	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm19	0.962	0.976	0.992	1.000	1.000	1.000	0.988
Farm20	0.617	0.619	0.621	0.622	0.619	0.615	0.619
Farm21	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm22	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm23	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm24	0.784	0.778	0.772	0.765	0.758	0.749	0.768
Farm25	0.713	0.706	0.698	0.691	0.684	0.677	0.695
Farm26	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm27	0.741	0.734	0.726	0.718	0.707	0.697	0.721
Farm28	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm29	0.870	0.853	0.849	0.845	0.841	0.836	0.849
Average	0.906	0.904	0.902	0.900	0.897	0.893	0.900
Max	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Min	0.617	0.619	0.621	0.622	0.619	0.615	0.619

The table reports the lower bound input reducing technical efficiency scores at various α-levels.

Table 3   Upper bound technical efficiency scores at various α-cut levels

	Upper Bound Membership Function Value $(E_j)^U_{\alpha i}$						
DMU	θ_1	$\theta_{0.8}$	$\theta_{0.6}$	$\theta_{0.4}$	$\theta_{0.2}$	θ_0	Average
Farm1	0.918	0.927	0.936	0.945	0.954	0.962	0.940
Farm2	0.864	0.867	0.866	0.864	0.863	0.861	0.864
Farm3	0.892	0.906	0.920	0.933	0.945	0.958	0.926
Farm4	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm5	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm6	0.967	0.972	0.974	0.977	0.979	0.981	0.975
Farm7	0.919	0.926	0.933	0.939	0.946	0.952	0.936
Farm8	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm9	0.906	0.911	0.911	0.911	0.911	0.912	0.910
Farm10	0.856	0.860	0.864	0.868	0.872	0.876	0.866
Farm11	0.982	0.982	0.982	0.982	0.983	0.983	0.982
Farm12	0.874	0.871	0.868	0.865	0.862	0.859	0.866
Farm13	0.994	0.997	0.999	1.000	1.000	1.000	0.998
Farm14	0.769	0.772	0.777	0.797	0.815	0.831	0.794
Farm15	0.833	0.823	0.812	0.802	0.793	0.792	0.809
Farm16	0.920	0.919	0.915	0.911	0.907	0.903	0.912
Farm17	0.883	0.880	0.875	0.871	0.866	0.862	0.873
Farm18	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm19	0.962	0.948	0.936	0.923	0.911	0.900	0.930
Farm20	0.617	0.635	0.666	0.693	0.717	0.738	0.678
Farm21	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm22	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm23	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm24	0.784	0.790	0.795	0.799	0.803	0.808	0.797
Farm25	0.713	0.719	0.726	0.733	0.740	0.749	0.730
Farm26	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm27	0.741	0.749	0.756	0.764	0.771	0.778	0.760
Farm28	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Farm29	0.870	0.890	0.908	0.923	0.937	0.949	0.913
Average	0.906	0.908	0.911	0.914	0.916	0.919	0.912
Max	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Min	0.617	0.635	0.666	0.693	0.717	0.738	0.678

The table reports the upper bound input reducing technical efficiency scores at various α-levels.

the ranking of scores of the conventional DEA model in order to discriminate which decision-making units are sensitive to the variation of the inputs/ output variable measurement inaccuracy. We use the Chen and Klein (1997) ranking method to compute an index, I, for ranking fuzzy numbers as:

$$I_j = \frac{\sum_{i=0}^{N} \left((E_j)^U_{\alpha i} - c \right)}{\left[\sum_{i=0}^{N} \left((E_j)^U_{\alpha i} - c \right) - \sum_{i=0}^{N} \left((E_j)^L_{\alpha i} - d \right) \right]}, N \to \infty, \tag{8}$$

where $c = \min_{i,j} \left\{ (E_{ji})^L_{\alpha i} \right\}$ and $d = \max_{i,j} \left\{ (E_{ji})^U_{\alpha i} \right\}$. The lower bound and upper bound efficiency indices are represented by $(E_{ji})^L_{\alpha i}$ and $(E_{ji})^U_{\alpha i}$. A larger index
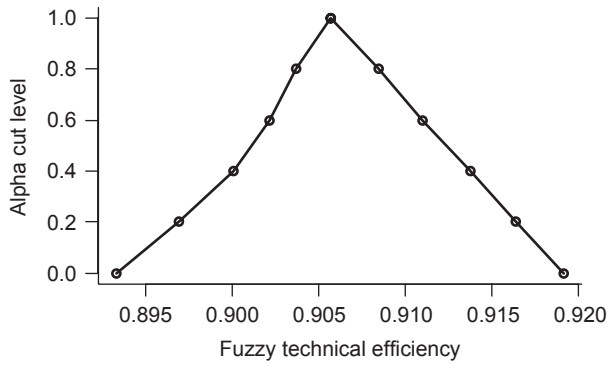
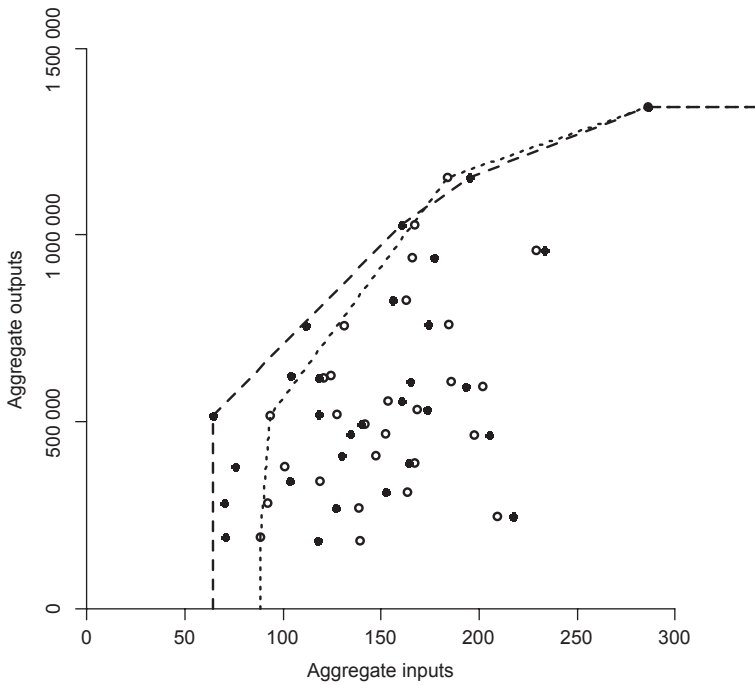**Figure 2** The empirical average triangular fuzzy efficiency scores for the 29 dairy operations.

Figure 3 Best practice frontiers at α-level = 0. The dotted line represent the lower variable returns to scale frontier at α-level = 0. The dashed line represent upper variable returns to scale frontier at α-level = 0.

indicates the fuzzy number is more preferred. The Chen–Kleins method is used to compute the ranking indices for each farm. The ranking is compared with a ranking of the crisp technical efficiency indices from the classical DEA model, and the results are presented in Table 4. The Chen–Klein ranking index gives similar results compared with the ranking of crisp technical efficiency scores. The Spearman rank correlation of the two ranking methods

Table 4 Ranking of the crisp and fuzzy efficiency scores

Rank	DMU	Chen–Klein index	CCR technical efficiency
1	Farm4	1.000	1.000
2	Farm5	1.000	1.000
3	Farm8	1.000	1.000
4	Farm18	1.000	1.000
5	Farm21	1.000	1.000
6	Farm22	1.000	1.000
7	Farm23	1.000	1.000
8	Farm26	1.000	1.000
9	Farm28	1.000	1.000
10	Farm13	0.964	0.962
11	Farm11	0.954	0.982
12	Farm6	0.954	0.994
13	Farm19	0.877	0.967
14	Farm16	0.788	0.920
15	Farm7	0.759	0.919
16	Farm1	0.755	0.918
17	Farm9	0.729	0.906
18	Farm3	0.695	0.883
19	Farm17	0.679	0.892
20	Farm12	0.671	0.874
21	Farm29	0.660	0.870
22	Farm2	0.628	0.864
23	Farm10	0.616	0.856
24	Farm15	0.583	0.833
25	Farm24	0.433	0.784
26	Farm14	0.422	0.769
27	Farm27	0.335	0.741
28	Farm25	0.267	0.713
29	Farm20	0.134	0.617

is 0.99 and is significant at <1%. Figure 2A illustrates the strong correlation between the Chen–Klein ranking index and the crisp technical efficiency scores.

6. Conclusions

The main objective of this paper was to introduce fuzzy DEA models by literature review and application as an alternative for analysing the productive efficiency of agricultural entities in an uncertain environment. Fuzzy DEA models were found to be applicable when expert judgment or environmental variables (linguistic variables) need to be incorporated into the conventional DEA model, when there are missing data and when the measurement of the data is imprecise.

An empirical example of symmetrical triangular membership functions was used to illustrate the application of fuzzy DEA to a group of 29 dairy farms in Pennsylvania. The α-cut level approach was used to convert the fuzzy DEA scores into crisp scores. The fuzzy DEA model was able to discriminate the farms whose efficiency performance is sensitive to variation

in the inputs/outputs. Compared to the classical DEA model, results from the fuzzy DEA model allow for a determination of robustness and might lead to recommendations that are more rigorous.

We conclude by arguing here that it will be interesting to apply empirical fuzzy DEA models in the field of agricultural economics using the α-cut level approach. Given the incomplete knowledge of input and output measures often used in DEA models, fuzzy DEA models will provide agricultural economists with an additional tool for efficiency analysis. Uncertainty always exists in human thinking and judgment. Research in efficiency and productivity analysis should apply recent advancements in DEA that address current concerns. Fuzzy DEA can play an important role for performance evaluation of decision-making units when data are imprecise.

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Appendix

TE (LN,LB,CW,MK,BT) $= Min\,\theta$

subject to:

Constraints
$$
\begin{cases}
\theta(\alpha LN^m_{io} + (1-\alpha)LN^l_{io}) \geq \sum_{j=1}^{34} \lambda_j(\alpha LN^m_{ij} + (1-\alpha)LN^l_{ij}), \\[2mm]
\theta(\alpha LB^m_{io} + (1-\alpha)LB^l_{io}) \geq \sum_{j=1}^{34} \lambda_j(\alpha LB^m_{ij} + (1-\alpha)LB^l_{ij}), \\[2mm]
\theta(\alpha CW^m_{io} + (1-\alpha)CW^l_{io}) \geq \sum_{j=1}^{34} \lambda_j(\alpha CW^m_{ij} + (1-\alpha)CW^l_{ij}), \\[2mm]
\theta(\alpha MK^m_{i0} + (1-\alpha)MK^l_{io}) \leq \sum_{j=1}^{34} \lambda_j(\alpha MK^m_{ij} + (1-\alpha)MK^l_{ij}), \\[2mm]
\theta(\alpha BF^m_{io} + (1-\alpha)BF^l_{io}) \leq \sum_{j=1}^{34} \lambda_j(\alpha BF^m_{ij} + (1-\alpha)BF^l_{ij}), \\[2mm]
\sum \lambda_j = 1,\ \lambda_j \geq 0, \theta_j \geq 0
\end{cases}
$$

TE (LN,LB,CW,MK,BT) $= Min\,\theta$

subject to:

Constraints
$$
\begin{cases}
\theta(\alpha LN^m_{io} + (1-\alpha)LN^u_{io}) \geq \sum_{j=1}^{34} \lambda_j(\alpha LN^m_{ij} + (1-\alpha)LN^u_{ij}), \\[2mm]
\theta(\alpha LB^m_{io} + (1-\alpha)LB^u_{io}) \geq \sum_{j=1}^{34} \lambda_j(\alpha LB^m_{ij} + (1-\alpha)LB^u_{ij}), \\[2mm]
\theta(\alpha CW^m_{io} + (1-\alpha)CW^u_{io}) \geq \sum_{j=1}^{34} \lambda_j(\alpha CW^m_{ij} + (1-\alpha)CW^u_{ij}), \\[2mm]
\theta(\alpha MK^m_{io} + (1-\alpha)MK^u_{io}) \leq \sum_{j=1}^{34} \lambda_j(\alpha MK^m_{ij} + (1-\alpha)MK^u_{ij}), \\[2mm]
\theta(\alpha BF^m_{io} + (1-\alpha)BF^u_{io}) \leq \sum_{j=1}^{34} \lambda_j(\alpha BF^m_{ij} + (1-\alpha)BF^u_{ij}), \\[2mm]
\sum \lambda_j = 1, \lambda_j \geq 0, \theta_j \geq 0
\end{cases}
$$

where LN = Land, LB = Labour, CW = Cows, MK = Milk and BF = Butterfat, $0 \leq \alpha \leq 1$ is the α-cut level, $0 < \theta \leq 1$ is the efficiency index, subscripts *l*, *m* and *u* indicate the lower, centre and upper bounds of the fuzzy number.