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THE LEONTIEF DYNAMIC INVERSE: METHODOLOGICAL ISSUES

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ABSTRACT. *The article is an expose' of the literature on the so-called Dynamic Inverse of the open dynamic input-output model. It first briefly describes the underlining assumptions and properties of the model. This is followed by a comprehensive discussion of the intricacies, controversies and difficulties of the dynamic model in practical applications.*

1.INTRODUCTION

From the outset, I wish to reveal that the article assumes some basic knowledge of the input-output scenario and no effort is made here to make an exposition of the input-output system and the range of assumptions on which the whole theory is built[1]. The paper first introduces the dynamic input-output system and the concomitant notion of the dynamic inverse. Stipulated solution methods and the associated controversies are then discussed at length.

2. THE DYNAMIC INPUT-OUTPUT SYSTEM

We shall first define the economic meanings of the variables and then introduce the open dynamic input-output model.

Let

X_t denote the n sectoral outputs

d_t represent deliveries to final demand or consumption

A_t stand for the matrix of technical coefficients and

B_t for the capital coefficient matrix, all in year t.

It is assumed all through that d_t does not include that part of output going to capital formation; and capital goods produced in year t are used in the production process in year t+1.

The workings of the economy are explained by the equation

$$X_t = A_t X_t + B_{t+1}(X_{t+1} - X_t) + d_t \qquad \text{... (1)}$$

where t= 0,1,...,m.

The first term on the right-hand side shows intermediate demand for goods by the industries; the second determines the allocation of inputs to investment; X_t and d_t are as defined earlier.

Letting $G_t = I - A_t + B_{t+1}$, it follows from equation (1) that

$$G_t X_t - B_{t+1} X_{t+1} = d_t \qquad \text{... (2)}$$

3. LEONTIEF'S BACKWARD INTEGRATION METHOD

The method pursues the aim of, *inter alia*, finding X_t starting from given or estimated A_t, B_t and d_t. One may think of iteratively proceeding by ordinary matrix inversion assuming some initial condition X_0. However, economic reality has revealed that B_t is usually singular as there are only a few sectors that produce capital goods implying that many rows of B_t are zero.

In view of overcoming this difficulty, W. Leontief has designed and applied the so-called Backward Integration Method [6, pp. 17-46] to the American economy (1947-1958). This procedure is discussed hereunder.

Equation (2) above generates a system of inter-locked equations that determine the path followed by the national economy in a period of m+1 years. In matrix form, it becomes

$$\begin{bmatrix} G-B & & & \\ & G-B & & \\ & & \ddots & \\ & & & G-B \\ & & & G \end{bmatrix} \begin{bmatrix} X_o \\ X_1 \\ \vdots \\ X_{m-1} \\ X_m \end{bmatrix} = \begin{bmatrix} d_o \\ d_1 \\ \vdots \\ d_{m-1} \\ d_m \end{bmatrix} \quad \ldots \; (3)$$

The method of analysis therefore assumes $X_{m+1}=0$. It is also assumed that A and B are constant over a period of m+1 years but not an absolute requirement. As elaborated by Leontief, technological changes can be introduced if we want to. The unknown Xs are computed starting with the last equation $GX_m = d_m$; substitute this solution into the second from the last and continue in that fashion until the solution of X_0 is obtained. This is what Leontief calls the Backward Integration Method. In matrix notation, the solution will be

$$\begin{bmatrix} X_o \\ X_1 \\ \vdots \\ X_{m-1} \\ X_m \end{bmatrix} = \begin{bmatrix} G^{-1} & \ldots & R^{m-2} G^{-1} & R^{m-1} G^{-1} & R^{m} G^{-1} \\ & & \vdots & & \\ & \ldots\ldots\ldots & G^{-1} & RG-1 & G^2G^{-1} \\ \cdot & & \ldots & G^{-1} & RG^{-1} \\ & & & \ldots\ldots & G^{-1} \end{bmatrix} \begin{bmatrix} d_o \\ d_1 \\ \vdots \\ d_{m-1} \\ d_m \end{bmatrix} \quad \ldots \; (3)$$

Here R stands for $G^{-1}B$. The matrix on the right-hand side of (4), which is in fact the inverse of the matrix on the left-hand side of (3), is called Leontief's Dynamic Inverse.

4. CONVERGENCE OF THE DYNAMIC INVERSE

By definition, the matrix on the left-hand side of (3) is infinite since there is no limit on the time horizon $t=m$ or $t=0$. The existence and convergence of its inverse is of practical importance. Leontief extends his analysis and includes the mathematical requirements for the existence of a convergent inverse [6, pp. 38-39]. These conditions are:

i) G^{-1} exists

ii) the so-called Frobenius theorem requires that all the eigenvalues of R fall within the unit circle.

Condition (i) is normally satisfied. A closer examination of the second condition is pertinent.

$$
\begin{aligned}
\mathbf{R} &= \mathbf{G}^{-1}\,\mathbf{B} \\
&= (\mathbf{I}-\mathbf{A}+\mathbf{B})^{-1}\,\mathbf{B} \\
&= [(\mathbf{I}-\mathbf{A})\ \{\mathbf{I}+(\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{B}\}]^{-1}\,\mathbf{B} \\
&= \{\mathbf{I}+(\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{B}\}^{-1}
\end{aligned}
$$

Setting $\mathbf{Q} = (\mathbf{I}-\mathbf{A})^{-1}$,

$$\mathbf{R} = (\mathbf{I}+\mathbf{QB})^{-1}$$

Hence, if QB has eigenvalues e_i, those of R will be e_i/He_i which shows $|e_i/He_i| < 1$ except when $e_i \leq -0.5$. In case some of the eigenvalues fall below -0.5, the problem is resolved by manipulating [6, p.40] the time lag of matrix B so that we get a spectrum of less than 1 for matrix R.

5. ECONOMIC INTERPRETATION OF THE DYNAMIC INVERSE

It may be recalled that in the open static version of the model, the j^{th} column of the inverse, $(I-A)^{-1}$, specifies the direct and indirect input requirements generated by one unit of the j^{th} product for final demand. From equation (4) the same is true for the dynamic inverse but these input requirements are distributed backwards in time.

For example, if we were to deliver one unit of final demand in the year m, then G^{-1} shows the input requirements to be fulfilled in the year m; RG^{-1} stipulates the input requirements to be delivered in the preceding year m-1; $R^2 G^{-1}$ specifies the input requirements to be provided in the year m-2, etc,. The last term $R^m G^{-1}$ determines the input quantities required in the year 0. As in the static inverse, the row shows the industry producing the input and the column indicates the industry making the supply for final consumption.

Given this background, it can be justified that the static inverse, $(I-A)^{-1}$, is in fact an aggregation of the dynamic inverse [2, p.160]. Consider the last column of the dynamic inverse in equation (4). For purposes of analytical convenience, let the terminal year m=0 and go backwards infinitely in time to effect the addition. It can be shown (see Appendix I for the verification) that each column of the infinite dynamic inverse adds to the static inverse $(I-A)^{-1}$. That is,

$$G^{-1} + RG^{-1} + R^2G^{-1} + \dots + R^tG^{-1} + \dots = (I-A)^{-1}$$

Hence, each column of the infinite dynamic inverse adds to the static inverse $(I-A)^{-1}$.

6. AN ALTERNATIVE APPROACH - FORWARD INTEGRATION

Kendrick pinpoints a drawback with Leontief's procedure and puts forward an alternative approach to the problem [4, pp. 693-696]. The shortcoming in the backward integration method is that Leontief assumes X_{m+1}, output corresponding to the year following the terminal year m, is zero. The assumption has resulted in negative inputs to investment in his empirical investigations toward the final year [6, p. 20].

It is therefore suggested that X_{m+1} be set at $\hat{H}X_m$ where $\hat{H}$ is a diagonal matrix designating the envisaged rates of growth in the different sectors for the year m+1 and proceed with the backward substitution procedure [4, pp. 694-695]. The computation is worked out in an iterative manner for various choices of X_{m+1} until the computed X_o converges toward the prevailing conditions in year t=0. As can be anticipated, the calculation is not going to be easy particularly if the number of economic sectors involved is large.

In case of this latter difficulty, Kendrick proposed a forward integration approach starting with exogenously determined initial conditions [4, pp. 695-696]. The ensuing lines feature this alternative approach.

The procedure begins with the partition of matrix B that the first n_1 rows are non-zero and the remaining $(n-n_1)$ rows are zero. Corresponding transformations are made on A, G, X_t and d_t. The partitioned forms are thus

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix} \; ; \; X_t = \begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix} \; ; \; A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad d_t = \begin{bmatrix} d_t^1 \\ d_t^2 \end{bmatrix}$$

Following the above rearrangements, equation (2) is expressed as:

$$G_{11}X^1_t + G_{12}X^2_t - B_{11}X^1_{t+1} - B_{12}X^2_{t+1} = d^1_t \qquad \ldots (5)$$

$$G_{21}x^1_t + G_{22}X^2_t = d^2_t \qquad \ldots (6)$$

We can now solve for X^2_t in terms of X^1_t from equation (6); substitute the solution into equation (5) and obtain the solution of X^1_{t+1}. The derivation (see Appendix II for the proof) shows

$$X^1_{t+1} = [B_{12}G^{-1}_{22}G_{21} - B_{11}]^{-1} [d^1_t - G_{12}G^{-1}_{22}d^2_t + B_{12}G^{-1}_{22}d^2_{t+1} - (G_{11} - G_{12}G^{-1}_{22}G_{21})\, X^1_t] \qquad \ldots (7)$$

With a given initial condition X_o, this last equation can be integrated forward to determine X^1_t for all $t = 0,1,\ldots, m$ as d_t is exogenously determined. Equation (6) is then utilized to obtain X^2_t.

The forward integration method was also analyzed from the point of the notion of the so-called Penrose's generalized inverse and justified in an article by Kreijger and Neudecker [5, pp. 505-507].

7. THE GENERALIZED VERSION OF LEONTIEF'S DYNAMIC INVERSE

The forward iteration method has overcome the commonly acknowledged singularity problem. It has also been possible to incorporate given initial conditions into the system to ensure consistency. In doing so however, Leontief's intertemporal multiplier properties discussed in section 4 have been skipped from the analysis [8, p. 641].

In view of these considerations, A. Schinnar provides a restatement of model (1), procedures for including initial output levels, and some computational techniques for use in practical planning [8, pp. 641-653]. The analysis remains within the bounds of the backward iteration approach as intended by Leontief. Equation (1) is rewritten in the form

$$X_t - RX_{t+1} = C_t$$

where

$$R = G^{-1} \text{ or } (I-A+B)^{-1}B \text{ and}$$

$$C_t = (I-A+B)^{-1}\, d_t \text{ or } G^{-1}\, d_t$$

Using matrix representation, equation (8) reduces to

$$\begin{bmatrix} I & -R & & & \\ & I & -R & & \\ & & \ddots & & \\ & & & I & -R \\ & & & & R \end{bmatrix} \begin{bmatrix} X_o \\ X_1 \\ \vdots \\ X_{m-1} \\ X_m \end{bmatrix} + \begin{bmatrix} O \\ O \\ \vdots \\ O \\ -R \end{bmatrix} X_{m+1} = \begin{bmatrix} C_o \\ C_1 \\ \vdots \\ C_m \end{bmatrix} \quad \ldots \quad (9)$$

If we set $X_{m+1} \neq 0$, ordinary matrix inversion does not give a solution. A generalized inverse solution [8, Appendix] is sought. Upon premultiplying (9) by

$$\begin{bmatrix} I-R & & & \\ & I-R & & \\ & & \ddots & \\ & & & I-R \\ & & & I \end{bmatrix} = \begin{bmatrix} I & R \ldots R^{m} \\ & I \ldots R^{m-1} \\ & \vdots \\ & R \\ & I \end{bmatrix}$$

The general Leontief solution becomes

$$\begin{bmatrix} X_o \\ X_1 \\ \vdots \\ X_m \end{bmatrix} = \begin{bmatrix} I & R & \ldots & R^{m} \\ & I & \ldots & R^{m-1} \\ & & & \vdots \\ & & & R \\ & & & I \end{bmatrix} \begin{bmatrix} C_o \\ C_1 \\ \vdots \\ C_m \end{bmatrix} + \begin{bmatrix} R^{m+1} \\ R^{m} \\ \vdots \\ R \end{bmatrix} \ldots \quad (10)$$

where X_{m+1} is arbitrary. In the original Leontief system X_{m+1} was assumed zero.

Two interesting properties of the dynamic inverse deserve special considerations.

1. The dynamic inverse enjoys a property similar to that of the inverse of the Leontief matrix in the open static model [10, pp. 142-150] [3, p. 301]. We know under certain conditions, $(I-A)^{-1} = 1 + A + A^2 + \ldots + A^n$ for n sufficiently large. In the case of the dynamic inverse , first observe that for $t \geq m$, the matrix

$$\begin{bmatrix} O & R & & \\ & O & R & \\ & & \ddots & \\ & & & R \\ & & & O \end{bmatrix}^{t} \to 0$$

thus showing a spectrum of less than 1. Upon the so-called Neumann series expansion, consider the simple case m=3 for our exposition, then

$$\begin{bmatrix} I-R & & \\ & I-R & \\ & & I \end{bmatrix}^{-1} = \left[\begin{bmatrix} I & & \\ & I & \\ & & I \end{bmatrix} - \begin{bmatrix} O & R & \\ & O & R \\ & & O \end{bmatrix}\right]^{-1}$$

$$= \begin{bmatrix} I & & \\ & I & \\ & & I \end{bmatrix} + \begin{bmatrix} O & R & \\ & O & R \\ & & O \end{bmatrix} + \begin{bmatrix} O & R & \\ & O & R \\ & & O \end{bmatrix}^{2} + \begin{bmatrix} O & R & \\ & O & R \\ & & O \end{bmatrix}^{3} + \ldots$$

$$= \begin{bmatrix} I & & \\ & I & \\ & & I \end{bmatrix} + \begin{bmatrix} O & R & \\ & O & R \\ & & O \end{bmatrix} + \begin{bmatrix} O & O & R^2 \\ O & O & O \\ & & O \end{bmatrix} + O + \ldots$$

$$= \begin{bmatrix} I & R & R^2 \\ & I & R \\ & & I \end{bmatrix}$$

2. In line with the famous Frobenius theorem R^{m+1} approaches $d^{m+1}\, r_1\, r_2$ where r_1 and r_2 are respectively right and left eigenvectors of R associated with the largest eigenvalue d such that $r_2 r_1 = 1$. Substitution in equation (10) yields

$$X_0 = C_0 + RC_1 + \ldots + R^m C^m + d^{m+1}\,(r_2\, X_{m+1})\, r_1 \qquad \ldots (11)$$

for sufficiently large values of m.

It is therefore argued by Schinnar that X_{m+1} is fixed at zero by Leontief may not be conceived as the major source of inconsistency with initial conditions. Changes in the value of X_{m+1} will only have a scalar effect on X_o. In lieu of the forward iteration method and the cumbersome iterative but backward integration, Schinnar proceeds with initial consistency as follows.

Let X_o be an identified initial condition. From equation (11) then

$$R^{m+1} X_{m+1} = X_o - (C_o + R\, C_1 + \dots + R^m\, C_m) \quad \dots (12)$$

As R_{m+1} is not of full rank, solution of X_{m+1} is sought using the notion of generalized inverses. Hence

$$X_{m+1} = \bar{P}\,(X_o - C_o - R\, C_1 \dots - R_m\, C_m) + (I - \bar{P}\, P)^Y \dots (13)$$

where P stands for R^{m+1}, $\bar{P}$ for the unique Moore-Penrose generalized inverse of R^{m+1} and Y is an arbitrary vector.

The choice of Y should be such that $X_1, X_2, \dots, X_m \geq 0$ and this requires

$$\begin{bmatrix} R^{m+1} \\ R^m \\ \cdot \\ \cdot \\ \cdot \\ R \end{bmatrix} X_{m+1} \geq - \begin{bmatrix} I & R & \dots & R^m \\ & I & \dots & R^{m-1} \\ & & & \cdot \\ & & & \cdot \\ & & I & R \\ & & & I \end{bmatrix} \begin{bmatrix} C_o \\ C_1 \\ \cdot \\ \cdot \\ \cdot \\ C_m \end{bmatrix} \quad \dots (14)$$

Feeding back (12) into (10), the remaining output levels $X_1, X_2, ..., X_m$ are obtained via backward substitution.

8. CONCLUSION

We began our task with the specification of the open dynamic input-output model. The plausibility of the Dynamic Inverse was then examined for projection purpose. W. Leontief stipulated the Backward Integration Method but it suffered the serious drawback $X_{m+1} = 0$. To overcome the shortcomings of Leontief algorithm, two attempts were thus made. The first is that of Kendrick's Forward Iteration Method starting with a given initial condition. This procedure, though elegant, was developed at the heavy cost of violating Leontief's inter-temporal dynamism. The controversy was finally settled by Schinnar consistent with Leontief's backward substitution method but with an identified initial condition and $X_{m+1} \neq 0$.

APPENDIX I

$$G^{-1} + RG^{-1} + R^2G^{-1} + ... + R^tG^{-1} + ...$$

$$= \sum_{0}^{\infty} [(I-A+B)^{-1} B]^t (I-A+B)^{-1}$$

$$= [I-(I-A+B)^{-1} B]^{-1} (I-A+B)^{-1}$$

$$= \{(I-A+B)^{-1} ((I-A+B)-B)\}^{-1}(I-A+B)^{-1}$$

$$= ((I-A+B)-B)^{-1}((I-A+B)^{-1}(I-A+B)^{-1}$$

$$= (I-A)^{-1}$$

APPENDIX II

Derivation of Equation (7), on page 7

First it is to noted that the equivalent form of equation (6) at time t+1 is

$G_{21}\, X^1{}_{t+1} + G_{22}\, X^2{}_{t+1} = d^2{}_{t+1}$ which implies $X^2{}_{t+1} = G^{-1}(d^2{}_{t+1} - G_{21}\, X^1{}_{t+1})$, as $X^2{}_t = G_{22}{}^{-1}(d^2{}_t - G_{21}\, X^1{}_t)$ at time t. Substituting these expressions of $X^2{}_t$ and $X^2{}_{t+1}$ into (5), we get

$$G_{11}X^1{}_t + G_{12}G_{22}{}^{-1}\,(d^2{}_t - G_{21}X^1{}_t) - B_{11}\, X^1{}_{t+1} - B_{12}\, G^{-1}{}_{22}\,(\, d^2{}_{t+1}\, G_{21}\, X^1{}_{t+1}\,) = d^1{}_t$$

If we effect the necessary multiplications, the result shows

$$G_{11}X^1t + G_{12}G_{22}{}^{-1}d^2{}_t - G_{12}G_{22}{}^{-1}G_{21}X^1{}_t - B_{11}X^1{}_{t+1} - B_{12}G_{22}{}^{-1}d^2{}_{t+1} + B_{12}\,G_{22}{}^{-1}G_{21}X^1{}_{t+1} = d^1{}_t$$

Fully keeping all the terms involving $X^1{}_{t+1}$ on the left hand side, collecting the rest on the right hand side and solving for $X^1{}_{t+1}$, we obtain

$$X^1{}_{t+1} = [B_{12}G_{22}{}^{-1}G_{21} - B_{11}]^{-1}\{(d^1{}_t - G_{22}{}^{-1}d^2{}_t + B_{12}G_{22}{}^{-1}d^2{}_{t+1}) - (G_{11} - G_{12}G_{22}{}^{-1}G_{21})X^1{}_t\}$$

NOTES

[1] The pioneer of input-output analysis is W. Leontief. An expository treatment of the problem together with most of his empirical findings is provided in [7].

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