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DUALITY THEORY FOR THE HOUSEHOLD

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Duality Theory for the Household
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Introduction

Many questions confront every economist who analyzes consumer behavior. The most basic of these include the theoretical structure and the econometric methodology. A comprehensive discussion of neoclassical consumer choice is contained in Barten and Böhm (1982). Deaton (1986) addresses econometric issues for analyzing the relatively naïve, neoclassical demand model. Nadiri (1982) presents a comprehensive survey of neoclassical production theory. Jorgenson (1986) addressed the econometric modeling for producer behavior. These four chapters in previous volumes of this series of handbooks form an essential background for the material that appears in this chapter, and are highly recommended to all readers of this chapter. I hope to build upon, rather than either repeat or translate, any of these works, or any others, in this chapter.

The purpose of this chapter is to present a clear statement of the current status of the theory of consumer choice. The focus is on developing an internally consistent, self-contained framework for the theoretical and empirical analysis of consumer preferences, household production activities, quality attributes, produced nonmarket commodities, and both static and dynamic environments for consumer demand analysis. Outlines of the underlying individual components essential for this framework are presented first. Then a
synthesized model of consumer choice in a static environment is developed to the level and extent possible given the current state of knowledge in this field. This static framework is extended to consumer choice problems in an intertemporal environment. Within the dynamic context, consumer’s expectations for future prices, incomes, asset returns, durable goods in household consumption, and naïve and rational habit formation become essential to the analysis at each stage of the discussion.

The chapter is organized in the following way. Section 1 outlines the main concepts and results of the neoclassical theory of consumer choice that forms the underlying framework, or skeleton, for the generalizations of the subsequent sections. Section 2 develops the theory of household production and connects this modeling framework to the neoclassical model, illustrating important special cases, including hedonic price functions, Becker’s model of household production, Gorman’s and Lancaster’s characteristics model of quality, and the Fisher-Shell repackaging model. Section 3 discusses dynamic versions of neoclassical consumer choice theory, with an emphasis on the ways that consumers’ form expectations about their future economic environment. This section analyzes models of myopic, adaptive, quasi-rational, and rational expectations, as well as perfect foresight. Section 4 presents the economic theory of intertemporal choice in a household production framework with durable stocks. In empirical applications, durable goods can be interpreted variously as stocks of unobservable consumption habits, holdings of durable goods such as housing, automobiles, and/or household appliances, or the current state of knowledge of the time path of the quality attributes of consumer goods. When consumption habits are part of the empirical model, naïve versus rational habit
Duality Theory for the Household

formation becomes a relevant topic of analysis. The final section summarizes the main results and briefly discusses some potential avenues for future theoretical and empirical research in this area.

The emphasis throughout the chapter is the development and analysis of an internally consistent and valid duality for each version consumer choice theory. The goal is to develop and motivate a general, logically consistent modeling framework that provides applied economic and econometric analysts a deeper understanding of the relationships among primary concepts of interest – consumers’ utility and ordinary market demand functions – and what can be reasonably described as secondary functions – indirect utility, expenditure and compensated demand functions – in each of their numerous forms and applications.

1. Neoclassical Demand Theory

Neoclassical consumer choice theory begins with the set of bundles of consumer goods that can be selected by a consuming household, $X$, a subset of a separable topological space. In this and the next section, we take $X$ to be a subset of a finite dimensional Euclidean space. However, later in the chapter, $X$ is best described in terms of (Lebesgue) measurable functions from the interval $[0, T]$ onto a finite dimensional Euclidean space.

Associated with the set $X$ is a binary preference relation, $\succeq$. The notation "$x \succeq y$" means the consumption bundle $x$ is at least as preferred as $y$. The relation $\succeq$ is endowed with properties that ensure that consumer choices are logically consistent. The following is a standard representation:
Jeffrey T. LaFrance

(i) **reflexivity;** \( \forall x \in X, x \succeq x \); 

(ii) **transitivity;** \( \forall x, y, z \in X, x \succeq y \text{ and } y \succeq z \Rightarrow x \succeq z \); 

(iii) **completeness;** \( \forall x, y \in X, \text{ either } x \succeq y \text{ or } y \succeq x \); 

(iv) **closure;** \( \forall x \in X, \text{ the sets } \{x^l \in X : x \succeq x^l\} \text{ and } \{x^l \in X : x \succeq x^l\} \text{ are closed.} \)

These properties imply that \( \succeq \) is a complete ordering on \( X \) and that there exists a continuous utility function, \( u: X \rightarrow \mathbb{R} \), such that \( \forall x, x^l \in X, u(x) \geq u(x^l) \) if and only if \( x \succeq x^l \) (Bowen (1968); Debreu (1954, 1959, 1964); Eilenberg (1941); Rader (1963)). Letting \( \succ \) denote the binary relation “strictly preferred to”, so that \( x \succ y \) means \( x \succeq y \) and not \( y \succeq x \), the following assumptions usually are added to (i)-(iv):

(v) **nonsatiation;** \( \exists x \in X \ni x \succ x^l \ \forall x^l \in X ; \)

(vi) **strict convexity;** \( x \succeq x^l \text{ and } t \in (0,1) \Rightarrow (tx + (1-t)x^l) \succ x^l ; \)

(vii) **survival;** \( \inf \{ p^l x: x \in X \} < m \); and

(viii) \( X \) is convex and bounded from below by \( 0 \), i.e., \( x \in X \Rightarrow x_i \geq 0 \ \forall i \).

In addition to continuity, properties i-viii imply that the utility function is strictly quasi-concave (Arrow and Enthoven (1961)).

The consumer’s decision problem is to choose a bundle of market goods from the set \( X \) that is maximal for \( \succeq \), given market prices, \( p \in \mathbb{R}^n_+ \), and income, \( m \in \mathbb{R}_+ \). Given
Duality Theory for the Household

properties i-viii, this can be represented as \( \sup\{u(x): x \in X, p'x \leq m\} \).\(^1\) Under these conditions, the utility-maximizing demand set is nonempty and a singleton and the budget constraint is satisfied with equality at the optimal choice for the consumption bundle. The utility maximizing quantities demanded, \( x = h(p,m) \), are known as the *Marshallian ordinary demand functions*. Marshallian demands are positive-valued and have the following properties:

(M.1) \( 0^\circ \) homogeneity in \((p,m)\); \( h(p,m) \equiv h(tp,tm) \forall t \geq 0 \);

(M.2) adding up; \( p'h(p,m) \equiv m \); and

(M.3) symmetry and negativity; the matrix of substitution effects,

\[
S \equiv \left[ \frac{\partial h(p,m)}{\partial p'} + \frac{\partial h(p,m)}{\partial m} h(p,m)' \right],
\]

is symmetric and negative semidefinite, provided that \( S \) exists and is continuous.

The maximum level of utility given prices \( p \) and income \( m \), \( v(p,m) \equiv u[h(p,m)] \), is the *indirect utility function*. Under i-viii, the indirect utility function has the following properties:

(V.1) continuous and quasiconvex in \((p,m)\);

(V.2) decreasing and strictly quasiconvex in \( p \);

---

\(^1\) The notation “\( \sup \)” denotes the supremum, or least upper bound, of the objective function on the associated set. Since the utility function is continuous, if the set \( X \) is closed and bounded from below and \( p \gg 0 \), then we can replace “\( \sup \)” with “\( \max \)”. Similarly, the notation “\( \inf \)” denotes the infimum, or greatest lower bound, of the objective function on the choice set. If the choice set is compact (closed and bounded) and the objective function is continuous, then we can replace “\( \inf \)” with “\( \min \)”. 

increasing in $m$;

(V.4) $0^\circ$ homogeneous in $(p,m)$;

(V.5) Roy's identity,

$$h(p,m) = -\left( \frac{\partial v(p,m)/\partial p}{\partial v(p,m)/\partial m} \right),$$

provided the right-hand side is well-defined.

The problem of maximizing utility subject to a budget constraint is associated with the converse problem of minimizing the total expenditure that is necessary to obtain a fixed level of utility, $u$, given market prices $p$, $\inf\{p'x : x \in X, u(x) \geq u\}$. The expenditure minimizing demands, $g(p,u)$, are known as the *Hicksian compensated demand functions*. Hicksian demands are positive valued and have the following properties:

(H.1) $0^\circ$ homogeneous in $p$;

(H.2) the Slutsky equations;

$$\left[ \frac{\partial g(p,u)}{\partial p'} \right] = \left[ \frac{\partial h(p,e(p,u))}{\partial p'} + \frac{\partial h(p,e(p,u))}{\partial m} h(p,e(p,u))' \right]$$

is symmetric and negative semidefinite, provided the derivatives exist and are continuous.

The expenditure function, $e(p,u) = p'g(p,u)$, has the following properties:

(E.1) continuous in $(p,u)$;

(E.3) increasing, $1^\circ$ homogeneous, and concave in $p$;

(E.4) increasing in $u$; and
Shephard's Lemma,

\[ g(p,u) \equiv \frac{\partial e(p,u)}{\partial p} , \]

provided the derivatives on the right exist.

A large body of theoretical and empirical literature exists for the neoclassical model of consumer choice. Much of this literature is based on the observation that \( e(p,u) \) and \( v(p,m) \) are inverse functions with respect to their \( n+1 \)st arguments, yielding, \textit{inter alia}, the following set of identities:

(1.1) \[ e[p,v(p,m)] \equiv m ; \]

(1.2) \[ v[p,e(p,u)] \equiv u ; \]

(1.3) \[ g(p,u) \equiv h[p,e(p,u)] ; \text{and} \]

(1.4) \[ h(p,m) \equiv g[p,v(p,m)] . \]

However, the neoclassical model has few empirical implications, embodied in the sign and symmetry of the substitution effects due to changes in the market prices of the goods \( x \), and leaves all variances in consumption behavior not explained by prices and income to differences in tastes and preferences. The neoclassical model also is entirely static. Finally, the neoclassical model does not readily accommodate technological change, the introduction of new goods in the market, or changes in the quality or characteristics of the goods that are available.

These considerations led to extensions of the neoclassical model of consumer choice. Among these extensions, the most widely employed is the theory of household
production. The seminal references are Becker (1965), Gorman (1956), and Lancaster (1966, 1971). Household production theory integrates the neoclassical theory of the consumer with that of the firm. The theory of the firm relates to that part of household decision making that is concerned with the efficient use of market goods, household time, and capital as inputs in the production of utility-yielding non-market commodities. The model posits that market goods and household time are combined via production processes analogous to the production functions of the theory of the firm to produce various commodities from which utility is obtained directly. Household production theory advances the neoclassical model by admitting analyses of issues like the number of family members in the work force, time as a constraining factor in consumption choices, quality changes among goods, durable goods in consumption, and consumer responses to the introduction of new goods.

2. The Theory of Household Production

In this section, we present a model of consumer choice that is sufficiently general and rich to account for many of the concerns summarized above. We first need some preliminary definitions and notation. Let \( x \in \mathbb{R}^n \) denote market goods and time used by the household, let \( b \in \mathbb{R}^p \) be a vector of parameters associated with the market goods, objectively measured and quantifiable by all economic agents, and let \( z \in \mathbb{R}^m \) be a vector of utility bearing commodities or service flows desired by the household and produced from \( x \). We assume that there is a household production relationship for each household relating \( x \) to \( z \), and this relationship depends explicitly on the parameters, \( b \). For given \( b \), let
$T(b) \subseteq \mathbb{R}^{m+n}$ denote a joint production possibilities set and let $y = [x' z']' \in T(b)$ denote a feasible vector of goods and commodities. For each possible $b$, the properties of $T(b)$ associated with a well-defined joint production function are (Rockafellar (1970); Jorgenson and Lau(1974)):

(i) origin; $\mathbf{0} \in T(b)$;

(ii) bounded; $\forall \, i, \, y \in T(b)$ and $|y_j| < \infty \forall \, j \neq i \Rightarrow |y_i| < \infty$,

(iii) closure; $y^n \in T(b) \forall \, n$ and $y^n \to y \Rightarrow y \in T(b)$;

(iv) convexity; $y, y^1 \in T(b)$ and $t \in [0, 1] \Rightarrow ty + (1-t)y^1 \in T(b)$;

(v) monotonicity; $\exists \, i \ni y \in T(b), \, y'_j = y_j \forall \, j \neq i$, and $y'_i \leq y_i \Rightarrow y' \in T(b)$.

Given i-v, we define the production function by

\[(2.1) \quad -F(y_{-i}, b) = \sup\{y_i; \, y \in T(b)\},\]

where $y_{-i}$ is the subvector of elements excluding the $i^{th}$ with $i$ chosen to satisfy v. Then $\forall \, y \in T(b)$, we have $y_i + F(y_{-i}, b) \leq 0$. For a given value of $b$, the function $F(\cdot, b)$ is closed (lower semicontinuous), proper ($F(y_{-i}, b) < +\infty$ for at least one $y_{-i}$ and $F(y_{-i}, b) > -\infty \forall \, y_{-i}$), and convex. Monotonicity in at least one element of $y$ (i.e., free disposal of $y_i$) is equivalent to applicability of the implicit function theorem to the transformation $G(y, b) = 0$, which defines the boundary of $T(b)$, to obtain the form $(2.1)$. Free disposal of all elements of $y$ implies monotonicity of $G(\cdot, b)$ in $y$.

Defining the epigraph of $F(\cdot, b)$ to be the set

\[(2.2) \quad T^*(b) = \{ y \in \mathbb{R}^{m+n}; \, -y_i \geq F(y_{-i}, b)\},\]
it follows immediately from the definition of $F(\cdot, b)$ that $T^*(b) \equiv T(b)$. Therefore, since a convex function is defined by its epigraph — equivalently, a closed convex function is the pointwise supremum of all affine functions that are majorized by it (Rockafellar (1970), Theorem 12.1), while a closed convex set is the intersection of all of the closed half spaces defined by its supporting hyperplanes — the properties of $F(\cdot, b)$ imply the properties of $T(b)$, and conversely.

Hence, let the goods/commodities/qualities efficient transformation frontier be defined by the implicit function $G(x, z, b) = 0$. We interpret $G(\cdot)$ to be a joint household production function with inputs $x$, outputs $z$, and parameters $b$. $G(\cdot, b)$ is convex in $(x, z)$, increasing in $z$, decreasing in $x$, and without loss in generality, strictly increasing in $z_1$. For given $b$, the feasible goods/commodities production possibilities set is defined in terms of $G(\cdot, b)$ by

\begin{equation}
T(b) = \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m : G(x, z, b) \leq 0\}.
\end{equation}

For fixed $b$, $T(b)$ is non-empty, closed, and convex; $G(0, 0, b) = 0$; and $G(x, z, b) = 0$ and $z \gg 0$ imply that $x \geq 0$, where $x \geq 0$ means $x_j \geq 0 \ \forall \ j$ and $x \neq 0$.

We assume that the correspondence $T(b)$ is continuous over the set of parameter vectors, $B \subset \mathbb{R}^s$, and that boundedness and closure of $T(b)$ hold throughout $B$. These conditions ensure that $G(x, z, \cdot)$ is continuous in $b \in B$, which can be demonstrated in the following way. Define $F(x, z, b)$ by

\begin{equation}
-F(x, z, b) = \sup \{z_i : (x, z) \in T(b)\}.
\end{equation}
Let $G(x,z,b) = z_1 + F(x,z,b)$ if $(x^n, z^n) \in T(b^n) \forall n$ and $(x^n, z^n, b^n) \rightarrow (x, z, b)$, then $(x, z) \in T(b)$ by the continuity of $T(\cdot)$ in $b$. Uniqueness of the supremum implies $z_1 \leq -F(x, z, b)$. On the other hand, $z_1'' = -F(x^n, z^n, b^n) \forall n$ implies that $z_1 \geq -F(x, z, b)$. Continuity of $G(x, z, \cdot)$ follows immediately. This means simply that the boundary of the feasible set $T(b)$ is connected and contained in $T(b)$, and that small changes in $b$ do not induce large changes in the boundary of $T(b)$. Therefore the greatest possible output of $z_1$ does not change much either.

The above conditions on the set $T(b)$ are standard in the general theory of the firm and do not exclude cases where the commodity vector $z$ includes some or all of the market goods. For example, if $m = n$ and $z_i \equiv x_i \forall i$, then the model reduces to the neoclassical framework where market goods are the desired commodities from which utility is derived directly. More generally, for any pair $i$ and $j$ such that $z_i \equiv x_j$, we can simply incorporate the function $z_i - x_j$ into the definition of the transformation function $G(\cdot)$.

An important aspect of the manner in which we have set up this model is the way in which time allocation tacitly enters the decision problem. If $t_0$ is the vector of labor times supplied to the market and $w$ the vector of market wage rates received for this labor, then the budget constraint has the form $p' x_n \leq y + w' t_0$, where $y$ is non-labor income. Let $t$ denote the vector of household times used in the production of nonmarket commodities. Then $x = [x', t']'$ and the joint household production function tacitly depends upon $t$. Finally, if $T$ is the vector of time endowments, then $T - t - t_0$ is a vector of leisure.
times and is tacitly included as part of the vector \( z \). The vector of time constraints take the form \( t + t_0 = T \) (Becker (1965); Deaton and Muellbauer (1980); Pollak and Wachter (1975)). Substituting \( t_0 = T - t \) into the budget constraint gives \( p'_t x = w'(T - t) \), or equivalently, \( p'_t x_t + w't \leq y + w'T \equiv m \), where \( m \) is the household's full income (Becker (1965)).

2.1 Static Consumer Choice Theory with Household Production

In addition to the joint transformation function relating goods, commodities, and qualities, we assume that there exists a continuous, quasiconcave utility function defined over the space of commodities, \( u(z) \), such that \( u(z) \geq u(z') \) if and only if \( z \succeq z' \), where \( \succeq \) is the binary preference relation defined over commodities. The consumer's decision problem is taken to be to seek a combination of market goods and household time, \( x \), that will produce the vector of commodities, \( z \), that maximizes utility, \( u(z) \), subject to a budget constraint, \( p'x \leq m \). Here \( p \) is an \( n \)-vector of market prices, defined by \( p = (p'_{x_t}, w')' \) and \( m = y + w'T \) is the household's full income. In addition to the standard budget constraint, the choice problem is subject to the constraint that the vector of commodities produced from the market goods and household time is feasible, \( G(x, z, b) \leq 0 \). This problem is related to the neoclassical consumer choice problem by the following theorem, which permits the translation of the utility function defined over produced nonmarket commodities to an induced utility function defined over market goods and time.

**Theorem 1:** For each \( (x, z) \in T(b)^o \), the interior of \( T(b) \), suppose that the set of feasible commodity bundles, \( W(x, b) = \{ z \in \mathbb{R}^m: G(x, z, b) \leq 0 \} \), has a non-empty
Then the induced utility function on the space of market goods,

\[ u^*: \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}, \text{ defined by} \]

\[ u^*(x,b) \equiv \sup \{ u(z) : z \in W(x,b) \}, \]

is strictly quasiconcave in \( x \).

**PROOF:** Fix two feasible goods vectors, \( x \) and \( x^1 \) such that \( u^*(x,b) = k \) and \( u^*(x^1,b) \geq k \) for some real number \( k \). Define \( x^2 = tx + (1-t)x^1 \) for some \( t \in (0,1) \). Pick vectors \( z, z^1 \) and \( z^2 \) to satisfy \( G(z,x,b) \leq 0, \ G(z^1,x^1,b) \leq 0, \ G(z^2,x^2,b) \leq 0, \ u(z) = u^*(x,b), \ u(z^1) = u^*(x^1,b), \) and \( u(z^2) = u^*(x^2,b) \). The strict convexity of \( G(\cdot,b) \) implies that \( G[tz + (1-t)z^1, x^1, b] < 0 \); hence \( tz + (1-t)z^1 \) is a feasible commodity bundle. On the other hand, strict quasiconcavity of \( u(\cdot) \) implies that \( u[tz + (1-t)z^1] > u(z) \). Consequently, \( u^*(x^2,b) = u(z^2) \geq u[tz + (1-t)z^1] > u(z) = k. \)

This result permits the household production model to be translated into the standard neoclassical model of consumer choice. The consumer choice problem of maximizing \( u(z) \) subject to the constraints \( \rho'x \leq m \) and \( G(x,z,b) \leq 0 \) thus can be represented in terms of the simpler problem of choosing \( x \) to maximize \( u^*(x,b) \) subject to \( \rho'x \leq m \).

Quasiconcavity of \( u^*(\cdot,b) \) implies that the resulting demands for the goods \( x \) have the standard properties of neoclassical demand functions. However, these demands also relay information regarding both preferences and consumption technology (Pollak and Wachter (1975); Barnett (1977)).

Under some general regularity conditions on \( u \) and \( G \), the following theorem de-
scribes the basic properties of the function $u^*(x,b)$ and the nature of the dual information on preferences and household consumption technology relayed by it. With little loss in generality, we restrict our attention to the case of strictly positive commodity consumption bundles.

**Theorem 2:** Suppose that $u(\cdot)$ is strictly quasiconcave and continuously differentiable, $G(\cdot)$ is strictly convex in $z$ and continuously differentiable in $(x,z,b)$, $W(x,b)$ has a non-empty interior, $u(\cdot)$ is nonsatiated on $T(b)$, and the optimal commodity vector satisfies $z(x,b) \gg 0$. Then $u^*(\cdot)$ has the following properties:

1. $u^*(\cdot, b)$ is increasing in $x$;
2. $u^*(\cdot)$ is continuously differentiable in $(x,b)$;
3. $\frac{\partial u^*}{\partial x} = -\left(\frac{\partial u}{\partial z_1}\right) \cdot \left(\frac{\partial G}{\partial x}/\frac{\partial G}{\partial z_1}\right)$;
4. $\frac{\partial u^*}{\partial b} = -\left(\frac{\partial u}{\partial z_1}\right) \cdot \left(\frac{\partial G}{\partial b}/\frac{\partial G}{\partial z_1}\right)$;
5. $\text{sgn} \left(\frac{\partial u^*}{\partial b_k}\right) = -\text{sgn} \left(\frac{\partial G}{\partial b_k}\right) \forall k$;
6. The preference map defined by $u^*(\cdot)$ is invariant to increasing transformations of $u(\cdot)$ and $G(\cdot)$.

**Proof:** Let the Lagrangean for the constrained maximization problem be

$$L = u(z) - \mu G(x,z,b),$$

where $\mu \geq 0$ is a Lagrange multiplier. Strict quasiconcavity of $u(\cdot)$ and strict convexity of $G(\cdot)$ in $z$ imply that the necessary and sufficient conditions for an interior constrained maximum are

$$\frac{\partial L}{\partial z} = \frac{\partial u}{\partial z} - \mu \frac{\partial G}{\partial z} = 0,$$
Duality Theory for the Household

\[ \frac{\partial L}{\partial \mu} = -G(x,z,b) = 0 , \]

where the second condition follows from nonsatiation of \( u \) throughout \( T(b) \).

Substituting the solution functions, \( z(x,b) \), into the first order conditions generates identities in a neighborhood of the point \( (x,b) \). Define the indirect objective function by

\[ u^*(x,b) \equiv u[z(x,b)] \] and the constraint identity by \( G[x,z(x,b),b] \equiv 0 \). Substituting these and the optimal solution function for the Lagrange multiplier, \( \mu (x,b) \), into the Lagrangean gives \( u^*(x,b) \equiv L(x,b) \). Differentiating with respect to \( x \) then gives

\[ \frac{\partial u^*}{\partial x} = (\frac{\partial u}{\partial z} - \mu \frac{\partial G}{\partial z}) \frac{\partial z}{\partial x} - (\frac{\partial \mu}{\partial x}) G - \mu \frac{\partial G}{\partial x} . \]

The first two terms on the right-hand side vanish and the Lagrange multiplier can be written as

\[ \mu = \frac{\partial u/\partial z_i}{\partial G/\partial z_j} > 0 . \]

Combining the right-hand-side expressions gives,

\[ \frac{\partial u^*}{\partial x} = -(\frac{\partial u}{\partial z_i}) \cdot (\frac{\partial G}{\partial x})(\partial G/\partial z_j) . \]

A completely analogous argument applies to \( \frac{\partial u^*}{\partial b} \), verifying (2.a) - (2.e).

To demonstrate (2.f), let \( w : \mathbb{R} \to \mathbb{R} \) and \( v : \mathbb{R} \to \mathbb{R} \) be arbitrary strictly increasing, continuously differentiable functions. Define the set

\[ W^*(x,b) \equiv \{ z \in \mathbb{R}^m : v[G(x,z,b)] \leq v(0) \} . \]

Then \( W^*(x,b) \equiv W(x,b) \); that is, \( z \in W(x,b) \) if and only if \( z \in W^*(x,b) \) and \( u(z) \) achieves a maximum in \( W^*(x,b) \) at the same point as it does in \( W(x,b) \). Similarly, \( w[u(z)] \) achieves a maximum on either \( W(x,b) \) or \( W^*(x,b) \) if and only if \( u(z) \) does. Finally, let

\[ L^* = w[u(z)] + \mu \{ v(0) - v[G(x,z,b)] \} . \]
The first-order conditions for an interior constrained maximum now are
\[
\frac{\partial L^*}{\partial z} = w^* \frac{\partial u}{\partial z} - \mu^* v^* \frac{\partial G}{\partial z} = 0,
\]
\[
\frac{\partial L^*}{\partial \mu^*} = v(0) - v[G(x, z, b)] = 0.
\]
Since \( v^{-1}[v(0)] = 0 \), the latter is equivalent to \( G(x, z, b) = 0 \). The former can be written as
\[
(w^* \frac{\partial u}{\partial z_i})/(w^* \frac{\partial u}{\partial z_i}) = (\mu^* v^* \frac{\partial G}{\partial z_i})/(\mu^* v^* \frac{\partial G}{\partial z_i}) \quad \forall \, i,
\]
which after canceling \( (w^*/w^*) \) and \( (\mu^* v^*/\mu^* v^*) \) is equivalent to the conditions implied by
\[
\frac{\partial u}{\partial z} - \mu \frac{\partial G}{\partial z} = 0.
\]

Several additional properties of the consumer choice model that uses \( u^*(x, b) \) as its starting point can be derived from the duality theory of the neoclassical model. Consider the problem,
\[
(2.1.1) \quad v^*(p, m, b) \equiv \sup\{u^*(x, b): p'x \leq m, x \in \mathbb{R}^n\}.
\]
This generates a vector of ordinary market demands, \( x^* = h^*(p, m, b) \), a Lagrange multiplier, \( \lambda^*(p, m, b) \), and an indirect utility function, \( v^*(p, m, b) \equiv u^*[h^*(p, m, b), b] \). Then consider an artificial two-stage formulation of the problem,
\[
(2.1.2) \quad v(p, m, b) \equiv \sup\{u(z): p'x \leq m, G(x, z, b) \leq 0\}
\]
\[
\equiv \sup\{\sup\{u(z): G(x, z, b) \leq 0\}: p'x \leq m\}.
\]
This yields a vector of ordinary market demands, \( x = h(p, m, b) \), a vector of nonmarket commodity demands, \( z = f(p, m, b) \), a Lagrange multiplier, \( \lambda(p, m, b) \), and an indirect utility function, \( v(p, m, b) \equiv u[f(p, m, b)] \). By the uniqueness of the supremum, we have
Duality Theory for the Household

\[ v(p, m, b) \equiv v^*(p, m, b). \]

This simple observation leads directly to the following result, which for brevity is stated in terms of \( v(\cdot) \).

**Theorem 3.** Under the conditions of Theorem 2, \( v(p, m, b) \) is continuously differentiable, homogeneous of degree zero in \((p, m)\), increasing in \( m \), quasiconvex and decreasing in \( p \), and

\[
\frac{\partial v}{\partial m} = \lambda(p, m, b); \tag{3.a}
\]

\[
h(p, m, b) \equiv -\left(\frac{\partial v}{\partial p}\right)\left(\frac{\partial v}{\partial m}\right) \text{ so long as the right-hand-side is well-defined}; \tag{3.b}
\]

\[
\frac{\partial v}{\partial b} \equiv -\lambda \frac{\partial G}{\partial b} \equiv \frac{\partial u^*}{\partial b}; \tag{3.c}
\]

\[
G[h(p, m, b), f(p, m, b), b] \equiv 0; \text{ and} \tag{3.d}
\]

\[
p'h(p, m, b) \equiv m. \tag{3.e}
\]

**PROOF:** Continuous differentiability of \( v(\cdot) \) follows from the continuous differentiability of \( u(\cdot) \) and \( G(\cdot) \), strict quasiconcavity of \( u(\cdot) \), and strict convexity of \( G(\cdot, b) \). Zero degree homogeneity follows from the fact that the budget set does not change if we multiply both sides by a positive constant, \( p'x \leq m \) if and only if \( tp'x \leq tm \forall t > 0 \). For monotonicity, if \( p \geq p^o \), then \( B = \{x: p'x \leq m\} \subset B^o = \{x: (p^o)'x \leq m\} \), and the maximum of \( u^*(x, b) \) over \( B^o \) can be no less than the maximum over \( B \). Hence \( v(\cdot, m, b) \) is decreasing in \( p \). The proof of monotonicity in \( m \) is of the same nature.

For quasiconvexity in \( p \), fix \( p \) and \( p^1 \) such that \( v(p, m, b) \leq k \) and \( v(p^1, m, b) \leq k \)
for some real number \( k \). Define \( p^2 = tp + (1-t)p^1 \) for \( t \in [0,1] \). If \( tp'x + (1-t)(p^1)'x \leq m \) then \( p'x \leq m \), \( (p^1)'x \leq m \), or both. Otherwise, \( p'x > m \Rightarrow tp'x > tm \) and \( (p^1)'x > m \Rightarrow (1-t)(p^1)'x > (1-t)m \). Summing the last inequalities in each of case implies \( tp'x + (1-t)(p^1)'x > m \), which is a contradiction. Therefore,

\[
v(p^2, m, b) = \sup\{u^*(x, b) \colon (p^2)'x \leq m\} \leq \max\{v(p, m, b), v(p^1, m, b)\} \leq k .
\]

Properties a - c are the result of the envelope theorem, d and e follow from the nonsatiation of \( u(\cdot) \).

This theorem simply states that there will be one and only one maximum level of utility, and the associated choice functions are invariant to the manner in which we choose to view the consumer choice problem, provided of course that the problem is well-behaved. In addition, it shows how the joint preference and household production technology information contained in \( u^*(\cdot) \) is transmitted to the indirect utility function \( v(\cdot) \) and hence the demand functions \( h(\cdot) \).

Analogous to the neoclassical consumer choice model, the converse to the utility maximization problem is the expenditure minimization problem,

\[
(2.1.4) \quad e(p, u, b) = \inf\{p'x \colon u^*(x, b) \geq u, \ x \in \mathbb{R}_+^n\}.
\]

This generates a vector of Hicksian compensated demands, \( x = g(p, u, b) \), a Lagrange multiplier, \( \mu(p, u, b) \), and an expenditure function, \( e(p, u, b) \equiv p'g(p, u, b) \). Our next result relates the indirect utility function to the expenditure function.

**Theorem 4:** Under the conditions of theorem 2, the expenditure function is in-
creasing, \(1^\circ\) homogeneous, and concave in \(p\); increasing in \(u\); continuously differentiable in \((p, u, b)\); and,

\[(4.a) \quad \frac{\partial e(p,u,b)}{\partial p} = g(p,u,b);\]

\[(4.b) \quad \frac{\partial e(p,u,b)}{\partial u} = \mu(p,u,b);\]

\[(4.c) \quad \frac{\partial e(p,u,b)}{\partial b} = -\mu(p,u,b) \cdot \partial u^*(g(p,u,b),b)/\partial b;\]

\[(4.d) \quad e[p,v(p,m,b),b] = m;\]

\[(4.e) \quad v[p,e(p,u,b),b] = u;\]

\[(4.f) \quad g[p,v(p,m,b),b] = h(p,m,b);\]

\[(4.g) \quad h[p,e(p,u,b),b] = g(p,u,b);\]

\[(4.h) \quad \mu(p,u,b) = \lambda[p,e(p,u,b),b]^{-1};\]

\[(4.i) \quad \lambda(p,m,b) = \mu[p,v(p,m,b),b]^{-1}.\]

**PROOF:** With the exception of c, these are all straightforward duality results proved in the same manner as in the neoclassical utility theory model. Property c follows from the envelope theorem.

Thus, the static consumer choice model with household production inherits all of the properties of the indirect utility function and expenditure function from the neoclassical model. The relationship between the Marshallian ordinary demands and Hicksian compensated demands with respect to the parameters \(b\) are summarized in the next theorem.

**Theorem 5:** The Marshallian demand functions are positive-valued, continuous and \(0^\circ\) homogeneous in \((p, m)\), and so long as the associated derivatives exist and
are continuous,

\[
(5.a) \quad \frac{\partial g(p,u,b)}{\partial b'} = \frac{\partial h(p,e(p,u,b),b)}{\partial b'} + \frac{\partial h(p,e(p,u,b),b)}{\partial m} \cdot \frac{\partial e(p,u,b)}{\partial b'}, \text{ and}
\]

\[
(5.b) \quad p' \frac{\partial h(p,m,b)}{\partial b'} \equiv 0'.
\]

**PROOF:** These results follow directly from differentiating 4.g and 3.e with respect to \(b\). □

### 2.2 Hedonic Price Functions

An alternative approach to the consumer choice problem with a household production function as an added constraint is obtained by focusing on the cost of obtaining a given vector of commodities from market goods and household time given market prices. This perspective is called the method of *hedonic price functions* (Lucas (1975); Muellbauer (1974); Rosen (1974)). To develop this approach in the current framework, we require a version of the Shephard-Uzawa-McFadden duality theorem for cost functions.

**Theorem 6:** For a feasible commodity vector \(z\), the cost function,

\[
c(p,z,b) \equiv \inf\{p'x: G(x,z,b) \leq 0, \ x \in \mathbb{R}_+^n\},
\]

is continuous in \((p,z,b)\), \(1^o\) homogeneous, increasing and concave in \(p\), increasing and convex in \(z\). When the inputs requirements set

\[
X(z,b) = \{x \in \mathbb{R}_+^n: G(x,z,b) \leq 0\}
\]

is strictly convex and \(\partial c(p,z,b)/\partial p\) exists, \(c(p,z,b)\) obeys Shephard’s Lemma (Shephard (1953)),

\[
G[\partial c(p,z,b)/\partial p,z,b] \equiv 0.
\]

When \(G\) is differentiable in \(z\) the ratio of marginal costs of two commodities is
equal to the marginal rate of transformation between them,
\[
(\partial c/\partial z_i)/(\partial c/\partial z_{i'}) = (\partial G/\partial z_i)/(\partial G/\partial z_{i'}) \quad \forall \ i, i'.
\]

PROOF: Convexity of \( T(b) \) implies convexity of \( X(z, b) \). To see this, let \( x^0 \) and \( x^1 \) be any two points such that \( G(x^0, z, b) \leq 0 \) and \( G(x^1, z, b) \leq 0 \). Define \( x^2 = tx^0 + (1 - t)x^1 \) for \( t \in [0, 1] \). Then \( G(x^2, z, b) \leq 0 \), since \( z = tz + (1 - t)z \). Similarly, closure of \( T(b) \) implies closure of \( X(x, b) \). To see this, let \( \{z^k\} \) and \( \{x^k\} \) be any two sequences such that \( z^k = z \ \forall k, \ x^k \to x \), and \( G(x^k, z^k, b) \leq 0 \ \forall k \). Then closure of \( T(b) \) implies that \( G(x, z, b) \leq 0 \); hence \( X(z, b) \) is closed. Decreasing monotonicity of \( G \) in \( x \) implies free disposal, since \( x \geq x^0 \) and \( G(x^0, z, b) = 0 \Rightarrow G(x, z, b) \leq 0 \). The nonemptiness of \( T(b) \) and the feasibility of \( z \) imply \( X(z, b) \neq \emptyset \). Theorems 1 and 2 of Uzawa (1964) and Lemma 1 of McFadden (1973) follow from this set of hypotheses, proving continuity, monotonicity, \( 1^\circ \) homogeneity, and concavity in \( p \).

To prove convexity in \( z \), fix two feasible outputs \( z^0 \) and \( z^1 \), and define \( z^2 = tz^0 + (1 - t)z^1 \) for \( t \in [0, 1] \). Let \( p'x^0 = c(p, z^0, b) \) and \( p'x^1 = c(p, z^1, b) \). Then \( p'x^2 = c(p, z^2, b) \leq p'x \ \forall x \in X(z^2, b) \). In particular, let \( x = tx^0 + (1 - t)x^1 \). Convexity of \( T(b) \Rightarrow (x, z^2) \in T(b) \). Therefore, since \( x \) is feasible but not necessarily optimal for \( z^2 \),
\[
c(p, z^2, b) \leq p'x = tp'x^0 + (1 - t)p'x^1 = tc(p, z^0, b) + (1 - t)c(p, z^1, b).
\]

When \( G(\cdot, z, b) \) is strictly convex in \( x \), so that the input requirements sets are strictly convex, Shephard's Lemma is obtained from the primal-dual function
\[
\phi(p, x, z, b) = p'x - c(p, z, b).
\]
\( \phi(\cdot) \) is non-negative for all \( x \in X(z,b) \) and attains a minimum at \( x = x(p,z,b) \), the cost-minimizing bundle of market goods. If \( \partial c/\partial p \) exists, minimizing \( \phi(\cdot) \) with respect to \( p \) requires

\[
\frac{\partial \phi(p,x,z,b)}{\partial p} = x - \frac{\partial c(p,z,b)}{\partial p} = 0.
\]

An additional property of the cost function is that if \( G(\cdot, b) \) exhibits constant returns to scale with respect to \( x \) and \( z \), so that \( G(x,z,b) = 0 \Rightarrow G(tx,tz,b) = 0 \forall t \geq 0 \), then \( c(p,z,b) \) is 1° homogeneous in \( z \) (Hall (1973)). By Euler's theorem we then have

\[
(2.2.1) \quad c(p,z,b) = \frac{\partial c(p,z,b)}{\partial z} z.
\]

The hedonic price for the \( i^{th} \) commodity is the marginal cost of its production, \( \rho_i(p,z,b) = \frac{\partial c(p,z,b)}{\partial z_i} \). Under constant returns to scale, then, we can represent the consumer's choice problem as

\[
(2.2.2) \quad \sup \{ u(z) : \rho'z \leq m, \ z \in \mathbb{R}^m \}.
\]

Now, suppose that we define the implicit commodity prices conditionally on the optimal level of commodity consumption, \( z^* = f(p,m,b) \), and then solve the household's choice problem given \( \rho^* = \rho(p,z^*,b) \). These conditional shadow prices define the hyperplane that separates the projection of the production possibility set onto the \( m \)-dimensional commodity subspace from the upper contour set of the utility function. Under constant returns to scale, the commodity demand vector, \( z^* = z(\rho^*,m,b) \), possesses all of the properties of neoclassical demand functions with respect to \( (\rho^*, m) \) (Barnett (1977)). That is, taking \( \rho^* \) as a vector of constants associated only with the separating hyperplane, the
functions \( z(\rho^*, m, b) \) are 0º homogeneous in \((\rho^*, m)\), satisfy the adding up condition, 
\[ \rho^* z(\rho^*, m, b) \equiv m, \]
and obey the Slutsky sign and symmetry conditions with respect to \( \rho^* \) and income \( m \), \( \partial z/\partial \rho^* + (\partial z/\partial m) z' \) is symmetric and negative semidefinite.

But the unconditional commodity shadow prices are defined by
\[ (2.2.3) \quad \rho(p, m, b) \equiv \rho^*(p, f(p, m, b), b), \]
so that we have the identity
\[ (2.2.4) \quad z(\rho(p, m, b), m, b) \equiv f(p, m, b). \]

Therefore, the hedonic prices for the non-market commodities and the commodity demands are necessarily simultaneously determined. Consequently, simply estimating either \( \rho^* = \rho(p, z^*, b) \) or \( z^* = z(\rho^*, m, b) \) with standard techniques leads to biased and inconsistent empirical results (Pollak and Wachter (1975)). Moreover, without constant returns to scale,
\[ (2.2.5) \quad m \equiv p'h(p, m, b) \equiv c[p, f(p, m, b), b] \neq \rho(p, m, b)'f(p, m, b), \]
and the above results for \( z(\rho^*, m, b) \) no longer hold. Even with constant returns to scale, the hedonic price functions relay information regarding both consumer preferences and the household production function.

The simultaneity between the hedonic price functions and the commodity demand functions is overcome when the household production function displays both constant returns to scale and nonjoint production. When both of these conditions are satisfied, the joint cost function takes the additively separable form (Hall (1973); Muellbauer (1974); Samuelson (1966)).
(2.2.6) \[ c(p, z, b) = \bar{c}(p, b)'z, \]

where \( \bar{c}(p, b) \) is the cost of producing a unit of the \( i^{th} \) commodity. In this case, \( \rho(p, z, b) \equiv \bar{c}(p, b) \) is independent of \( z \), so that constant returns to scale and nonjoint production imply a linear budget constraint in commodities space. As long as the Jacobian matrix for the hedonic price functions is of full rank, that is, \( \min(m, n) = \text{rank}[\partial \bar{c}/\partial p'] \), the shadow price functions can be locally inverted to give price functions of the form \( p = \varphi(\rho, b) \). This is the form in which nearly all empirical analyses of hedonic price functions have been undertaken, with the commodity shadow prices \( \rho \) estimated as constants in a linear or log-linear regression equation.

However, Pollak and Wachter point out that nonjointness is a restrictive assumption since it implies that the time spent in household production activities cannot yield utility except in terms of the amount of leisure time that is reduced by these activities. Moreover, we expect a priori that different households have different consumption technologies, and hence, different implicit price equations even if we are willing to impose constant returns to scale and nonjoint production. This leads to different unit cost functions for the commodity outputs and different hedonic price relationships for different households.

Additionally, in any model of equilibrium price relationships, demand and supply conditions combine in the marketplace to create market clearing prices (Rosen (1974); Lucas (1975)). The implicit prices for quality calculated from an hedonic price equation therefore represent the marginal conditions equating supply and demand, mapping ob-
served market quantities, prices, and qualities into a single point in the space of producer
costs and consumer preferences. Consequently, these relationships yield information
about consumer preferences only in the sense of market equilibrium conditions. They do
not contain any information regarding the direction or size of quantity or price changes
that are likely to result from changes in the quality levels contained in market goods.

2.3 Special Cases

The Gorman/Lancaster model of product characteristics arises when the household pro-
duction function can be decomposed into the linear system

\[ z_i = \sum_{j=1}^{n} b_j x_j, \quad i = 1, \ldots, m \]

where \( s = m \cdot n \). The utility function over goods and qualities is then of the form

\[ u^*(x, b) = u\left(\sum_{j=1}^{n} b_{1j} x_j, \sum_{j=1}^{n} b_{2j} x_j, \ldots, \sum_{j=1}^{n} b_{mj} x_j\right) \]

The Muth/Becker/Michael model results when \( G(x, z, b) = 0 \) is nonjoint, so that each
commodity is produced by an individual production function of the form

\[ z_i = f_i(x_{(i)}, b_{(i)}), \quad i = 1, \ldots, m, \]

where \( x = \sum_{i=1}^{m} x_{(i)} \) and \( b = [b_{11}', b_{12}', \ldots, b_{(m)}]' \). In this case, the utility function for goods has
the form

\[ u^*(x, b) = u\left(f_1(x_{(1)}, b_{(1)}), \ldots, f_m(x_{(m)}, b_{(m)})\right) \]

The additional property of constant returns to scale, advocated by Muth as merely a ques-
tion of definitions, generates demand-side hedonic price equations discussed above.

Many applications of the hedonic price function model of quality employ transla-
tion and scaling methods (Pollak and Wales (1981)) to produce a utility function defined over goods and qualities in the scaled form

\[ u^*(x, b) = u(\varphi_1(b_{11})x_1, \cdots, \varphi_n(b_{nn})x_n), \]

or in the translated form

\[ u^*(x, b) = u(x_1 + \varphi_1(b_{11}), \cdots, x_n + \varphi_n(b_{nn})), \]

or a combination of the two. In the former case, the ordinary demand functions take the form

\[ x_i = h^i(p_i/\varphi_1(b_{11}), \cdots, p_n/\varphi_n(b_{nn}), m)/\varphi_i(b_{ii}), \]

which is commonly called the Fisher-Shell repackaging model (Fisher and Shell (1971)). In the latter case, the ordinary demand functions take the form

\[ x_i = h^i(p, m + \sum_{i=1}^n p_i\varphi_i(b_{ii}))/\varphi_i(b_{ii}), \]

(Hanemann (1980); Pollak and Wales (1981)). In both cases, one implication is that preferences are weakly separable in the partition \{((x_1, b_{11}), \cdots, (x_n, b_{nn}))\}. However, this will not be the case if goods are scaled by an \(n \times n\) matrix, say \(A(b) = [\varphi_y(b)]\), that is nonsingular and has nonzero off-diagonal elements. Then the Marshallian ordinary demands take the form (Samuelson (1948), p. 137),

\[ x = A(b)^{-1}h(A(b)^{-1}p, m). \]

The Gorman/Lancaster characteristics model is simply a special case of (2.3.9).

3. Intertemporal Models of Consumer Choice

Many studies of food consumption use time series data. The static neoclassical model of
consumer choice has been extended to accommodate the analysis of household decisions over time. In this section, we discuss models of intertemporal consumer choice that combine the models of the previous sections with the dynamic nature of household decision problems. Initially, we consider a model that mirrors the static neoclassical theory of consumer choice through additively separable preferences across points in time. The processes that consumers use to form expectations about the future values of economic factors such as prices and incomes are important aspects of dynamic models. Intertemporal models of household production are considered next, including the purchase, use, depreciation, maintenance, and replacement of durable household goods. Since some of these durable stocks can be interpreted as consumption habits, naïve versus rational models of habit formation is an important issue in this framework.

In continuous time, the simplest form of the neoclassical intertemporal consumer choice model considers a consuming household which chooses the time path of consumption for the vector of goods, \( x(t) \geq 0 \), to

\[
\text{(3.1)} \quad \max_{x(t)} U = \int_0^T u(x(t))e^{-\rho t}dt 
\]

subject to the intertemporal budget constraint,

\[
\text{(3.2)} \quad \frac{dM(t)}{dt} = -\rho(t)'x(t)e^{-rt}, \quad M(0) = M_0, \quad M(t) \geq 0 \quad \forall \ t \in [0, T],
\]

where \( u(\cdot) \) is the instantaneous flow of utility from consumption, \( \rho \) is the consuming household’s rate of time discount, or impatience, \( r \) is the real market discount rate, at
which the consumer can freely borrow or lend, $M_0$ is the household’s initial wealth plus the discounted present value of its full income stream, and $p(t)$ is the vector of market prices for the goods $x(t)$ in period $t$.\(^3\) We assume throughout that $u(\cdot)$ is twice continuously differentiable, strictly increasing \((\partial u(x) / \partial x \geq 0)\) and strongly concave \((\partial^2 u(x) / \partial x \partial x' \text{ is negative definite}) \forall x \in \mathbb{R}^n.\(^4\)

### 3.1 Perfect Foresight

In this subsection, we assume that the consuming household has complete information regarding all past, present, and future prices, incomes, and other relevant economic variables. The Hamiltonian for this problem can be written as

\[
(3.1.1) \quad H = u(x(t))e^{-\rho t} - \lambda(t)p(t)'x(t)e^{-\rho t},
\]

where $\lambda(t)$ is the shadow price, or costate variable for the equation of motion for household wealth. The first-order necessary and sufficient conditions for the maximum principle are

\[
(3.1.2) \quad \partial H / \partial x = e^{-\rho t} \partial u / \partial x - e^{-\rho t} \lambda p \leq 0, \quad x \geq 0, \quad x' \partial H / \partial x = 0 \forall t \in [0, T],
\]

\(^2\) A constant discount rate for consumers over the life cycle simplifies matters and plays only a minor role in the results of this section. A fixed and finite planning horizon is a strong hypothesis, but time and space preclude a more detailed analysis of the issues presented by it here.

\(^3\) We use the continuous time maximum principle to study solutions to this problem. The solution applied throughout this section is known in the operations research and economic dynamics literature as open loop with feedback. Although this concept differs from the closed loop, or dynamic programming, solution concept, it permits a much sharper focus on the role that expectations play in consumption models.

\(^4\) Additive separability, twice continuous differentiability, and quasiconcavity of $U$ imply that $u(\cdot)$ is concave. This can be proven for discrete time with a finite planning horizon by a simple extension of the arguments in Gorman (1970). This argument then can be extended to continuous time by passing to the limit via increasingly small time increments and then appealing to the continuity of the Hessian matrix. It also can be shown that concavity of $u(\cdot)$ is necessary for the existence of an optimal consumption path. Strong concavity, in turn, implies that the optimal path is unique.
Consider an interior solution for \( x \forall t \in [0, T] \). Then first-order condition (3.1.2) implies

\[
(3.1.5) \quad x = u_x^{-1}(e^{(\rho-r)T}\lambda p),
\]

where \( u_x^{-1}(\cdot) \) is the \( n \)-vector inverse of \( u_x(\cdot) \). The strict monotonicity of \( u(\cdot) \) combined with strictly positive prices \( p \) requires that \( \lambda(t) > 0 \forall t \in [0, T] \). Because the Hamiltonian does not depend on current wealth, \( M(t) \), equation (3.1.3) implies \( \lambda(t) \) is constant over the entire planning horizon, \( \lambda(t) = \lambda_0 \forall t \in [0, T] \). Therefore, multiplying by \( e^{-rt}p(t) \) and integrating with respect to \( t \) produces a defining relationship for the wealth shadow price, \( \lambda_0(M_0 - M_T, \rho, r, T) \),

\[
(3.1.6) \quad M_0 - M_T = \int_0^T e^{-rt}p(t)u_x^{-1}(e^{(\rho-r)t}\lambda_0(M_0 - M_T, \rho, r, T) \cdot p(t))dt.
\]

The integral form of (3.1.6) implies that the optimal solution for \( \lambda_0 \) depends on all prices at all points in time, but that for given \( t \) and any finite change in \( p(t) \), with prices remaining unchanged at all other times, \( \partial \lambda_0 / \partial p(t) \equiv 0 \).\(^5\)

Substituting \( \lambda_0(M_0 - M_T, \rho, r, T) \) into (3.1.5) gives the optimal demands at time \( t \),

\[
(3.1.7) \quad x^*(t) \equiv u_x^{-1}(e^{(\rho-r)t}\lambda_0(M_0 - M_T, \rho, r, T) \cdot p(t)).
\]

In contrast to the static model of the previous section, the neoclassical dynamic model

\(^5\) More generally, \( \lambda(\cdot) \) does not vary with any absolutely bounded changes in prices on any subset of \([0, T]\) that has Lebesgue measure zero.
with perfect foresight has a matrix of instantaneous uncompensated price slopes that is symmetric and negative definite,

\[(3.1.8) \quad \partial x^* / \partial p^* = e^{(\rho - r)\lambda_0} u_{xx}^{-1}.\]

This difference in the symmetry properties of static and dynamic consumer choice models is the result of the intertemporal allocation of expenditure and is not due to perfect foresight or income smoothing, *per se*. The difference is due to the integral form of the budget constraint on total household wealth in the dynamic framework. The additive structure of intertemporal preferences implies that the flow of utility in any given instant is perfectly substitutable for utility flows at every other instant. Consequently, a change in market prices at a single point in time generates substitution effects which are perceptible at the given instant but are imperceptibly spread across the consumption bundles in all other times. Even in the simplest of dynamic contexts, therefore, the ubiquitously applied and tested Slutsky symmetry and negativity conditions of static consumer choice theory do not transcend to models in which wealth, rather than current income, is the constraint on consumption choices.

Continuing this line of inquiry, the marginal wealth effects on the demands at each \( t \in [0, T] \) satisfy

\[(3.1.9) \quad \partial x^* / \partial M_0 = e^{(\rho - r)\lambda_0} u_{xx}^{-1} p \cdot \partial \lambda_0 / \partial M_0.\]

By differentiating both sides of (3.1.6) with respect to \( M_0 \), combining the result obtained on the right-hand-side with (3.1.8), regrouping, canceling common terms, and distributing the integral, we have
(3.1.10) \[ \frac{\partial \lambda_0}{\partial M_0} = 1/ \int_0^T e^{(\rho - 2r)t} p' u_{xx}^{-1} \rho_0 \frac{p'}{M_0} \, dt < 0, \]

where the inequality on the far right follows from the (strong) concavity of \( u(\cdot) \) and \( p(t) > 0 \) \( \forall \ t \in [0, T] \).

The maximal level of cumulative discounted utility is defined by

(3.1.11) \[ V(M_0 - M_T, \rho, r, T) \equiv \int_0^T u(x^*) e^{-\rho t} \, dt. \]

Differentiating \( V \) with respect to \( M_0 \) gives

(3.1.12) \[ \frac{\partial V}{\partial M_0} \equiv \int_0^T \frac{\partial u}{\partial x^*} \frac{\partial x^*}{\partial M_0} e^{-\rho t} \, dt \]

\[ \equiv \int_0^T e^{-rt} \lambda_0 p' u_{xx}^{-1} p \cdot \frac{\partial \lambda_0}{\partial M_0} \, dt \]

\[ \equiv \lambda_0 \int_0^T e^{-rt} p' u_{xx}^{-1} \rho_0 \, dt / \int_0^T e^{-rt} p' u_{xx}^{-1} \rho_0 \, dt \equiv \lambda_0 > 0, \]

which is a direct intertemporal analogue to the envelope theorem (LaFrance and Barney (1991)). Following the same steps, but applied to \( M_T \) implies

(3.1.13) \[ \frac{\partial \lambda_0}{\partial M_T} = 1/ \int_0^T e^{(\rho - 2r)t} p' u_{xx}^{-1} \rho_0 \frac{p'}{M_T} \, dt > 0, \]

and

(3.1.14) \[ \frac{\partial V}{\partial M_T} \equiv -\lambda_0 < 0. \]

As a consequence, in the absence of any bequeath motive, the optimal terminal wealth vanishes. This is the intertemporal analogue to the static budget identity when preferences are nonsatiable. Note, however, that \( V(\cdot) \) is (strongly) concave in the household’s initial wealth as a direct consequence of the (strong) concavity of \( u(\cdot) \) in \( x \). This contrasts with
the static model where the marginal utility of money may be constant, increasing or decreasing due to the ordinality of preferences.

To relate the intertemporal model more closely to the static framework, define total consumption expenditures at time \( t \) by \( m(t) = p(t)'x(t) \) and consider the static optimization problem of maximizing \( u(x) \) subject to \( x \geq 0 \) and \( p'x \leq m \). Let \( \tilde{\lambda} \) denote the shadow price for the static budget constraint. The first-order conditions for an interior solution are \( u_x = \tilde{\lambda} \) and \( p_x'x = m \), which produce the static neoclassical demand functions, \( x = h(p, m) \).

In the static problem, we take \( m \) as given, and calculate the comparative statics for \( x \) and \( \tilde{\lambda} \) from

\[
\frac{\partial x}{\partial p'} + \frac{\partial x}{\partial m} \cdot x' = \tilde{\lambda} \left[ u_x^{-1} - \frac{u_{xx}^{-1}p'u_{xx}'}{p'u_{xx}^{-1}} \right] - \frac{u_{xx}^{-1}p'x'}{p'u_{xx}^{-1}} \cdot \frac{u_x^{-1}p}{p'u_{xx}^{-1}}. 
\]

These in turn give the static Slutsky matrix,

\[
S = \frac{\partial x}{\partial p} + \frac{\partial x}{\partial m} \cdot x' = \tilde{\lambda} \left[ u_x^{-1} - \frac{u_{xx}^{-1}p'u_{xx}'}{p'u_{xx}^{-1}} \right],
\]

a symmetric, negative semidefinite, rank \( n-1 \) matrix.

Now, let \( v(p, m) = u(h(p, m)) \) be the indirect utility function for the static problem, which defines the instantaneous flow of maximal utility subject to the static instantaneous budget constraint. Then, because of the additively separable structure of intertemporal preferences, the dynamic consumption problem can be represented equivalently as
(3.1.17) \[ \text{maximize} \int_0^T v(p,m)e^{-rt}dt \quad \text{subject to} \quad \int_0^T me^{-rt}dt \leq M_0, \ m \geq 0 \ \forall \ t \in [0,T]. \]

The first-order conditions for an optimal (interior) solution are

(3.1.18) \[ \frac{\partial v}{\partial m} \equiv \lambda(t) = e^{(\rho-r)t} \lambda, \]

(3.1.19) \[ \dot{\lambda} = 0, \]

(3.1.20) \[ \int_0^T me^{-rt}dt = M_0, \]

where the identity in the center of (3.1.18) is due to the (static) envelope theorem. Total expenditure therefore is not predetermined in each period, but rather is jointly determined with quantities and is smoothed over time to equate the present value of the marginal utility of money across all points in time,

(3.1.21) \[ \frac{\partial v}{\partial m^*} \equiv \lambda = e^{(\rho-r)t} \lambda_0, \]

where \(m^*(t)\) denotes the optimal level of total consumption expenditures \(\forall t \in [0,T]\.\)

Applying Roy’s identity to the static problem then gives

(3.1.22) \[ x^*(t) \equiv h(p(t),m^*(t)) = -\frac{\partial v(p(t),m^*(t))}{\partial m} / \frac{\partial v(p(t),m^*(t))}{\partial m} \ \forall \ t \in [0,T]. \]

Taking the vector product of (3.1.22) with \(p\), multiplying by \(e^{-rt}\) and integrating with respect to \(t\), and utilizing (3.1.18) and (3.1.19) generates two alternative defining relationships for \(\lambda_0\),

\[ \text{In other words, the optimal flow of consumption expenditures generally depends on the parameters of the utility function and market prices at time } t, \text{ as well as initial wealth, individual and market discount rates, and the optimal value of the shadow price for the wealth constraint. Thus, except for models with myopic expectations and with } \rho = r, \text{ total consumption expenditures can not be treated as exogenous (Engel, Hendry, and Richard (1983)) in empirical models of intertemporal consumer choice.} \]
(3.1.23) \[ \lambda_0 = \frac{1}{M_0} \int_0^T e^{-p(t)} \nu_p(p(t), m^*(t)) \, dt \]

where the second identity follows from zero degree homogeneity of \( \nu(\cdot) \) in \((p, m)\) and the inequality follows from the fact that \( \nu(\cdot) \) is strictly increasing in \( m \). As before, we conclude that \( \lambda_0 \) is invariant to all absolutely bounded changes in prices on subsets of \([0, T]\) with Lebesgue measure zero. Hence, differentiating (3.1.21) with respect to \( p \) implies

(3.1.24) \[ \frac{\partial m^*}{\partial p} = -\frac{\frac{\partial^2 \nu(p, m^*)}{\partial m^2} / \partial m}{\partial^2 \nu(p, m^*) / \partial m^2}. \]

Instantaneous symmetry is obtained by differentiating (3.1.22) with respect to \( p \), substituting (3.1.24) into the result, and canceling vanishing terms due to (3.1.21) and (3.1.22),

(3.1.25) \[ \frac{\partial x^*}{\partial p'} = \frac{\frac{\partial^2 v}{\partial p^2}}{\partial v} + \frac{\partial^2 v}{\partial m^2} \left( \frac{\partial m^*}{\partial p'} \right) + \frac{\partial v}{\partial m} \left[ \frac{\partial^2 v}{\partial m^2} + \frac{\partial^2 v}{\partial m^2 \cdot \partial p} \right] \]

\[ = \frac{1}{\frac{\partial v}{\partial m}} \left[ \frac{\partial^2 v}{\partial m \partial p} \cdot \frac{\partial^2 v}{\partial m^2} \right], \]

a symmetric, matrix. It is also worthwhile to contrast (3.1.25) with its static neoclassical counterpart, which has the asymmetric form

(3.1.26) \[ \frac{\partial x^*}{\partial p'} = \frac{1}{\frac{\partial v}{\partial m}} \left[ \frac{\partial^2 v}{\partial p^2} \cdot \frac{\partial^2 v}{\partial m^2} \right]. \]
3.2 Myopic Expectations

The opposite of full information regarding all future economic values on the part of consuming households is myopic expectations. In this case, the household is modeled as if it expects no change in relative prices of goods or services throughout its planning horizon, i.e., $p(t) = p_0 \forall t \geq 0$. This assumption plays an important part in many contemporary dynamic economic models (e.g., Cooper and McLaren (1980); McLaren and Cooper (1980); Epstein (1981, 1982); and Epstein and Denny (1983)). One drawback is the apparent contradiction between the level of sophistication that individuals are presumed to use to formulate their economic plans versus the manner in which they formulate and update their expectations about future events. As pointed out by Epstein and Denny (1983, pp. 649-650), “Current prices are … expected to persist indefinitely. As the base period changes and new prices … are observed, the [decision maker] revises its expectations and its previous plans. Thus only the $t = 0$ portion of the plan … is carried out in general.”

One unfortunate implication of the myopic expectations hypothesis is that economic decision makers are infinitely forward looking when they design their optimal consumption plans, but are totally myopic when they formulate their expectations about their future economic environment. Nevertheless, prior to moving on to more general and robust models of household expectations formation, it will prove useful to identify the economic structure and duality of the intertemporal consumer choice problem with myopic expectations. The primary reason for this is that several properties of the myopic expectations framework generalize in straightforward ways to the more general situations that we shall consider later.
Under myopic expectations, the model and solution approach of the previous section continues to apply, with the caveat that \( p(t) \) is replaced by \( p_0 \) at all points in time. This apparently minor change significantly alters many of the conclusions drawn for the case of perfect foresight.

We begin with the question of Slutsky symmetry in this context. Specifically, the goal of the following developments is to demonstrate that there is no short-run, instantaneous analogue for the static Slutsky symmetry condition in dynamic models. However, we will also show that a dynamic analogue to static Slutsky symmetry exists, but it takes the form of an \( n \times n \) matrix of integral terms.

The first step is to note that the shadow price for the budget constraint now satisfies the condition

\[
M_0 \equiv \int_0^T e^{-rt} p_0' u_x^{-1} \left( e^{(\rho - r)t} \lambda_0(p_0, M_0, \rho, r, T) p_0 \right) dt,
\]

while, since \( p_0 \) is presumed constant over the planning horizon, (3.1.8) now has the form

\[
(3.2.2) \quad \frac{\partial x^*(t)}{\partial p_0'} = e^{(\rho - r)t} u_x^{-1} \left[ \lambda_0 I + p_0 \frac{\partial \lambda_0}{\partial p_0'} \right].
\]

Differentiating the intertemporal budget identity, \( \int_0^T e^{-rt} p_0' x^*(t) dt \equiv M_0 \), with respect to \( p_0 \) then implies that

\[
(3.2.3) \quad \int_0^T e^{-rt} \frac{\partial x^*(t)}{\partial p_0'} p_0 dt = -\int_0^T e^{-rt} x^*(t) dt.
\]

Next, we pre-multiply (3.2.2) by \( e^{-rt} p_0' \), integrate over \( t \), combine the results with (3.2.3), and solve for \( \frac{\partial \lambda_0}{\partial p_0} \), all of which gives
Duality Theory for the Household

(3.2.4) \[ \frac{\partial \lambda_0}{\partial p_0} \equiv - \frac{\int_0^T e^{-\tau t} (x_0^* + u_{xx}^{-1} u_x) dt}{p_0 \left( \int_0^T e^{(\rho - 2\tau) t} u_{xx}^{-1} dt \right) p_0} . \]

This completes the first step, which was to derive the response of the shadow price for initial wealth to changes in relative prices.

The second step is to proceed along similar lines of reasoning, but now with respect to changes in the initial level of wealth. In particular, the analogue to (3.1.9) now has the form,

(3.2.5) \[ \frac{\partial x^*(t)}{\partial M_0} \equiv \frac{e^{(\rho - r) \tau} u_{xx}^{-1} p_0}{p_0 \left( \int_0^T e^{(\rho - 2\tau) t} u_{xx}^{-1} dt \right) p_0} . \]

We then combine (3.2.2) with (3.2.4) to generate the matrix of instantaneous uncompensated (ordinary demand) price derivatives as

(3.2.6) \[ \frac{\partial x^*(t)}{\partial p_0'} + \frac{\partial x^*(t)}{\partial M_0} x^*(t) \left[ \lambda_0 \left[ u_{xx}^{-1} - u_{xx}^{-1} p_0 p_0' \int_0^T e^{(\rho - 2\tau) t} u_{xx}^{-1} dt \right] \right] \equiv \frac{u_{xx}^{-1} p_0 \int_0^T e^{-\tau t} x^*(t) x^*(t) dt \left[ \int_0^T e^{(\rho - 2\tau) t} u_{xx}^{-1} dt \right] p_0}{p_0' \left( \int_0^T e^{(\rho - 2\tau) t} u_{xx}^{-1} dt \right) p_0} . \]

Simple inspection of this matrix equation shows that, in contrast to the case of perfect foresight, \( \frac{\partial x^*(t)}{\partial p_0} \) is not a symmetric negative definite matrix.

Now, by combining (3.2.5) and (3.2.6), we find the instantaneous “wealth-compensated” substitution matrix,

(3.2.7) \[ \frac{\partial x^*(t)}{\partial p_0'} + \frac{\partial x^*(t)}{\partial M_0} x^*(t) \left[ \lambda_0 \left[ u_{xx}^{-1} - u_{xx}^{-1} p_0 p_0' \int_0^T e^{(\rho - 2\tau) t} u_{xx}^{-1} dt \right] \right] \equiv \frac{u_{xx}^{-1} p_0 \int_0^T e^{-\tau t} x^*(t) x^*(t) dt \left[ \int_0^T e^{(\rho - 2\tau) t} u_{xx}^{-1} dt \right] p_0}{p_0' \left( \int_0^T e^{(\rho - 2\tau) t} u_{xx}^{-1} dt \right) p_0} . \]
Again, this matrix is neither symmetric nor negative semidefinite.

Finally, using the identity \( x^*(t) \equiv h(p_0, m^*(t)) \), the instantaneous “income-compensated” substitution matrix is given by

\[
\frac{\partial x^*(t)}{\partial p_0'} + \frac{\partial h(p_0, m^*(t))}{\partial m} x^*(t)' \equiv 0
\]

\[
e^{(\rho-r)t} \left\{ \sum_{s} \left[ u_{sx}^{-1} p_0^{-1} p_0' \left( \int_{0}^{T} e^{(\rho-2r)\tau} u_{sx}^{-1} d\tau \right) p_0' \left( \int_{0}^{T} e^{(\rho-2r)\tau} u_{sx}^{-1} d\tau \right) p_0 \right] - u_{sx}^{-1} p_0 \left( \int_{0}^{T} e^{(\rho-2r)\tau} x^*(\tau) d\tau \right) p_0' \left( \int_{0}^{T} e^{(\rho-2r)\tau} x^*(\tau) d\tau \right) p_0 \right\} + \frac{u_{sx}^{-1} p_0 x^* (t)'}{p_0' u_{sx} p_0},
\]

which also is neither symmetric nor negative semidefinite.

Thus, the primary mainstay of static consumer choice theory — the Slutsky symmetry and negativity condition — does not have any short-run (instantaneous) counterpart in dynamic contexts. As we shall see in the sequel, this important result carries over to dynamic models of consumer choice in which individuals form expectations for the values of economic factors that influence their future environments.

Nevertheless, since the solution to the consumer’s choice problem is the result of a maximization exercise subject to a linear budget constraint, we know instinctively that there must be some kind of symmetry inherent in the problem’s optimal solution. Indeed, such a symmetry condition does exist, although it has a nonstandard form relative to the static model, and in practice would be extremely difficult, if not impossible, to either empirically implement or test.

To show this, we begin heuristically and constructively by multiplying both sides of equations (3.2.5) and (3.2.6) by \( e^{-rt} \) and integrating over \([0, T]\) to obtain.
Next, multiplying (3.2.9) by \( \left[ \int_0^T e^{-rt} x^* (t) dt \right]' \) and adding the result to (3.2.10) produces an \( n \times n \) matrix of integral equations, which gives us the discounted present value of wealth-compensated cross-price substitution terms,

\[
\int_0^T e^{-rt} \frac{\partial x^* (t)}{\partial \rho_0} dt = \lambda_0 \left[ \int_0^T e^{(\rho - 2r)t} u_{xx}^{-1} dt - \frac{\left( \int_0^T e^{(\rho - 2r)t} u_{xx}^{-1} dt \right) p_0}{\rho_0} \left( \int_0^T e^{(\rho - 2r)t} u_{xx}^{-1} dt \right) p_0 \right] \\
- \frac{\left( \int_0^T e^{(\rho - 2r)t} u_{xx}^{-1} dt \right) p_0}{\rho_0} \left( \int_0^T e^{-rt} x^* (t) dt \right)'.
\]

The \( n \times n \) matrix on the right-hand-side of (3.2.11) is clearly symmetric, negative semidefinite, and has rank \( n-1 \). It turns out that this matrix of discounted present values of wealth compensated cross-price substitution terms is precisely the dynamic analogue to the static matrix of Slutsky symmetry terms. We this fact directly in the course of develop-
oping the duality arguments that follow next.⁷

First, we define the maximal level of discounted utility flows, subject to the wealth constraint, by

\[(3.2.12) \quad V(p_0, M_0) \equiv \sup_{u(x)} \left\{ \int_0^T e^{-\rho t} u(x) dt: \int_0^T e^{-\rho t} p_0' x dt = M_0 \right\}, \]

where the equality constraint follows from monotonicity of \(u(\cdot)\). We call \(V(p_0, M_0)\) the dynamic indirect utility function.⁸ Under myopic expectations, the dynamic indirect utility function has properties that are intertemporal analogues to those of the static indirect utility function. That is, \(V(p_0, M_0)\) is:

(DV.1) twice continuously differentiable in \((p_0, M_0)\);

(DV.2) decreasing and quasiconvex in \(p_0\);

(DV.3) strictly increasing and strongly concave in \(M_0\); and

(DV.4) \(0^\circ\) homogeneous in \((p_0, M_0)\); and

(DV.5) satisfies the Dynamic Envelope Theorem,

\[\partial V(p_0, M_0)/\partial p_0 \equiv -\lambda_0(p_0, M_0) \int_0^T e^{-\rho t} h(p_0, M_0, t) dt \ll 0,\]

\[\partial V(p_0, M_0)/\partial M_0 \equiv \lambda_0(p_0, M_0) > 0,\]

---

⁷ A simple, heuristic argument for the validity of (3.2.11) as the dynamic Slutsky substitution matrix is the following. Let \(U_{ss}^{-1} \equiv \int_0^T e^{(r-z)\rho} u_{ss}^{-1} dt\) and note that this \(n \times n\) matrix is negative definite and defines, in a sense, the “inverse Hessian” matrix that determines how changes in consumption choices due to changes in relative prices are allocated over the life cycle. Direct substitution into (3.2.11) gives

\[S \equiv \lambda_0 \left[ U_{ss}^{-1} - (p'_0 U_{ss}^{-1} U_{ss}^{-1} p_0) p'_0 U_{ss}^{-1} \right],\]

which has exactly the form of the static neoclassical Slutsky substitution matrix.

⁸ The function \(V(\cdot)\) also depends upon the discount rates, \(\rho\) and \(r\), and the length of the planning horizon, \(T\). Since these parameters are not the central focus of our discussion, they have been suppressed as arguments to reduce the notational burden.
Duality Theory for the Household

and the Dynamic Roy’s Identity,

\[-\left(\frac{\partial V(p_0, M_0)/\partial p_0}{\partial V(p_0, M_0)/\partial M_0}\right) \equiv \int_0^T e^{-\rho t} h(p_0, M_0, t) dt,
\]

where \(x^*(t) \equiv h(p_0, M_0, t)\) is the \(n\)-vector of dynamic ordinary Marshallian demands at time \(t\).

Twice continuous differentiability of \(V(\cdot)\) follows from strict monotonicity and twice continuous differentiability of \(u(\cdot)\). Strict monotonicity, and strong concavity in \(M_0\) follow from the adaptation, without change, of (3.1.10) and (3.1.12) to the present situation. Monotonicity in \(p_0\) also follows from the monotonicity of \(u(\cdot)\) and the fact that the intertemporal budget set contracts as prices increase. Quasiconvexity is demonstrated in precisely the same manner as for a static problem. Homogeneity follows from the fact that the wealth constraint, \(p_0' \int_0^T e^{-\rho t} x(t) dt = M_0\), is invariant to proportional changes in all prices and initial wealth.

In a very general context, including both equality and inequality constraints and a countable number of switch points over the planning horizon, the dynamic envelope theorem is demonstrated by LaFrance and Barney (1991). Their argument is complex and involved and will not be reproduced here. However, it is pedagogically useful to verify DV.5 by direct calculation to lend heuristic support for the dynamic envelope theorem results that are presented below. This is accomplished simply by differentiating

\[V(p_0, M_0) \equiv \int_0^T e^{-\rho t} u(h(p_0, M_0, t)) dt\]

with respect to \(p_0\) and \(M_0\), substituting the first-order conditions into the resulting expres-
sions, grouping terms, and integrating over the planning horizon, to obtain

\begin{equation}
(3.2.13) \quad \frac{\partial V(p_0, M_0)}{\partial p_0} \equiv \int_0^T e^{-\rho t} \frac{\partial h_1}{\partial p_0} u_x dt
\end{equation}

\begin{align*}
&= \left[ e^{-\rho t} \left( e^{(\rho - r)t} \right) \lambda_0 u_1 - \lambda_0 u_x \right] \left( \int_0^T e^{(\rho - 2r)t} u_x^1 dt \right) p_0' \frac{h_1 (\int_0^T e^{(\rho - r)t} u_x^1 dt) p_0}{p_0' (\int_0^T e^{(\rho - 2r)t} u_x^1 dt) p_0} \right) e^{(\rho - r)t} \lambda_0 p_0 dt \\
&= \lambda_0^2 \left( \int_0^T e^{(\rho - 2r)t} u_x^1 dt \right) p_0
\end{align*}

\begin{align*}
&= \lambda_0^2 \left( \int_0^T e^{(\rho - 2r)t} u_x^1 dt \right) p_0 p_0' \left( \int_0^T e^{(\rho - 2r)t} u_x^1 dt \right) p_0 + \lambda_0 \left( \int_0^T e^{(\rho - r)t} h dt \right) p_0' \left( \int_0^T e^{(\rho - 2r)t} u_x^1 dt \right) p_0 \\
&= - \lambda_0 \int_0^T e^{(\rho - r)t} h(p_0, M_0, t) dt,
\end{align*}

and

\begin{equation}
(3.2.14) \quad \frac{\partial V(p_0, M_0)}{\partial M_0} \equiv \int_0^T e^{-\rho t} u_x' \frac{\partial h}{\partial M_0} dt
\end{equation}

\begin{align*}
&= \left[ e^{-\rho t} \left( e^{(\rho - r)t} \lambda_0 p_0' \right) \right] \left( \int_0^T e^{(\rho - 2r)t} u_x^1 p_0 \left( \int_0^T e^{(\rho - 2r)t} u_x^1 dt \right) p_0 \right) dt \equiv \lambda_0.
\end{align*}

The converse of the intertemporal utility maximization problem is the problem of minimizing the discounted present value of consumption expenditures subject to the constraint that the discounted cumulative flow of utility is no lower than a given value, \( U_0 \),

\begin{equation}
(3.2.15) \quad E(p_0, U_0) \equiv \inf_{\{x(t)\}} \left\{ \int_0^T e^{(\rho - r)t} p_0' x dt : \int_0^T e^{(\rho - r)t} u(x) dt \geq U_0 \right\}.
\end{equation}
We call this the *dynamic expenditure function*. The dynamic expenditure function is:

(DE.1) *twice continuously differentiable, strictly increasing, 1° homogeneous, and concave in* $p_0$;

(DE.2) *twice continuously differentiable, strictly increasing and strongly convex in* $U_0$; and

(DE.3) *satisfies the Dynamic Envelope Theorem,*

$$\frac{\partial E(p_0, U_0)}{\partial p_0} \equiv \int_0^T e^{-\rho t} g(p_0, U_0, t) dt, \text{ and}$$

$$\frac{\partial E(p_0, U_0)}{\partial U_0} \equiv \mu_0(p_0, U_0) > 0,$$

where $x^*(t) = g(p_0, U_0, t)$ is the vector of wealth-compensated dynamic Hicksian demands at time $t$ and $\mu_0(p_0, U_0)$ is the shadow price for the intertemporal utility constraint.

To lay the groundwork for our analysis of more general models in later sections, we develop these properties and the intertemporal duality between the dynamic indirect utility and expenditure functions for the present, simple case of myopic expectations. Toward this end, let $U(0) = U_0$, $dU(t) / dt = -e^{-\rho t} u(x) \forall t \in [0, T]$, and redefine the constraint on the discounted present value of total utility flows as an inequality restriction, $U(T) = U(0) - \int_0^T e^{-\rho t} u(x) dt \geq 0$. Then the Hamiltonian for the dynamic expenditure minimization problem is

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9 This transformation converts the consumer’s intertemporal expenditure minimization problem from an isoperimetric calculus of variations problem into a standard optimal control problem. The latter form is convenient for generating comparative dynamics results and the properties of the optimal solution path. The former, to which we will return momentarily, is useful for analyzing dynamic duality.
(3.2.16) \[ H = e^{-\rho t} p'_0 x - \mu e^{-\rho t} u(x), \]
and the first-order necessary and sufficient conditions for an interior optimal path are:

(3.2.17) \[ \frac{\partial H}{\partial x} = e^{-\rho t} p'_0 - \mu e^{-\rho t} \frac{\partial u}{\partial x} = 0; \]

(3.2.18) \[ \frac{\partial H}{\partial U} = 0 = -\dot{\mu}; \]

(3.2.19) \[ \frac{\partial H}{\partial \mu} = -e^{-\rho t} U = \dot{U}, \quad U(T) \equiv U_0 - \int^T_0 e^{-\rho t} u dt \geq 0. \]

It is easy to see that strict monotonicity of \( u(\cdot) \) implies that \( U(T) = 0 \) since otherwise the discounted present value of expenditures could be lowered without violating the inequality constraint on the present value of discounted utility flows. It also follows from the properties of \( u(\cdot) \) that the optimal path is unique. As in the case of dynamic utility maximization, condition (3.2.18) implies that the shadow price is constant throughout the planning horizon, \( \mu(t) = \mu_0 \forall t \in [0, T] \).

Let \( x^*(t) \equiv g(p_0, U_0, t) \) denote the optimal dynamic Hicksian demands at time \( t \) and let \( \mu_0(p_0, U_0) > 0 \) denote the optimal shadow price for the intertemporal utility constraint. We can verify DE.1 – DE.6 by direct calculation. We begin by first differentiating (3.2.17) with respect to \( p_0 \), and solving for \( \frac{\partial g}{\partial p_0}' \),

(3.2.20) \[ \frac{\partial g}{\partial p_0}' = \mu_0^{-1} u^{-1}_x \left[ e^{(\rho-r)t} I - u_s \frac{\partial \mu_0}{\partial p_0} \right]. \]

We then can differentiate the identity \( \int^T_0 e^{-\rho t} u(g(p_0, U_0, t)) dt = U_0 \) with respect to \( p_0 \) to get \( \int^T_0 e^{-\rho t} (\frac{\partial g}{\partial p_0}) u_s dt \equiv 0 \), transpose both sides of (3.2.20), post-multiply by \( e^{-\rho t} u_s \), integrate over the planning horizon, and solve for \( \frac{\partial \mu_0}{\partial p_0} \),
Duality Theory for the Household

\[(3.2.21) \quad \frac{\partial \mu_0}{\partial p_0} \equiv \left[ \int_0^T e^{-rt} u_{xx}^{-1} u_x dt \right] \equiv \mu_0 \frac{\left( \int_0^T e^{(\rho-2r)t} u_{xx}^{-1} dt \right) p_0}{p_0^2 \left( \int_0^T e^{(\rho-2r)t} u_{xx}^{-1} dt \right) p_0}. \]

Our next step is to substitute the right-hand-side of (3.2.21) into (3.2.20), which gives the instantaneous wealth-compensated matrix of cross-price substitution effects as

\[(3.2.22) \quad \frac{\partial g}{\partial p_0} \equiv e^{(\rho-r)t} \mu_0^{-1} u_{xx}^{-1} \left[ \left( \int_0^T e^{(\rho-2r)t} u_{xx}^{-1} dt \right) p_0 \right]. \]

We can now verify the dynamic analogue to Hotelling’s/Shephard’s Lemma as represented by the first identity in DE.3. By definition of the dynamic expenditure function,

\[E(p_0, U_0) \equiv \int_0^T e^{-\rho t} p_0' g(p_0, U_0, t) dt, \] we have

\[(3.2.23) \quad \frac{\partial E(p_0, U_0)}{\partial p_0} \equiv \int_0^T e^{-\rho t} \left( \frac{\partial g'}{\partial p_0} \right) p_0 + g \right) dt \]

\[= \int_0^T e^{-\rho t} \left[ \frac{\partial g'}{\partial p_0} \right] e^{(\rho-r)t} \mu_0 u_x + g \right] dt \]

\[= \mu_0 \int_0^T e^{-\rho t} \frac{\partial g'}{\partial p_0} u_x dt + \int_0^T e^{-\rho t} g dt \]

\[= \int_0^T e^{-\rho t} g dt \rightarrow 0. \]

Thus, by the converse to Euler’s theorem, the dynamic expenditure function is linearly homogeneous in \(p_0\). Moreover, since the right-hand-side of (3.2.22) is continuous, \(E(p_0, U_0)\) is twice continuously differentiable in \(p_0\).

Next, although concavity in \(p_0\) can be demonstrated with the same arguments as are used for the static neoclassical model, it is useful to verify this directly. Differentiat-
ing (3.2.23) with respect to \( p_0 \), using (3.2.22) for the right-hand-side integrand, we have

\[
(3.2.24) \quad \frac{\partial^2 E(p_0, U_0)}{\partial p_0 \partial p'_0} = \int_0^T e^{-rt} \frac{\partial g}{\partial p_0'} dt \\
\equiv \int_0^T e^{(p-2r)t} u_0^{-1} \left[ u_0^{-1} - \frac{u_0^{-1} p_0 p_0' \left( \int_0^T e^{(p-2r)t} u_0^{-1} dt \right)}{p_0' \left( \int_0^T e^{(p-2r)t} u_0^{-1} dt \right) p_0} \right] dt \\
\equiv \mu_0^{-1} \left[ \int_0^T e^{(p-2r)t} u_0^{-1} dt - \frac{\left( \int_0^T e^{(p-2r)t} u_0^{-1} dt \right) p_0 p_0' \left( \int_0^T e^{(p-2r)t} u_0^{-1} dt \right)}{p_0' \left( \int_0^T e^{(p-2r)t} u_0^{-1} dt \right) p_0} \right].
\]

Since \( \mu_0 > 0 \) and \( U_{xx}^{-1} = \int_0^T e^{(p-2r)t} u_0^{-1} dt \) is symmetric, negative definite, the Hessian matrix for \( E(p_0, U_0) \) is negative semidefinite with rank \( n-1 \).

This completes the verification of DE.1 and the first half of DE.3. We shall return to (3.2.24) in a moment to verify that it is in fact the symmetric, negative semidefinite, rank \( n-1 \) wealth-compensated Slutsky matrix given in (3.2.11) above.

The steps required to verify DE.2 are similar. First, differentiating (3.2.17) with respect to \( U_0 \) implies

\[
(3.2.25) \quad \frac{\partial g}{\partial U_0} \equiv -\mu_0^{-1} u_0^{-1} \frac{\partial u_0}{\partial U_0}.
\]

Second, by differentiating the identity for the discounted present value of total utility flows, \( \int_0^T e^{-p_0 t} u(g(p_0, U_0, t)) dt \equiv U_0 \), with respect to \( U_0 \), we have \( \int_0^T e^{-p_0 t} u'_t \frac{\partial g}{\partial U_0} dt \equiv 1 \).

Therefore, premultiplying (3.2.25) by \( e^{-p_0 t} u'_t \), integrating over \( t \), and using the first-order condition (3.2.17) to replace \( u_0 \), we obtain
Duality Theory for the Household

(3.2.26) \[ \frac{\partial \mu}{\partial U_0} = \frac{-\mu_0}{\int_0^t e^{-rt} u'_{xx} u_x dt} = \frac{-\mu_0^3}{p_0 \left( \int_0^t e^{-rt} u'_{xx} dt \right) p_0} > 0, \]

where the inequality on the far right follows from strong concavity of \( u(\cdot) \) and \( p_0 \neq 0 \).

Third, substituting (3.2.26) into (3.2.25) gives

(3.2.27) \[ \frac{\partial g}{\partial U_0} \equiv \frac{u'_{xx} u_x}{\int_0^t e^{-r \tau} u'_{xx} u_x d \tau} \equiv \frac{e^{(\rho-r)\tau} \mu_0^{-1} u_{xx}^{-1} p_0}{p_0' \left( \int_0^t e^{(\rho-2r)\tau} u_{xx}^{-1} d \tau \right) p_0}. \]

Finally, differentiating the dynamic expenditure function with respect to \( U_0 \) gives

(3.2.28) \[ \frac{\partial E(p_0, U_0)}{\partial U_0} \equiv \int_0^T e^{-rt} p_0' \frac{\partial g}{\partial U_0} dt \]

\[ \equiv \int_0^T e^{-rt} p_0' \frac{e^{(\rho-r)\tau} \mu_0^{-1} u_{xx}^{-1} p_0}{p_0' \left( \int_0^t e^{(\rho-2r)\tau} u_{xx}^{-1} d \tau \right) p_0} dt \equiv \mu_0 > 0. \]

Inspection of (3.2.28) and (3.2.26) then shows us that \( \partial^2 E(p_0, U_0) / \partial U_0^2 > 0 \), thus completing the verification of DE.2 and the second half of DE.3.

The duality between the dynamic indirect utility function and the dynamic expenditure function can be established most directly by viewing them as problems in the classical theory of the calculus of variations (e.g., Clegg (1968), pp. 117-121). Recalling the strict monotonicity of \( u(\cdot) \) and noting that \( p_0' x \) is strictly decreasing in at least one \( x_i \) if \( p_0 \neq 0 \), the utility maximization and expenditure minimization problems can be restated in the isoperimetric form

\[ V(p_0, M_0) \equiv \sup_{x(t)} \left\{ \int_0^T e^{-\rho t} u(x(t)) dt : \int_0^T e^{-rt} p_0' x(t) dt = M_0 \right\}. \]
A well-known result in the theory of the calculus of variations is that, for isoperimetric control problems, the solutions to the two problems are equivalent throughout the entire optimal path if \( M_0 = E(p_0, U_0) \), or equivalently, if \( U_0 = V(p_0, M_0) \). This equivalence is analogous to the duality in static models of consumer choice, \( M_0 \equiv E(p_0, V(p_0, M_0)) \) and \( U_0 \equiv V(p_0, E(p_0, U_0)) \), except that now all definitions are in terms of the discounted present values of consumption expenditures and utility flows. Several conclusions follow directly from this fact, each generating the dynamic analogue to a corresponding duality property in the static theory:

\[
\begin{align*}
(3.2.29) \quad \lambda_0(p_0, E(p_0, U_0)) & \equiv \mu_0(p_0, U_0)^{-1}; \\
(3.2.30) \quad \mu_0(p_0, V(p_0, M_0)) & \equiv \lambda_0(p_0, M_0)^{-1}; \\
(3.2.31) \quad g(p_0, U_0, t) & \equiv h(p_0, E(p_0, U_0), t); \\
(3.2.32) \quad h(p_0, M_0, t) & \equiv g(p_0, V(p_0, M_0), t); \\
(3.2.33) \quad \frac{\partial E(p_0, U_0)}{\partial p} & \equiv \int_0^T e^{-rt} g(p_0, U_0, t) dt \\
& \equiv \int_0^T e^{-rt} h(p_0, E(p_0, U_0), t) dt \\
& \equiv -\frac{\partial V(p_0, E(p_0, U_0))}{\partial p} / \partial M^t; \\
(3.2.34) \quad \frac{\partial g(p_0, U_0, t)}{\partial p_0} \\
\end{align*}
\]
Equation (3.2.34) defines the instantaneous Slutsky substitution matrix. The first matrix on the right-hand-side denotes the instantaneous price effects on the ordinary demands at each point in time and the second right-hand-side matrix denotes the wealth effects. However, it is (3.2.35) and not (3.2.34) that is symmetric and negative semidefinite. Even in this simplest of possible dynamic contexts, therefore, caution is advisable when interpreting hypothesis tests for “Slutsky symmetry and negativity” and other strictures of the static theory. Also note that the identities (3.2.29) and (3.2.35) establish the validity of equation (3.2.11) as the dynamic Slutsky substitution matrix.

Finally, if consumers look ahead with respect to their future economic environment when designing their consumption plans, the manner in which they form expectations is a critical determinant of observable behavior. This topic is the focus of the following subsection.
3.3 Other Forecasting Rules

In the neoclassical model of competition, market prices are invariant to the purchasing and consumption choices of the individual. However, this does not imply that consumers are incapable of learning about market price mechanisms or of forming expectations about their future economic environment. Perfect foresight and myopic expectations are but two extreme possibilities among an uncountable number of alternative forecasting rules that may be reasonable hypotheses in a model of consumption behavior. In this subsection, therefore, we analyze models in which consuming households employ forecasting rules for predicting their future economic conditions when they formulate their dynamic consumption plans. Important members of the class of rules we consider are adaptive, rational, and quasi-rational expectations. Notwithstanding the previous subsection’s detailed analysis of myopic expectations, rather than treat each of these special cases separately, we attempt to embed all of these hypotheses as special cases within a general, unifying framework.

Clearly, future incomes, rates of return on assets, and market rates of interest at which the individual can borrow or lend are important economic variables affecting future opportunity sets. However, the basic questions, arguments, and conclusions arising from expectations formation processes are most clear and simplest to present when we focus on forecasting market prices.

To motivate the models and the solution approach that we shall consider throughout this section, assume that prices follow some form of filtered diffusion, say,

\[(3.3.1) \quad dp(t) = \alpha(p(t), t)dt + \beta(p(t), t)dz(t),\]
where $dz(t) \sim i.i.d. N(0_n, I_n dt)$. Let \( \{F_t\} \) denote an increasing sequence of $\sigma$-algebras defining a filtration for the pre-visible stochastic process $p(t)$, with $F_t \subseteq F_s$ for $t \leq s$.

Conditional on $F_0$, $\forall \ t \geq 0$, we then have

\[
E[dp(t) | F_0] = E[\alpha(p(t), t) | F_0] dt,
\]

so that

\[
\frac{d}{dt} E[p(t) | F_0] = E[\alpha(p(t), t) | F_0], \ E[p(0) | F_0] = p_0
\]

defines a system of $n$ ordinary differential equations with initial condition $p_0$.

Under standard regularity conditions (e.g., those that will be maintained throughout this section of the chapter), we can denote the unique solution to this system of differential equations as a smooth function of the initial conditions, say $\phi(p_0, t)$. Note that the general (i.e., unspecified) dependence of $\phi(\cdot)$ on $t$ admits all sorts of expectations formation processes, as well as potential dependence of the expectations formation process on other economic variables and phenomena.

If (3.3.1) is the “true” data generating process for future price movements, then the rational expectations open loop with feedback solution for the household’s intertemporal choice problem can be written in the form

\[
\sup \left\{ \int_0^T e^{-\rho t} u(x(t)) dt : \int_0^T e^{-\rho t} \phi(p_0, t) x(t) dt = M_0 \right\}.
\]

However, it is entirely possible that consumers use forecasting models, including myopic, adaptive, or quasi-rational expectations, in formulating their perceptions for how the future economic environment is expected to evolve.
Regardless of the forecasting rule, it is essential to assume that the consuming household carries out only the initial instant of the optimal consumption plan. Once additional information becomes available regarding the realization of prices (and/or other relevant economic phenomena), the household updates its information set according to its filtering mechanism and designs a new intertemporal consumption plan. This is consistent with assumptions made in previous market demand analyses, as well as the manner in which a consumer would behave if it applied the closed loop, or stochastic dynamic programming, solution concept in formulating an intertemporal consumption plan. Of course, the open loop solution in general is not equivalent to the closed loop solution. In particular, with price uncertainty, because the dynamic budget constraint is bilinear in prices (a vector of uncontrolled state variables) and consumption purchases (a vector of control variables), it is impossible for the open loop and closed loop solutions to coincide – even when the instantaneous utility function is quadratic.

The primary advantages to studying open loop with feedback solutions to this class of dynamic consumer choice problems include the following: (1) It permits a direct and clear comparison with static model results, as well as those obtained for dynamic models with myopic expectations or perfect foresight. (2) The closed loop solution does not lend itself readily to any clearly stated or universal comparative statics or dynamics results, and in particular, is highly dependent on the specific structure of the consumer’s preference function as well as the data generating process for future economic phenomena. and (3) The open loop solution produces a well-defined, intuitively appealing, and easy to understand and interpret set of intertemporal duality results that can be readily re-
lated to those obtained for the static neoclassical model of consumer choice.

Therefore, let the system of ordinary differential equations,

\[
\dot{p}(t) = \psi(p(t), t), \quad p(0) = p_0,
\]

be the rule that the consumer is presumed to use to form expectations for future prices, where the “\(\cdot\)” over a variable (or vector of variables) denotes the ordinary time derivative. We assume throughout the discussion that \(\psi: \mathbb{R}_+^n \times \mathbb{R}_+ \to \mathbb{R}_+^n\) is twice continuously differentiable and \(\partial \psi / \partial p \neq 0\) throughout its domain. This implies the existence of a unique, twice continuously differentiable solution to the differential equation system which defines all future price forecasts as a function of the initial price vector, \(p_0\), and time, \(t\),

\[
(3.3.6) \quad p(t) \equiv \varphi(p_0, t) = p_0 + \int_0^t \psi(\varphi(p_0, \tau), \tau) d\tau.
\]

In addition to the above properties for \(\psi(\cdot)\), we shall assume that the solution to the forecasting rule generates strictly positive price forecasts, \(\varphi: \mathbb{R}_+^n \times \mathbb{R}_+ \to \mathbb{R}_+^n\).

It is a well-accepted stylized empirical fact that observed market prices tend to have common trends. In the present context, the most general statement of such a property is that the forecasting solution, \(\varphi(\cdot, t)\), is linearly homogeneous in \(p_0\). It turns out that this property is necessary and sufficient for the dynamic expenditure function to be linearly homogeneous in current prices in this model. This property can be stated equivalently in terms of the condition that \(\psi(\cdot, t)\) is homogeneous of degree one in \(p(t) \forall t \in [0, T]\).

**Lemma.** \(\varphi(\cdot, t)\) is 1º homogeneous in \(p_0 \forall (p_0, t) \in \mathbb{R}_+^n \times \mathbb{R}_+\), if and only if \(\psi(\cdot, t)\) is 1º homogeneous in \(p \forall (p, t) \in \mathbb{R}_+^n \times \mathbb{R}_+\).
PROOF: Suppose that \( \phi(p, t) \equiv \frac{\partial \phi(p, t)}{\partial p} p \quad \forall (p, t) \in \mathbb{R}_{+}^{n} \times \mathbb{R}_{+}. \) Then

\[
\frac{\partial \phi(p_0, t)}{\partial p_0} \equiv I + \int_{0}^{t} \frac{\partial \psi(\phi(p_0, \tau), \tau)}{\partial p'} \frac{\partial \phi(p_0, \tau)}{\partial p_0} d\tau
\]

\[
\Rightarrow \phi(p_0, t) \equiv \frac{\partial \phi(p_0, t)}{\partial p_0} p_0 \equiv p_0 + \int_{0}^{t} \frac{\partial \psi(\phi(p_0, \tau), \tau)}{\partial p'} \phi(p_0, \tau) d\tau
\]

\[
eq p_0 + \int_{0}^{t} \psi(\phi(p_0, \tau), \tau) d\tau.
\]

where the far right-hand-side is the definition of the far left-hand-side, while the middle identity follows from the linear homogeneity of \( \phi(, t) \) in \( p_0 \). Subtracting \( p_0 \) from the last two expressions implies

\[
\int_{0}^{t} \frac{\partial \psi(p(\tau), \tau)}{\partial p'} p(\tau) d\tau \equiv \int_{0}^{t} \psi(p(\tau), \tau) d\tau.
\]

By the fundamental theorem of calculus, differentiating both sides with respect to \( t \) gives

\[
\frac{\partial \psi(p(t), t)}{\partial p'} p(t) \equiv \psi(p(t), t).
\]

Hence, \( \psi(, t) \) is 1° homogeneous in \( p \) by the converse to Euler’s theorem, proving necessity.

We verify sufficiency by employing the method of successive approximations to solve the ordinary differential equation system (3.3.5). Each iteration begins with an approximate solution that is linearly homogeneous. We then show that this property is inherited by the subsequent iteration’s approximate solution. The proof is concluded by induction, and an appeal to the contraction mapping theorem, to verify that the sequence of iterations con-
Duality Theory for the Household

verges to a unique solution to the ordinary differential equation system that itself must be linearly homogeneous in $p_0$.

Let $\varphi^{(0)}(p_0, t) \equiv p_0$, which is trivially 1º homogeneous in $p_0$, and define

$$\varphi^{(1)}(p_0, t) \equiv p_0 + \int_0^t \psi(p_0, \tau)d\tau \equiv p_0 + \int_0^t \varphi^{(0)}(p_0, \tau, \tau)d\tau,$$

so that

$$\frac{\partial \varphi^{(1)}(p_0, t)'}{\partial p_0} p_0 \equiv p_0 + \int_0^t \frac{\partial \psi(p_0, \tau)}{\partial p_0} p_0 d\tau \equiv p_0 + \int_0^t \psi(p_0, \tau)d\tau \equiv \varphi^{(1)}(p_0, t),$$

which therefore also is 1º homogeneous in $p_0$. Proceeding by induction, if for any $i \geq 2$, we have

$$\frac{\partial \varphi^{(i-1)}(p_0, t)'}{\partial p_0} p_0 \equiv \varphi^{(i-1)}(p_0, t)$$

and we define

$$\varphi^{(i)}(p_0, t) \equiv p_0 + \int_0^t \psi(\varphi^{(i-1)}(p_0, \tau), \tau)d\tau,$$

then

$$\frac{\partial \varphi^{(i)}(p_0, t)'}{\partial p_0} p_0 \equiv p_0 + \int_0^t \frac{\partial \psi(\varphi^{(i-1)}(p_0, \tau), \tau)'}{\partial p} \cdot \frac{\partial \varphi^{(i-1)}(p_0, \tau)'}{\partial p} p_0 d\tau$$

$$\equiv p_0 + \int_0^t \frac{\partial \psi(\varphi^{(i-1)}(p_0, \tau), \tau)'}{\partial p} \cdot \varphi^{(i-1)}(p_0, \tau)d\tau$$

$$\equiv p_0 + \int_0^t \psi(\varphi^{(i-1)}(p_0, \tau), \tau)d\tau \equiv \varphi^{(i)}(p_0, t),$$

and $\varphi^{(i)}(p_0, t)$ is 1º homogeneous in $p_0 \ \forall \ i \geq 1$. It follows that the unique solution to the ordinary differential equations,
\[
\varphi(p_0, t) \equiv \lim_{t \to \infty} \varphi^{(i)}(p_0, t),
\]
also must be linearly homogeneous in \(p_0\).

For the remainder of the chapter, therefore, we assume that the forecasting rule, \(\varphi(p_0, t)\), is twice continuously differentiable in \((p_0, t)\), and increasing, positively linearly homogeneous, and concave in \(p_0\). As we shall see in the course of the discussion that follows, the last condition is an essential ingredient for concavity of the dynamic expenditure function.

When relative prices change over time and consumers form expectations for future price levels according to some rule that is consistent with (3.3.6), the defining equation for the wealth constraint’s shadow price takes the form

\[
(3.3.7) \quad \int_0^t e^{-\lambda(t-s)} \lambda_0(p_0, M_0) \varphi(p_0, t) \, dt \equiv M_0. \tag{3.3.7}
\]

Consequently, whenever consumers form price expectations in devising their consumption plans, market prices can not be exogenous in the empirical model.\(^\text{10}\) This can be seen most clearly in the case of rational expectations, where \(\varphi(p_0, t)\) equals the conditional mean of the price vector at time \(t\) given information available at time 0, so that the parameters of the marginal distribution for prices enter the conditional distribution for quantities given prices. The implication is that if consumers form expectations about their future economic environment as they develop consumption plans, then the expectation process must be modeled jointly with demand behavior to obtain consistent and efficient

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\(^{10}\) Hendry (1995), chapter 5, contains an exhaustive treatment of exogeneity in econometric models.
Duality Theory for the Household

Following the same logic as in the previous subsections, we obtain the instantaneous price and wealth effects on demands to be

\[
\frac{\partial h(p_0, M_0, t)}{\partial M_0} = e^{(\rho - r)t} u_{xx}^{-1} \varphi(p_0, t) \frac{\partial \lambda_0(p_0, M_0)}{\partial M_0},
\]

\[
\frac{\partial h(p_0, M_0, t)}{\partial p_0^0} = e^{(\rho - r)t} u_{xx}^{-1} \left[ \lambda(p_0, M_0) \frac{\partial \varphi(p_0, t)}{\partial p_0^0} + \varphi(p_0, t) \frac{\partial \lambda_0(p_0, M_0)}{\partial p_0^0} \right],
\]

while the impacts of a change in initial prices and wealth on the marginal utility of wealth are

\[
\frac{\partial \lambda_0(p_0, M_0)}{\partial p_0^0} \equiv - \frac{\left( \int_0^T e^{(\rho - 2r)t} (\partial \varphi^0 / \partial p_0^0) u_{xx}^{-1} \varphi dt + \int_0^T e^{-r} (\partial \varphi^0 / \partial p_0^0) h dt \right)}{\int_0^T e^{(\rho - 2r)t} \varphi u_{xx}^{-1} \varphi dt},
\]

\[
\frac{\partial \lambda_0(p_0, M_0)}{\partial M_0} \equiv - \frac{1}{\int_0^T e^{(\rho - 2r)t} \varphi u_{xx}^{-1} \varphi dt} < 0.
\]

It follows from the last two equations that \( \lambda_0 \) is \(-1^0\) homogeneous in \((p_0, M_0)\) if and only if \( \varphi \) is \(1^0\) homogeneous in \(p_0\). This implies that linear homogeneity of the price forecasting rule in current prices is necessary and sufficient for \(0^0\) homogeneity of the ordinary demand functions in current prices and wealth. In turn, this latter property is necessary and sufficient for linear homogeneity of the dynamic expenditure function in the initial price vector, \(p_0\).

In the price forecasting model, the intertemporal Slutsky matrix has the form

\[
S = \int_0^T e^{-r} \frac{\partial \varphi'}{\partial p_0} \frac{\partial h}{\partial p_0} dt + \left( \int_0^T e^{-r} \frac{\partial \varphi'}{\partial p_0} \frac{\partial h}{\partial M_0} dt \right) \times \left( \int_0^T e^{-r} \frac{\partial \varphi'}{\partial p_0} h dt \right).
\]
a symmetric, negative semidefinite matrix with rank no greater than \( n-1 \).\(^\text{11} \) Notwithstanding the effects of the additional terms \( \partial \phi' / \partial p_0 \), the relationship between the intertemporal Marshallian and wealth-compensated (Hicksian) demands remains the same as in the previous subsection.

However, the dynamic Slutsky matrix no longer is the Hessian matrix for the dynamic expenditure function. In particular, the dynamic envelope theorem now implies

\[
(3.3.13) \quad \frac{\partial E(p_0, U_0)}{\partial p_0} \equiv \int_0^T e^{-rt} \frac{\partial \phi(p_0, t)'}{\partial p_0} g(p_0, U_0, t) dt,
\]

where \( g(p_0, U_0, t) \) is the time \( t \) vector of Hicksian demands which solve the dynamic expenditure minimization problem

\[
(3.3.14) \quad E(p_0, U_0) \equiv \inf_{\{x(t)\}} \left\{ \int_0^T e^{-rt} \phi(p_0, t)' x(t) dt : \int_0^T e^{-rt} u(x(t)) dt = U_0 \right\}.
\]

Differentiating (3.3.9) with respect to \( p_0 \) therefore implies

\[
(3.3.15) \quad \frac{\partial^2 E(p_0, U_0)}{\partial p_0 \partial p_0'} \equiv \int_0^T e^{-rt} \frac{\partial \phi(p_0, t)'}{\partial p_0} \frac{\partial g(p_0, U_0, t)}{\partial p_0'} dt
\]

\[
+ \sum_{i=1}^n \int_0^T e^{-rt} g_i(p_0, U_0, t) \frac{\partial^2 \phi_i(p_0, t)}{\partial p_0 \partial p_0'} dt
\]

\(^{11} \) In fact, if the rank of \( \partial \phi / \partial p' \) is constant \( \forall t \in [0, T] \), then \( \text{rank}(S) = \min\{n-1, \text{rank}(\partial \phi / \partial p')\} \).
\[ S + \sum_{i=1}^{n} \int_{0}^{T} e^{-rt} g_i(p_0, U_0, t) \frac{\partial^2 \Phi_i(p_0, t)}{\partial p_i \partial p'_i} dt, \]

where \( \lambda_0(p_0, E(p_0, U_0)) \equiv 1/\mu_0(p_0, U_0) \) has been used on the right-hand-side. It follows that, in general, the dynamic expenditure function will be concave in \( p_0 \) only if all of the components of the price expectation rule are jointly concave in the initial price vector.

Introducing a general class of forecasting rules results in only minor changes to the duality between the dynamic indirect utility and expenditure functions. Writing the utility maximization and expenditure minimization problems in their isoperimetric forms for the present case,

\[(3.3.16)\quad V(p_0, M_0) \equiv \sup_{\{x(t)\}} \\left\{ \int_{0}^{T} e^{-\rho t} u(x(t)) dt : \int_{0}^{T} e^{-\rho t} \Phi(p_0, t)' x(t) dt = M_0 \right\},\]

\[(3.3.17)\quad E(p_0, U_0) \equiv \inf_{\{x(t)\}} \\left\{ \int_{0}^{T} e^{-\rho t} \Phi(p_0, t)' x(t) dt : \int_{0}^{T} e^{-\rho t} u(x(t)) dt = U_0 \right\},\]

it follows that \( M_0 \equiv E(p_0, V(p_0, M_0)) \) and \( U_0 \equiv V(p_0, E(p_0, U_0)) \). Consequently, (3.2.39) — (3.2.32) remain unchanged, (3.2.35) becomes (3.3.15), and (3.2.33) and (3.2.34), respectively, become:

\[(3.3.18)\quad \frac{\partial E(p_0, U_0)}{\partial p_0} \equiv \int_{0}^{T} e^{-\rho t} \frac{\partial \Phi(p_0, t)'}{\partial p_0} g(p_0, U_0, t) dt \]

\[\equiv \int_{0}^{T} e^{-\rho t} \frac{\partial \Phi(p_0, t)'}{\partial p_0} h(p_0, E(p_0, U_0), t) dt \equiv \frac{\partial V(p_0, E(p_0, U_0))}{\partial p_0};\]

and

\[(3.3.19)\quad \frac{\partial g(p_0, U_0, t)}{\partial p'_0} \equiv \frac{\partial h(p_0, E(p_0, U_0), t)}{\partial p'_0};\]
\[
\frac{\partial h(p_0, E(p_0, U_0), t)}{\partial M_0} + \frac{1}{\partial M_0} \left( \int_0^T e^{-\tau} \frac{\partial \varphi(p_0, \tau)}{\partial p_0} h(p_0, E(p_0, U_0), \tau) d\tau \right).
\]

Similarly, the dynamic envelope theorem for the indirect utility function previously given in (3.2.13) above now takes the form

\[ (3.3.20) \quad \frac{\partial V(p_0, M_o)}{\partial p_0} \equiv -\lambda_o \int_0^T e^{-\tau} \frac{\partial \varphi(p_0, t)}{\partial p_0} h(p_0, M_o, t) dt, \]

while (3.2.14) continues to be \( \frac{\partial V(p_0, M_o)}{\partial M_o} \equiv \lambda_o(p_0, M_o) \). Note the effect of initial prices on future price expectations. This plays a significant role in each of the above results, determining when the dynamic expenditure function is \( 1^o \) homogeneous and concave in prices, as well as the functional expressions for the dynamic envelope theorem and the instantaneous and intertemporal Slutsky equations.

4. Dynamic Household Production Theory

This section merges household production theory with the theory of consumer choice over time. In this context, it is natural to incorporate durable goods into the household’s production process. The basic model structure and variable definitions are analogous to previous sections, with \( x(t) \) an \( n \)-vector of flows of consumable market goods used at time \( t \), \( z(t) \) an \( m \)-vector of flows of nonmarket commodities produced by the household and which generate utility directly, and \( k(t) \) an \( \ell \)-vector of stocks of household durables, some of which may be interpreted as consumption habits. We continue to take the household’s objective to be to maximize the present value of discounted lifetime utility flows, but the flow of produced nonmarket commodities is now presumed to generate the flow of consumer satisfaction,
The efficient boundary of the household production possibility set for each point in time is defined by the joint consumables/durables/commodities transformation function
\begin{equation}
G(x(t), k(t), z(t), \beta, t) \leq 0,
\end{equation}
where \( \beta \) is an \( s \)-vector of quality characteristics of both the consumables and durables and the index \( t \) tacitly implies that the feasible household production possibilities set may vary over time.\(^{12}\) The rates of change in the household’s holdings of durable stocks are defined by the differential equations,
\begin{equation}
\dot{k}(t) = f(x(t), k(t), \gamma, t), \ k(0) = k_0, \text{ given ,}
\end{equation}
where \( \gamma \) is a vector of durable goods’ characteristics that affect the rates of accumulation and/or decay. The household’s life cycle budget constraint is defined by
\begin{equation}
M_0 = \int_0^T e^{-\gamma t} p(t) x(t) dt.
\end{equation}
We begin with a straightforward extension of theorem 2 to show that the derived instantaneous utility function defined over consumables, durables, qualities, and time,
\begin{equation}
\begin{aligned}
u^*(x, k, \beta, t) &\equiv \sup_{z \geq 0} \{ u(z) : G(x, k, z, \beta, t) \leq 0 \} \\
\end{aligned}
\end{equation}
is jointly strongly concave in \((x, k)\) and increasing (decreasing) in \( x_i \) or \( k_j \) if and only if \( f(\cdot) \) is increasing (decreasing) in the corresponding \( x_i \) or \( k_j \). This implies that the instantaneous

---

\(^{12}\) One possibility is that the characteristics of market goods vary over time with consumer expectations for these changes modeled similarly as for price expectations in section 3.3 above. This would imply that \( \beta \) in equation (4.2) tacitly represents goods characteristics at the initial date in the planning horizon, while the structure of \( f(\cdot) \) reflects the consuming household’s expectations for both future household production technology and goods qualities.
myopic indirect utility function,

\[ u(p, m, k, \beta, t) \equiv \sup_{x \geq 0} \left\{ u^*(x, k, \beta, t) : p'x \leq m \right\}, \]

is neoclassical in \((p, m)\), i.e., \(u(\cdot, k, \beta, t)\) is continuous and 0º homogeneous in \((p, m)\), increasing in \(m\), and decreasing and quasiconvex in \(p\). The corresponding myopic ordinary demands, \(x = \tilde{h}(p, m, k, \beta, t)\), therefore also possess all of the neoclassical properties, while reflecting the structure commonly known as na"ive habit formation.

Continuity of \(f(\cdot)\) and \(\tilde{h}(\cdot)\) implies that there is a unique solution for the time path of household durables holdings defined by

\[ \hat{k}(k_0, \beta, t) \equiv k_0 + \int_0^t f\left(\tilde{h}(p(\tau), m(\tau), \hat{k}(k_0, \beta, \tau), \beta, \tau), \hat{k}(k_0, \beta, \tau), \beta, \tau\right) d\tau. \]

Note that \(\hat{k}(k_0, \beta, t)\) depends upon all past prices and consumption expenditures. This implies the following for consumption models under naive habit formation:

(a) Current stocks of durables can not be weakly exogenous.

(b) Preferences are intertemporally inconsistent, i.e., current preferences depend on the entire history of past consumption choices.

(c) Consumers are assumed to understand the effects of changes in household durables on the solution to their instantaneous utility maximization problem, but to ignore this when planning for future consumption.

(d) If consumers are assumed to be naive regarding the influence of current consumption on future preferences and consumption possi-
Duality Theory for the Household

bilities, then only the assumption of myopic price expectations
avoids a logical contradiction regarding household planning and
foresight.

This essentially summarizes the current state of the art in empirical demand analy-
ysis, particularly with respect to foodstuffs and agricultural products. With very few ex-
ceptions, extant empirical demand analyses incorporate naïve habit formation and myopic
price expectations. There are, perhaps, many reasons for this. Household holdings of du-
rable stocks, including real capital items, are often unobservable, particularly when one is
using aggregate time series data. Consequently, lagged quantities demanded of the con-
sumable, nondurable goods are generally used to proxy these as well as other unobserv-
able trends in the data. Even with the simplification that results when lagged quantities
are used to proxy habits and other missing consumption trends, however, incorporating
rational habit persistence in demand is difficult and complicated (Browning). Rational
habit formation also suggests rational expectations, or at least expectations other than
myopic. This complicates the econometric analysis even further.

In this context, it is important to analyze the economics of dynamic models of
household production and consumption that includes expectations processes and a dy-
namic accounting of the effects of changes in the level of household durables. With re-
gard to future expectations, we will maintain our focus on prices, smooth expectation
rules, and the open loop with feedback control solution concept. However, the analysis of
this section could be readily extended to include future expectations for other economic
factors as deemed appropriate by the analyst. Also, our main focus continues to be estab-
lishing an internally consistent duality theory of the dynamic household production model.

When the household production technology is time dependent, e.g., as a result of technological change, the derived instantaneous utility function over consumable market goods and household stocks is a function of time, \( t \). Hence, to reduce the notational burden, define \( \tilde{u}(x, k, \beta, t) \equiv e^{-\rho t}u^*(x, k, \beta, t) \). The consumer’s decision problem now is to solve

\[
V(p_0, M_0, k_0, \beta, \gamma) = \sup \left\{ \int_0^T \tilde{u}(x, k, \beta, t)dt : M_0 = \int_0^T e^{-\rho t} \varphi'xdt, \dot{k} = f(x, k, \gamma, t), k(0) = k_0, k(T) \geq 0 \right\}.
\]

Let \( \omega \) be the \( \ell \)-vector of co-state variables (i.e., shadow prices) for the equations of motion for household durables and let \( \lambda \) be the co-state variable for the equation of motion for the present value of wealth. Then the Hamiltonian can be written as

\[
H = \tilde{u}(x, k, \beta, t) + \omega'f(x, k, \gamma, t) - \lambda e^{-\gamma t} \varphi'x.
\]

To simplify the discussion, we assume that (a) the Hamiltonian is jointly concave in \((x, k)\), (b) for each \( t \in [0, T] \), \( \tilde{u} + \omega'g \) is strictly increasing in \( x \) throughout an \( n+\ell+1 \)-dimensional open tube in the neighborhood of the optimal path, and (c) the optimal path satisfies \((x^*(t), k^*(t)) \to (0,0) \) \( \forall t \in [0, T] \). Then the necessary and sufficient first-order conditions plus the transversality conditions for the optimal path are:

\[
\frac{\partial H}{\partial x} = \frac{\partial \tilde{u}}{\partial x} + \frac{\partial f'}{\partial x} \omega - \lambda e^{-\gamma t}p = 0;
\]
Duality Theory for the Household

\( (4.11) \quad \frac{\partial H}{\partial k} = \frac{\partial \tilde{u}}{\partial k} + \frac{\partial f'}{\partial k} \omega = -\dot{\omega} ; \)

\( (4.12) \quad \frac{\partial H}{\partial M} = 0 = -\dot{\lambda} ; \)

\( (4.13) \quad \frac{\partial H}{\partial \omega} = f = \dot{k}, \quad k(0) = k_0, \quad k(T) \geq 0 ; \)

\( (4.14) \quad \frac{\partial H}{\partial \lambda} = -e^{-\gamma} \varphi' x = \dot{M}, \quad M(0) = M_0, \quad M(T) = 0 ; \) and

\( (4.15) \quad \omega_j(T) k_j(T) = 0 \quad \forall \quad j = 1, \ldots, \ell . \)

At time \( t \), the optimal Marshallian demands, stocks of household durable goods, and shadow prices are

\( (4.16) \quad x^*(t) \equiv h(p_0, M_0, k_0, \beta, \gamma, t) , \)

\( (4.17) \quad k^*(t) \equiv k(p_0, M_0, k_0, \beta, \gamma, t) , \)

\( (4.18) \quad \omega^*(t) \equiv \omega(p_0, M_0, k_0, \beta, \gamma, t) , \) and

\( (4.19) \quad \lambda^*(t) \equiv \lambda_0(p_0, M_0, k_0, \beta, \gamma) , \)

respectively. Differentiating the \textit{intertemporal budget identity}, \( \int_0^t e^{-\gamma} \varphi' h dt \equiv M_0 , \) with respect to \( p_0 \) and \( M_0 \) generates the \textit{intertemporal Cournot aggregation} and the \textit{intertemporal Engel aggregation}, respectively,

\( (4.20) \quad \int_0^t e^{-\gamma} \left[ \frac{\partial \varphi'}{\partial p_0} h + \frac{\partial h'}{\partial p_0} \varphi \right] dt \equiv 0 , \) and

\( (4.21) \quad \int_0^t e^{-\gamma} \varphi' \frac{\partial h}{\partial M_0} dt \equiv 1 . \)
Now, to minimize the notational clutter, define the following blocks of Hessian terms: 

\[ H_{xx} = \tilde{u}_{xx} + \sum_{i=1}^{\ell} \tilde{\omega}_i f_{x}^i ; \quad H_{sk} = \tilde{u}_{sk} + \sum_{i=1}^{\ell} \tilde{\omega}_i f_{x}^i ; \quad \text{and} \quad H_{kk} = \tilde{u}_{kk} + \sum_{i=1}^{\ell} \tilde{\omega}_i f_{x}^i . \]

Then, following exactly the same steps as in the previous section, we obtain

\[
\frac{\partial h}{\partial p_0} = H_{xx}^{-1} \left[ e^{-\tau t} \left( \lambda_0 \frac{\partial \varphi}{\partial p_0} + \varphi \frac{\partial \lambda_0}{\partial p_0'} \right) - H_{sk} \frac{\partial \kappa}{\partial p_0'} - \frac{\partial f'}{\partial x} \frac{\partial \omega}{\partial p_0'} \right],
\]

\[
\frac{\partial \lambda_0}{\partial p_0} = \frac{-\int_0^T e^{-\tau t} \left\{ \frac{\partial \varphi'}{\partial p_0} h + \left[ e^{-\tau t} \lambda_0 \frac{\partial \varphi'}{\partial p_0} - \frac{\partial \kappa'}{\partial p_0} H_{xx} - \frac{\partial \omega'}{\partial p_0} \frac{\partial f'}{\partial x} \right] H_{xx}^{-1} \varphi \right\} dt}{\int_0^T e^{-2\tau t} \varphi' H_{xx}^{-1} \varphi dt},
\]

\[
\frac{\partial h}{\partial M_0} = H_{xx}^{-1} \left[ e^{-\tau t} \varphi \frac{\partial \lambda_0}{\partial M_0} - H_{sk} \frac{\partial \kappa}{\partial M_0} - \frac{\partial f'}{\partial x} \frac{\partial \omega}{\partial M_0} \right],
\]

\[
\frac{\partial \lambda_0}{\partial M_0} = \frac{1 + \int_0^T e^{-\tau t} \varphi' H_{xx}^{-1} \left( H_{sk} \frac{\partial \kappa}{\partial M_0} + \frac{\partial f'}{\partial x} \frac{\partial \omega}{\partial M_0} \right) dt}{\int_0^T e^{-2\tau t} \varphi' H_{xx}^{-1} \varphi dt}, \quad \text{and}
\]

\[
\int_0^T e^{-\tau t} \frac{\partial \varphi'}{\partial p_0} \frac{\partial h}{\partial p_0'} dt + \left( \int_0^T e^{-\tau t} \frac{\partial \varphi'}{\partial p_0} \frac{\partial h}{\partial M_0'} dt \right) \left( \int_0^T e^{-\tau t} \frac{\partial \varphi'}{\partial p_0} h dt \right) = \]

\[
\lambda_0 \int_0^T e^{-2\tau t} \frac{\partial \varphi'}{\partial p_0} H_{xx}^{-1} \varphi dt - \left( \int_0^T e^{-2\tau t} \frac{\partial \varphi'}{\partial p_0} H_{xx}^{-1} \varphi dt \right) \left( \int_0^T e^{-2\tau t} \frac{\partial \varphi'}{\partial p_0} H_{xx}^{-1} \varphi dt \right) \]

\[
+ \left( \int_0^T e^{-2\tau t} \frac{\partial \varphi'}{\partial p_0} H_{xx}^{-1} \varphi dt \right) \left( \int_0^T e^{-2\tau t} \varphi' H_{xx}^{-1} \varphi dt \right) .
\]
Note, in particular, the added complexity of the intertemporal wealth-compensated price effects represented by the additional terms in the last three lines of (4.26). Even symmetry, much less negativity or homogeneity, of all but the first matrix on the right-hand-side is highly complex and extremely difficult to prove using the direct methods of the previous sections. Therefore, we will pursue another approach to the intertemporal duality of the dynamic household production and consumption problem (4.8).

This alternative way of looking at problems of this type has several advantages. The approach is simple and both heuristically and pedagogically appealing. It establishes a connection between the duality of static models and both discrete and continuous dynamic models, including the envelope theorem, adding up, homogeneity, symmetry and negativity, and the relationships between utility maximization and expenditure minimization. Finally, and at least as important as the simple and clear derivations, the arguments are valid for a very large class of problems — essentially all of optimal control theory.}

\[ + \left[ \int_0^T e^{-\tau} \phi' H_{xx}^{-1} \left( H_{xx} \frac{\partial \kappa}{\partial M_0} + \frac{\partial f'}{\partial x} \frac{\partial \omega}{\partial M_0} \right) dt \right] \left( \int_0^T e^{-\tau} \frac{\partial \phi'}{\partial p_0} h dt \right) \] 

\[ - \left[ \int_0^T e^{-\tau} \frac{\partial \phi'}{\partial p_0} H_{xx}^{-1} \left( H_{xx} \frac{\partial \kappa}{\partial M_0} + \frac{\partial f'}{\partial x} \frac{\partial \omega}{\partial M_0} \right) dt \right] \left( \int_0^T e^{-\tau} \frac{\partial \phi'}{\partial p_0} h dt \right) \]
We proceed by defining the “Lagrangean” function for (4.8) by

\[(4.27)\quad \ell_1 \equiv \int_0^T \bar{u}dt + \lambda \left( M_0 - \int_0^T e^{-\tau t} \phi' xdt \right) + \int_0^T \omega'(f - \dot{k})dt \]

\[
\equiv \int_0^T \left( \bar{u} - \lambda e^{-\tau t} \phi' x + \omega f + \omega' \dot{k} \right)dt + \lambda M_0 + \omega(0)k_0 - \omega(T)k(T),
\]

where the second line follows from integrating the term \(-\omega' \dot{k}\) by parts. Finding the pointwise maximum with respect to \(x\) of either the first or second lines of (4.27) reproduces first-order condition (4.10). Similarly, minimizing either expression for \(\ell_1\) with respect to \(\lambda\) reproduces (4.14) and also motivates the constant marginal utility of money condition given in (4.12). On the other hand, pointwise minimization of the first line of (4.27) with respect to \(\omega\) gives (4.13), while pointwise maximization of the second line with respect to \(k\) generates (4.11). Also note that when the first-order conditions are satisfied \(\forall t \in [0, T]\), the integrals of the constraints multiplied by their associated shadow prices vanish. This, in turn, implies that \(\ell_1^\#(p_0, M_0, k_0, \beta, \gamma) \equiv V(p_0, M_0, k_0, \beta, \gamma)\). From the second line of (4.27), this simple observation immediately generates the following pair of dynamic envelope theorem results:

\[(4.28)\quad \frac{\partial V}{\partial M_0} \equiv \lambda_0 > 0; \quad \text{and}\]

\[(4.29)\quad \frac{\partial V}{\partial k_0} \equiv \omega(p_0, M_0, k_0, \beta, \gamma, 0).\]

Several other dynamic envelope theorem results, as well as symmetry, curvature, and homogeneity properties also can be derived from the Lagrangean in (4.27). We do so in detail here for \(\partial V/\partial p_0\) to illustrate the basic logic. We follow this with a statement of
the properties of the dynamic indirect utility function for this problem. We then proceed with a brief development of the properties of the dynamic expenditure function. We conclude this section with a statement of the intertemporal duality for this problem.

We first proceed by substituting (4.16) — (4.19) into (4.27) to generate 

$$\ell^*_1(p_0, M_0, k_0, \beta, \gamma).$$

Then we differentiate the resulting expression term-by-term with respect to $p_0$, which gives

$$\frac{\partial \ell^*_1}{\partial p_0} = \theta \int_0^T \frac{\partial^2 \hat{u}}{\partial x} \left( -\lambda e^{-rt} \Phi + \frac{\partial^2 f}{\partial x} \right) dt + \int_0^T \frac{\partial \kappa}{\partial k} \left( \frac{\partial \hat{u}}{\partial k} + \frac{\partial f^\prime}{\partial k} \Omega + \Omega \right) dt$$

$$+ \int_0^T \frac{\partial \kappa}{\partial p_0} \left( M_0 - \int_0^T e^{-rt} \Phi^\prime hdt \right) - \lambda \int_0^T e^{-rt} \frac{\partial \Phi^\prime}{\partial p_0} hdt + \int_0^T \frac{\partial \omega^\prime}{\partial p_0} fdt$$

$$+ \int_0^T \frac{\partial^2 \omega}{\partial t} \hat{k} dt + \frac{\partial \omega(\cdot, 0)}{\partial p_0} k_0 = \frac{\partial \omega(\cdot, T)}{\partial p_0} \kappa(\cdot, T) - \frac{\partial \kappa(\cdot, T)}{\partial p_0} \omega(\cdot, T),$$

where the first three terms vanish by the first-order conditions (4.10) — (4.14) and the final term vanishes by the transversality conditions (4.15). Given the properties hypothesized for $\hat{u}(\cdot), f(\cdot),$ and $\Phi(\cdot)$, the Marshallian demands and the marginal utility of money are continuously differentiable, while the household durables and their shadow prices are twice continuously differentiable. Hence, by Young’s theorem, we can integrate the terms $(\partial^2 \omega/\partial t \partial p_0) \kappa \equiv (\partial^2 \omega/\partial p_0 \partial t) \kappa$ by parts, which gives

$$\int_0^T \frac{\partial^2 \omega}{\partial p_0 \partial t} \kappa dt \equiv \frac{\partial \omega(\cdot, T)^\prime}{\partial p_0} \kappa(\cdot, T) - \frac{\partial \omega(\cdot, 0)^\prime}{\partial p_0} k_0 - \int_0^T \frac{\partial \omega}{\partial p_0} \frac{\partial \kappa}{\partial t} dt.$$

Canceling the terms that vanish on the right-hand-side of (4.30) and substituting the right-
hand-side of (4.31) into (4.30) gives

\[
\frac{\partial \ell_1}{\partial p_0} = -\lambda_0 \int_0^T e^{-rt} \frac{\partial \Phi'}{\partial p_0} h dt + \int_0^T \frac{\partial \omega'}{\partial p_0} \left( f - \frac{\partial \kappa}{\partial t} \right) dt = -\lambda_0 \int_0^T e^{-rt} \frac{\partial \Phi'}{\partial p_0} h dt.
\]

Applying the identity \( \ell_1^*(p_0, M_0, k_0, \beta, \gamma) \equiv V(p_0, M_0, k_0, \beta, \gamma) \) we therefore can state the dynamic envelope theorem with respect to the initial price vector as

\[
\frac{\partial V}{\partial p_0} = \int_0^T \left( \frac{\partial h'}{\partial p_0} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial p_0} \frac{\partial \tilde{u}}{\partial k} \right) dt = -\lambda_0 \int_0^T e^{-rt} \frac{\partial \Phi'}{\partial p_0} h dt.
\]

We follow essentially the same steps for each of the other sets of parameters to obtain the following list of properties for the dynamic indirect utility function.

**Theorem 6.** The dynamic indirect utility function in (4.8) is twice continuously differentiable in \((p_0, M_0, k_0, \beta, \gamma)\) and satisfies

\[
\begin{align*}
(6.a) \quad & \frac{\partial V}{\partial p_0} = \int_0^T \left( \frac{\partial h'}{\partial p_0} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial p_0} \frac{\partial \tilde{u}}{\partial k} \right) dt = -\lambda_0 \int_0^T e^{-rt} \frac{\partial \Phi'}{\partial p_0} h dt; \\
(6.b) \quad & \frac{\partial V}{\partial M_0} = \int_0^T \left( \frac{\partial h'}{\partial M_0} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial M_0} \frac{\partial \tilde{u}}{\partial k} \right) dt = \lambda_0 > 0; \\
(6.c) \quad & \frac{\partial V}{\partial k_0} = \int_0^T \left( \frac{\partial h'}{\partial k_0} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial k_0} \frac{\partial \tilde{u}}{\partial k} \right) dt = \omega(\cdot, 0); \\
(6.d) \quad & \frac{\partial V}{\partial \beta} = \int_0^T \left( \frac{\partial h'}{\partial \beta} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial \beta} \frac{\partial \tilde{u}}{\partial k} \right) dt = \int_0^T \frac{\partial \tilde{u}}{\partial \beta} dt; \text{ and} \\
(6.e) \quad & \frac{\partial V}{\partial \gamma} = \int_0^T \left( \frac{\partial h'}{\partial \gamma} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial \gamma} \frac{\partial \tilde{u}}{\partial k} \right) dt = \int_0^T \frac{\partial f'}{\partial \gamma} \omega dt. \\
(6.f) \quad & \text{If } \Phi(\cdot, t) \text{ is } 1^\circ \text{ homogeneous in } p_0, \text{ then } V(\cdot, k_0, \beta, \gamma) \text{ is } 0^\circ \text{ homogeneous}
\end{align*}
\]
Duality Theory for the Household

in \((p_0, M_0)\).

(6.g) If \(\varphi(\cdot, t)\) is increasing and concave in \(p_0\), then \(V(\cdot, k_0, \beta, \gamma)\) is decreasing and quasiconvex in \(p_0\).

In addition, the dynamic Marshallian demand functions satisfy the intertemporal budget identity, Cournot aggregation, Engle aggregation, and Roy’s identity,

(6.h) \(\int_0^T e^{-rt} \varphi' h dt \equiv M_0\),

(6.i) \(\int_0^T e^{-rt} \left[ \frac{\partial \varphi'}{\partial p_0} h + \frac{\partial h'}{\partial p_0} \varphi \right] dt \equiv 0\),

(6.j) \(\int_0^T e^{-rt} \varphi' \frac{\partial h}{\partial M_0} dt \equiv 1\), and

(6.k) \(-\frac{\partial V}{\partial p_0} = \int_0^T e^{-rt} \frac{\partial \varphi'}{\partial p_0} h dt\), respectively.

The properties of the dynamic expenditure function,

\[ \mathcal{E}(p_0, U_0, k_0, \beta, \gamma) \]

\[ \equiv \inf \left\{ \int_0^T e^{-rt} \varphi' x dt : u_0 = \int_0^T \tilde{u}(x, k, \beta, t) dt, \dot{k} = f(x, k, \gamma, t), k(0) = k_0, k(T) \geq 0 \right\} \]

are derived in a similar way. We first define the Lagrangean function for the consumer’s intertemporal cost minimization problem as

(4.36) \[ \ell_2 = \int_0^T e^{-rt} \varphi' x dt + \mu \left( U_0 - \int_0^T \tilde{u} dt \right) + \int_0^T \psi' (\dot{k} - f) dt \]

\[ \equiv \int_0^T \left( e^{-rt} \varphi' x dt - \mu \tilde{u} - \psi' f - \psi' k \right) dt + \mu U_0 + \omega(T) \dot{k}(T) - \omega(0) \dot{k}_0, \]

where \(\mu\) is the shadow price on the discounted utility constraint, \(\psi\) is the vector of shadow
prices for the equations of motion for household durable goods, and as before, the second line of (4.36) is obtained by integrating the terms $\psi'k$ by parts. We continue to assume that $(x^*(t),k^*(t)) \rightarrow (0,0)$ $\forall t \in [0,T)$, as well as the previous regularity conditions for $\tilde{u}(\cdot)$ and $f(\cdot)$.

The necessary and sufficient first-order conditions for the unique optimal path now are:

\begin{align}
(4.37) \quad & e^{-\gamma} \varphi - \mu \frac{\partial \tilde{u}}{\partial x} - \frac{\partial f'}{\partial x} \psi = 0; \\
(4.38) \quad & \mu \frac{\partial \tilde{u}}{\partial k} + \frac{\partial f'}{\partial k} \psi + \psi = 0; \\
(4.39) \quad & \dot{\mu} = 0; \\
(4.40) \quad & f = \dot{k}, \ k(0) = k_0, \ k(T) \geq 0; \text{ and} \\
(4.41) \quad & U_0 = \int_0^T \tilde{u} \, dt ;
\end{align}

\begin{align}
\text{together with the transversality conditions} \\
(4.42) \quad & \psi_j(T)k_j(T) = 0 \quad \forall \ j = 1, \ldots, \ell .
\end{align}

At time $t$, the optimal Hicksian demands, stocks of household durable goods, and shadow prices are

\begin{align}
(4.43) \quad & x^*(t) \equiv g(p_0,U_0,k_0,\beta,\gamma,t) , \\
(4.44) \quad & k^*(t) \equiv \xi(p_0,U_0,k_0,\beta,\gamma,t) , \\
(4.45) \quad & \psi^*(t) \equiv \psi(p_0,U_0,k_0,\beta,\gamma,t) , \text{ and} \\
(4.46) \quad & \mu^*(t) \equiv \mu_0(p_0,U_0,k_0,\beta,\gamma) ,
\end{align}
respectively.

The first-order conditions imply that the optimal Lagrangean function and the dynamic expenditure function satisfy

\[ (4.47) \quad \ell_\gamma(p_0, U_0, k_0, \beta, \gamma) \equiv E(p_0, U_0, k_0, \beta, \gamma) \]

\[ \equiv \int_0^T e^{-rt} \phi(p_0, t) g(p_0, U_0, k_0, \beta, \gamma, t) dt . \]

This, in turn, when combined with the discounted utility constraint, \( U_0 \equiv \int_0^T \tilde{u}(\cdot) dt \), implies the following set of properties for \( E(p_0, U_0, k_0, \beta, \gamma) \):

**Theorem 7.** The dynamic expenditure function in (4.36) is twice continuously differentiable in \((p_0, U_0, k_0, \beta, \gamma)\) and satisfies

\[ (6.a) \quad \frac{\partial E}{\partial p_0} = \int_0^T e^{-rt} \frac{\partial \phi'}{\partial p_0} g dt ; \]

\[ (6.b) \quad \frac{\partial E}{\partial U_0} = \mu_0 > 0 ; \]

\[ (6.c) \quad \frac{\partial E}{\partial k_0} = -\psi(\cdot, 0) ; \]

\[ (6.d) \quad \frac{\partial E}{\partial \beta} = \int_0^T e^{-rt} \frac{\partial g'}{\partial \beta} \phi dt \equiv -\mu_0 \int_0^T \frac{\partial \tilde{u}}{\partial \beta} dt ; \text{ and} \]

\[ (6.e) \quad \frac{\partial E}{\partial \gamma} = \int_0^T e^{-rt} \frac{\partial g'}{\partial \gamma} \phi dt \equiv -\int_0^T \frac{\partial f'}{\partial \gamma} \psi dt . \]

\[ (6.f) \quad E(\cdot, U_0, k_0, \beta, \gamma) \text{ is } 1^\circ \text{ homogeneous in } p_0 \text{ if and only if } \phi(\cdot, t) \text{ is } 1^\circ \text{ homogeneous in } p_0. \]

\[ (6.g) \quad \text{If } \phi(\cdot, t) \text{ is increasing and concave in } p_0, \text{ then } E(\cdot, U_0, k_0, \beta, \gamma) \text{ is increas-} \]
ing and concave in $p_0$, with Hessian matrix defined by

$$\frac{\partial^2 E}{\partial p_0 \partial p_0'} = \int_0^T e^{-\xi t} \left( \sum_{i=1}^n g_i \frac{\partial^2 \phi_i'}{\partial p_0 \partial p_0'} + \frac{\partial \phi_i'}{\partial p_0} \frac{\partial g}{\partial p_0'} \right) dt.$$  

In addition, the dynamic Hicksian demand functions and the expenditure minimizing demands for household durables satisfy,

(6.h) $$\int_0^T \tilde{u}(g, \xi, \beta, t) dt \equiv U_0,$$

(6.i) $$\int_0^T \left( \frac{\partial g'}{\partial p_0} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \xi'}{\partial p_0} \frac{\partial \tilde{u}}{\partial k} \right) dt \equiv 0,$$

(6.j) $$\int_0^T \left( \frac{\partial g'}{\partial U_0} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \xi'}{\partial U_0} \frac{\partial \tilde{u}}{\partial k} \right) dt \equiv 1,$$

(6.k) $$\int_0^T \left( \frac{\partial g'}{\partial \beta} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \xi'}{\partial \beta} \frac{\partial \tilde{u}}{\partial k} + \frac{\partial \tilde{u}}{\partial \beta} \right) dt \equiv 0,$$ and

(6.l) $$\int_0^T \left( \frac{\partial g'}{\partial \gamma} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \xi'}{\partial \gamma} \frac{\partial \tilde{u}}{\partial k} \right) dt \equiv 0.$$

The final piece of the puzzle is to establish the dual relationship between the dynamic expenditure and indirect utility functions as inverses to each another with respect to their $n+1^{st}$ arguments. We now will show that if $U_0 = V(p_0, M_0, k_0, \beta, \gamma)$, then $E(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv M_0$; equivalently, if $M_0 = E(p_0, U_0, k_0, \beta, \gamma)$, then $V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv U_0$. Intuitively, this seems obvious. The minimum present value of discounted consumption expenditures necessary to obtain the maximum present value of discounted utility flows that can be afforded with initial wealth $M_0$ must be $M_0$. Utilizing the Lagrangeans in (4.27) and (4.36) makes the proofs nearly as obvious.
The intuition actually is quite simple. From theorem 4, recall that in the static case, \( \mu(p,u,b) \equiv 1/\lambda(p,e(p,u,b),b) \), and similarly, \( \lambda(p,m,b) \equiv 1/\mu(p,v(p,m),b) \), where \( \lambda \) is the marginal utility of money, \( \mu \) is the marginal cost of utility, and the first-order conditions for expenditure minimization and utility maximization are identical when income is set equal to expenditure. In the dynamic case, we will show that analogous properties hold, although the argument is slightly more involved.

Let \( M_0 = E(p_0, U_0, k_0, \beta, \gamma) \), and note that

\[
(4.48) \quad \ell_1^* (p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv \int_0^T \bar{u} \left( h(p_0, E(p_0, U_0, k_0, \beta, \gamma, t), k_0, \beta, \gamma, t), \kappa(p_0, E(p_0, U_0, k_0, \beta, \gamma, t), \beta, \gamma, t), \beta, t \right) dt
\]

\[
\equiv V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \geq U_0,
\]

where the inequality follows from maximization and the fact that \( U_0 \) is affordable. Writing out \( \ell_1^* \) explicitly, we have

\[
(4.49) \quad 0 \leq \ell_1^* - U_0 = \int_0^T \bar{u} dt + \lambda_0 \left( M_0 - \int_0^T e^{-\gamma t} \phi' h dt \right) + \int_0^T \omega'(f - \kappa) dt - U_0
\]

\[
= \lambda_0 \left\{ M_0 - \left[ \int_0^T e^{-\gamma t} \phi' h dt + (1/\lambda_0) \left( U_0 - \int_0^T \bar{u} dt \right) + \int_0^T \omega'(\omega/\lambda_0)'(\kappa - f) dt \right] \right\}
\]

\[
= \lambda_0 \left( M_0 - \hat{\ell}_2 \right) \leq 0,
\]

where \( \hat{\ell}_2 \) is the Lagrangean for the expenditure minimization problem evaluated along the utility maximizing path for \( x \) and \( k \) with \( \mu_0 = 1/\lambda_0 \) and \( \xi \equiv \omega \lambda_0 \).

The second inequality follows from the fact that this path is feasible, so that
\[ M_0 = \int_0^T e^{-rt} \varphi' g dt = \ell_2^*(p_0, U_0, k_0, \beta, \gamma) \leq \ell_2^* = \int_0^T e^{-rt} \varphi' h dt . \] It follows immediately that
\[ V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv U_0 . \]

The argument for \( E(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv M_0 \) is identical, with the roles of the dynamic expenditure and indirect utility functions interchanged.

We therefore have the following rather remarkable result. Only parts h and i of the next theorem are not immediately obvious from the previous developments. However, this pair of conclusions follows from: (a) the uniqueness of the optimal paths for the two problems; (b) the above relationships among the shadow prices; and (c) the fact that the first-order conditions for the two problems are equivalent \( \forall t \in [0, T] \). Hence, no further proof is necessary to establish the following.

**Theorem 8.** The dynamic indirect utility and expenditure functions for the intertemporal consumer choice problem with household production and non-static price expectations satisfy

\[ (8.a) \quad V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv U_0 ; \]
\[ (8.b) \quad E(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv M_0 ; \]
\[ (8.c) \quad \frac{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial M_0} \equiv \lambda_0(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \]
\[ \equiv \frac{1}{\mu_0(p_0, U_0, k_0, \beta, \gamma)} \equiv \frac{1}{\partial E(p_0, U_0, k_0, \beta, \gamma) / \partial U_0} ; \]
\[ (8.d) \quad \frac{\partial E(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial U_0} \equiv \mu_0(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \]
\[ \frac{1}{\lambda(p_0, M_0, k_0, \beta, \gamma)} \equiv \frac{1}{\partial V(p_0, M_0, k_0, \beta, \gamma)/\partial M_0}; \]

(8.e) \[ \frac{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial k_0} \equiv \omega(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma, 0) \]

\[ \equiv \psi(p_0, U_0, k_0, \beta, \gamma, 0) \quad \frac{\partial E(p_0, U_0, k_0, \beta, \gamma)}{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial U_0}; \]

(8.f) \[ \frac{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial \beta} \equiv - \frac{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial \beta}{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial U_0}; \]

(8.g) \[ \frac{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial \gamma} \equiv - \frac{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial \gamma}{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial U_0}; \]

(8.h) \[ \frac{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial p_0} \]

\[ \equiv \int_0^T e^{-\tau} \frac{\partial \psi(p_0, t)}{\partial p_0} h(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma, t) dt \]

\[ \equiv \int_0^T e^{-\tau} \frac{\partial \psi(p_0, t)}{\partial p_0} g(p_0, U_0, k_0, \beta, \gamma, t) dt \equiv \frac{\partial E(p_0, U_0, k_0, \beta, \gamma)}{\partial p_0}; \]

(8.f) \[ h(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma, t) \equiv g(p_0, U_0, k_0, \beta, \gamma, t) \forall t \in [0, T]; \]

(8.g) \[ g(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma, t) \equiv h(p_0, M_0, k_0, \beta, \gamma, t) \forall t \in [0, T]; \]

and

\[ \frac{\partial^2 E(p_0, U_0, k_0, \beta, \gamma)}{\partial p_0^2} \]

\[ \equiv \int_0^T e^{-\tau} \sum_{i=1}^n g_i(p_0, U_0, k_0, \beta, \gamma, t) \frac{\partial^2 \psi(p_0, t)}{\partial p_0 \partial p_0^*} dt \]

\[ + \int_0^T e^{-\tau} \sum_{i=1}^n \frac{\partial \psi(p_0, t)}{\partial p_0} \frac{\partial g(p_0, U_0, k_0, \beta, \gamma, t)}{\partial p_0^*} dt \]
\[ e^{-\eta} \sum_{i=1}^{n} h'(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma, t) \frac{\partial^2 \varphi(p_0, t)}{\partial p_0 \partial p_0'} \sigma_{i,j} \]

\[ + \int_{0}^{T} e^{-\eta} \frac{\partial \varphi(p_0, t)'}{\partial p_0} \frac{\partial h(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma, t)}{\partial p_0'} \sigma_{i,j} \]

\[ + \left( \int_{0}^{T} e^{-\eta} \frac{\partial \varphi(p_0, t)'}{\partial p_0} \frac{\partial h(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma, t)}{\partial M_0} \right) \times \]

\[ \left( \int_{0}^{T} e^{-\eta} \frac{\partial \varphi(p_0, t)'}{\partial p_0} h(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma, t) dt \right)' \]

is a symmetric $n \times n$ matrix with rank no greater than $n-1$ and is negative semidefinite if $\varphi(\cdot, t)$ is concave in $p_0$.

5. Discussion

The static neoclassical model provides a solid foundation for the host of generalizations to consumer choice theory considered in this chapter. The central core of the duality theory of the neoclassical model transcends the theory of household production, characteristics theory, and intertemporal consumer choice, models of consumer expectations for future values of important economic variables, durable goods, consumption habits, and changing household production technologies and/or goods qualities and characteristics.

This illustrates a robust theoretical framework. But the way that the duality theory manifests itself varies substantially across specifications. When intertemporal considerations are added, there no longer is any short-run, or instantaneous, counterpart to the static neoclassical model’s Slutsky symmetry and negativity conditions. Once the proper concept of substitution has been taken into account in a dynamic setting, however, the precise
nature of the symmetry condition becomes self evident. In addition, the standard homogeneity and curvature conditions of the static model are not necessarily satisfied in a dynamic framework. Again, however, once the influences of consumers’ expectations about future economic variables on their optimal plans have been identified, the conditions in which homogeneity and curvature are satisfied become apparent.

The analysis in this chapter shows that the naïve way that consumption habits and durable goods have traditionally been treated in empirical demand analysis suffers from several weaknesses, particularly when myopic expectations are not assumed. On the other hand, myopic expectations in dynamic consumption models is fraught with logical inconsistencies in its own right.

This raises several questions for future research in empirical demand analysis. How do we estimate rational consumption habits? Can we distinguish between changes in household durables and changes in consumption habits? Does habit formation exist? What other economic trends do consumption habits proxy for – e.g., changing demographics, changes in the distribution of income over time, nonlinearities in the income responses of goods’ demand, and/or structural change in the demand for foods and other goods? Can expectations be modeled successfully in empirical demand studies? In a dynamic setting with rational consumers, how do we correctly measure the consumer welfare effects of policy changes? If consumers use rational forecasting models and are intertemporal utility maximizers, do they respond to future uncertainty with open loop or closed loop consumption plans? Can these solution types be distinguished empirically?
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