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Non-Constant Discounting in Continuous  
Time

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## **Abstract**

This note derives the dynamic programming equation (DPE) to a differentiable Markov Perfect equilibrium in a problem with non-constant discounting and general functional forms. We begin with a discrete stage model and take the limit as the length of the stage goes to 0 to obtain the DPE corresponding to the continuous time problem. We characterize the multiplicity of equilibria under non-constant discounting and discuss the relation between a given equilibrium of that model and the unique equilibrium of a related problem with constant discounting. We calculate the bounds of the set of candidate steady states and we Pareto rank the equilibria.

# Non-constant discounting in continuous time\*

Larry Karp<sup>†</sup>

January 5, 2004

## Abstract

This note derives the dynamic programming equation (DPE) to a differentiable Markov Perfect equilibrium in a problem with non-constant discounting and general functional forms. We begin with a discrete stage model and take the limit as the length of the stage goes to 0 to obtain the DPE corresponding to the continuous time problem. We characterize the multiplicity of equilibria under non-constant discounting and discuss the relation between a given equilibrium of that model and the unique equilibrium of a related problem with constant discounting. We calculate the bounds of the set of candidate steady states and we Pareto rank the equilibria.

*Keywords:* hyperbolic discounting, time consistency, Markov equilibria, non-uniqueness, observational equivalence, Pareto efficiency

*JEL classification numbers* D83, L50

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# 1 Introduction

This note studies the set of differentiable Markov Perfect equilibria in a control problem with declining discount rates. The basic model is a discrete stage infinite horizon problem in which each stage lasts for  $\varepsilon$  units of time. The discount rate declines during the first  $n = \frac{T}{\varepsilon}$  periods, where  $n$  is an arbitrarily large but finite constant. After  $n$  periods the discount rate becomes constant. Harris and Laibson (2001) derive the Dynamic Programming Equation (DPE) under quasi-hyperbolic discounting (where  $n = 1$ ). Their method can also be used to obtain the DPE for the more general problem in this paper (where  $n$  is arbitrary). A solution to this DPE, i.e. a value function and a control rule, is indexed by the parameter  $\varepsilon$ . Formally taking the limit as  $\varepsilon \rightarrow 0$  yields a continuous time DPE that corresponds to the continuous time limit of the discrete stage problem.

Inspection of the DPE shows how non-constant discounting changes the control problem, relative to the case of constant discounting. A Markov equilibrium to the problem under non-constant discounting can be obtained by solving a control problem with constant discounting but a different objective function, jointly with a “side condition”. The necessary condition to the DPE leads to a differential equation that the optimal control must satisfy, an Euler equation. This equation shows in a transparent manner that (i) differentiable Markov Perfect equilibria (MPE) are generically non-unique, and that (ii) the solution under non-constant discounting is, in general, not observationally equivalent to the solution to a control problem with constant discounting (holding the utility function fixed). Under a restriction we show that more conservative decision rules (i.e. those that result in a higher steady state of the resource) Pareto dominate less conservative rules.

Most recent papers on hyperbolic discounting, including those that obtain explicit solutions, appear not to have recognized the multiplicity of differentiable MPE. Karp (2003) discusses the multiplicity of equilibria in the context of a discrete time model with quasi-hyperbolic discounting. The multiplicity arises because of an “incomplete transversality condition” in the infinite horizon problem. This circumstance is analogous to the incomplete transversality condition in differential games noted by Tsutsui and Mino (1990). This source of non-uniqueness is different than in Krusell and Smith (2003), which relies on decision rules that are step functions – and therefore not differentiable. Our construction of an equilibrium assumes differentiability. The Markov assumption eliminates the type of non-uniqueness that arises when strategies are supported by the use of threats, as in Laibson (1994).

The fact that the equilibrium decision rules under constant and non-constant discount rates are in general not observationally equivalent is by now well known; see, for example Barro (1999), Diamond and Koszegi (2003), Luttmer and Mariotti (2003). However, the generic non-uniqueness of Markov equilibria means that even if a particular decision rule under non-constant discounting is observationally equivalent to the rule under constant discounting, there exists some other equilibrium rule that is not observationally equivalent.

The next two sections describe the problem and the solution method, and present the results. Proofs are in the Appendix.

## 2 Model

The instantaneous discount rate at time  $s$  is a function  $r(s)$  for  $0 \leq s \leq T$ , and  $r(s) = \bar{r}$  (a constant) for  $s \geq T$ . We assume that  $r(s)$  is a non-increasing function, since this seems to be the most interesting case. The discount factor at time  $t$  used to evaluate a payoff at time  $\tau + t$ ,  $\tau \geq 0$ , is

$$\theta(\tau) = \exp\left(-\int_0^\tau r(s)ds\right).$$

This formulation includes the possibility that  $T = \infty$ , i.e. the discount rate converges to a constant asymptotically. For  $T = \infty$  this formulation also includes the possibility that  $\bar{r} = 0$ . Provided that instantaneous utility is finite, a sufficient condition for convergence of the payoff is

$$\int_0^\infty \theta(\tau)d\tau = \int_0^T \theta(\tau)d\tau + \theta(T) \int_T^\infty \exp(-\bar{r}s) ds < \infty. \quad (1)$$

For finite  $T$  the integral diverges when  $\bar{r} = 0$ . We view the case  $\bar{r} = 0$  as the limit where  $\bar{r} \rightarrow 0$  and  $T \rightarrow \infty$ . As  $\bar{r} \rightarrow 0$  it must be the case that  $T \rightarrow \infty$  sufficiently rapidly for the integral to converge.

The state variable  $S$  evolves according to

$$\dot{S} = f(S, x) \quad (2)$$

where  $x$  is the control variable. At time  $t$  the state  $S_t$  is predetermined. The flow of payoff is the concave function  $U(S, x)$ , which is increasing in  $x$ . The present discounted value of the stream of future payoffs evaluated at time  $t$  is

$$\int_0^\infty \theta(\tau)U(S_{t+\tau}, x_{t+\tau})d\tau. \quad (3)$$

This problem has an infinite horizon, regardless of the value of  $T$ .

One way to study this problem is to begin with a discrete stage approximation in which each period lasts for  $\varepsilon$  units of time, and all variables are constant within an interval. This approximation replaces equation (2) and the objective in (3) with

$$\begin{aligned} S_{t+\varepsilon} &= f(S_t, x_t) \varepsilon \\ \sum_{i=0}^{\infty} \theta(i\varepsilon) U(S_{t+i\varepsilon}, x_{t+i\varepsilon}) \varepsilon. \end{aligned} \tag{4}$$

For fixed  $T < \infty$  and  $\varepsilon > 0$ , the number of periods during which the discount rate is non-constant is  $n = \frac{T}{\varepsilon} < \infty$ . (We ignore the “integer problem”.) The fact that  $n < \infty$  makes it possible to use an extension of a method described in Harris and Laibson (2001) to obtain a rigorous and transparent derivation of the DPE. This DPE contains  $\varepsilon$  as an argument. Passing to the continuous time limit of this DPE is “formal”, in the sense that the procedure assumes that the endogenous functions are analytic in  $\varepsilon$ , so that the Taylor approximation is valid. The continuous time DPE contains  $T$  as an argument.

### 3 Results

The first subsection discusses the DPE for the general problem. The next subsection specializes this problem and discusses the issues of observational non-equivalence, non-uniqueness, and Pareto efficiency.

#### 3.1 The general problem

The Appendix derives the discrete time DPE and its continuous time limit. Here we describe the results, after introducing some notation. Let  $\chi(s)$  be the equilibrium control rule for the continuous time limit of the discrete stage problem;  $\chi(S)$  is a function to be determined. Define

$$H(S) \equiv U(S, \chi(S)), \tag{5}$$

the flow of payoff under the equilibrium rule. The function  $K(S)$  is defined as

$$K(S_t) \equiv \int_0^T \theta(\tau) (r(\tau) - \bar{r}) H(S_{t+\tau}) d\tau. \tag{6}$$

In writing the expression on the right side of (6) as a function of  $S_t$  we use the fact that  $x_\tau = \chi(S_\tau)$  in equilibrium. After substituting this function into equation (2) and using the fact that

$S_t$  is predetermined at  $t$ , the solution to equation (2) can be written as

$$S_{t+\tau} = g(\tau; S_t). \quad (7)$$

Using this expression, an alternative form of  $K$  is

$$K(S_t) \equiv \int_0^T \theta(\tau) (r(\tau) - \bar{r}) H(g(\tau; S_t)) d\tau. \quad (8)$$

The derivative of  $K$  is

$$K'(S_t) \equiv \int_0^T \theta(\tau) (r(\tau) - \bar{r}) H'(g(\tau; S_t)) \frac{\partial g(\tau; S_t)}{\partial S_t} d\tau, \quad (9)$$

where

$$H'(S) = U_x(\cdot) \chi'(S) + U_S(\cdot). \quad (10)$$

Define  $W(S_t)$  as the value function, i.e. as the equilibrium value of the payoff in expression (3) when the control rule  $\chi(S)$  is used and the initial condition is  $S_t$ . Our first result is:

**Proposition 1** *The limit as  $\varepsilon \rightarrow 0$  of the dynamic programming equation to the discrete stage problem in equation (4) is*

$$K(S_t) + \bar{r}W(S_t) = \max_x \{U(S_t, x) + W'(S_t)f(S_t, x)\}. \quad (11)$$

The parameter  $T$  (the amount of time during which the discount rate is falling) appears in the definition of the function  $K$ , in equation (8). Although we need a finite value of  $T$  to begin the derivation that leads to the DPE, we can study the problem with  $T = \infty$  by taking limits with respect to  $T$ . The presence of the function  $K(S_t)$  in the DPE is due to non-constant discounting.

Equations (8) and (11) provide a basis for a numerical solution to the continuous time problem. The solution consists of a function  $\chi(S)$  that maximizes  $U + W'f$ , and satisfies  $K + \bar{r}W = H + W'f$  and equation (8). The appendix discusses a numerical solution to this model.

The following is obvious from inspection of equation (11):

**Remark 1** *The MPE to the control problem with a non-constant discount rate is produced by solving the necessary conditions to the following control problem:*

$$\begin{aligned} \max \int_0^\infty e^{-\bar{r}\tau} (U(S_{t+\tau}, x_{t+\tau}) - K(S_{t+\tau})) d\tau \\ \text{subject to } \dot{S} = f(S, x). \end{aligned} \quad (12)$$



*In this problem, the decision-maker treats the function  $K(\cdot)$  as exogenous, since this function is determined by the behavior of “future decision-makers”, and the current decision-maker takes that behavior as given. The function  $K(\cdot)$  is given by equation (8), the “side condition” alluded to in the Introduction.*

Since  $K(\cdot)$  is endogenous, we cannot guarantee that the control problem in equation (12) satisfies any of the standard sufficiency conditions. Therefore, our analysis relies on necessary conditions.

Under plausible circumstances, the equilibrium flow of payoff is positive and increasing in the stock ( $H > 0$  and  $H' > 0$ ) along the equilibrium trajectory. In this case,  $K > 0$ , and  $K' > 0$ . These inequalities imply that a MPE can be generated by solving a control problem with constant discounting and a negative and decreasing stock amenity,  $-K$ . Other things equal, this negative stock amenity reduces the trajectory of the state variable. If the decision-maker (for the original problem with the decreasing discount rate) were able to make binding commitments she would reduce future consumption over an interval of time in order to allow the stock to grow, thereby making higher future consumption possible. In the Markov equilibrium, where the decision maker cannot make commitments, she has an incentive to reduce the future stock in an effort to induce future decision makers to reduce their consumption. The negative amenity value,  $-K$ , in the control problem (12) reflects this incentive.

The “transversality condition at infinity” for this problem is  $\lim_{t \rightarrow \infty} \theta(t) W'(S_t) = 0$  (Kamihigashi 2001). We assume that this condition is satisfied by requiring that the state converge to a steady state, a solution to  $f(S, \chi(S)) = 0$ . Asymptotic stability of a steady state,  $S_\infty$ , requires

$$z(S_\infty) \equiv f_S + f_x \chi' < 0 \text{ where } f(S_\infty, \chi(S_\infty)) = 0. \quad (13)$$

Standard manipulations of equation (11) lead to the Euler equation. For the general model this Euler equation is not revealing, so we only provide the equation for a special case in the next subsection. We define a candidate steady state  $S_\infty$  as a value of  $S$  that satisfies both the Euler equation (not shown) evaluated at the steady state, and the stability condition (equation (13)). For a particular steady state  $S_\infty$  that satisfies both of these conditions, define  $\chi(S; S_\infty)$  as the function that solves the Euler equation and that drives the state to  $S_\infty$  (thereby satisfying the transversality condition at infinity). We refer to  $\chi(S; S_\infty)$  as a “candidate” equilibrium, since – as we remarked above – we use only the necessary conditions to construct the function.<sup>1</sup>

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<sup>1</sup>The policy function  $\chi(S; S_\infty)$  may be defined only for a subset of state space. A similar circumstance arises

## 3.2 A specialization

Here we consider a special case where it is easy to compare the necessary condition under constant and non-constant discounting:  $U_S \equiv 0$  and  $f(S, x) = F(S) - x$  (i.e.,  $f_x \equiv -1$ ). These restrictions are appropriate in a resource problem where the flow of utility is independent of the stock, and consumption reduces the stock linearly. In this special case, standard manipulations of the DPE, equation (11), result in the Euler equation (a “modified Ramsey Rule”):

$$\eta(x_t) \frac{\dot{x}}{x} = F'(S_t) - \left( \bar{r} + \frac{K'(S_t)}{U'(x_t)} \right) \quad (14)$$

where  $\eta = -\frac{U''}{U'}x$  is the elasticity of marginal utility,  $K'$  is given by equation (9), and  $\bar{r} + \frac{K'(S_t)}{U'(x_t)}$  is the effective discount rate. If the discount rate is constant ( $r(s) \equiv \bar{r}$  for all  $s$ ) then  $K'(S) \equiv 0$  and equation (14) reproduces the standard Ramsey Rule. With non-constant discounting (in a MPE), the “current regulator” takes into account the effect of current consumption on future values of the state, and the resulting changes in future consumption and in future marginal utility. The function  $K'(S)$  incorporates these reactions by “future regulators”.

### 3.2.1 Observational non-equivalence

In special cases of this model, Barro (1999) and Cropper and Laibson (1999) note that the (linear) Markov equilibrium is observationally equivalent to the equilibrium under constant discounting. That is, there is a constant discount rate that gives rise to the same equilibrium control rule as the linear Markov rule under non-constant discounting. It is well known that observational equivalence (between constant and non-constant discounting) does not hold in general.

Equation (14) implies the following

**Remark 2** *A necessary and sufficient condition for observational equivalence between the models of constant and non-constant discounting in the special case where  $U_S \equiv 0$  and  $f_x \equiv -1$  is*

$$\frac{K'(S_t)}{U'(x_t)} = \int_0^T \theta(\tau) (r(\tau) - \bar{r}) \frac{U'(t+\tau)}{U'(t)} \chi'(S_{t+\tau}) \frac{\partial g(\tau; S_t)}{\partial S_t} d\tau = \text{constant}. \quad (15)$$

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in differential games where the incomplete transversality condition leads to a continuum of candidate decision rules (Tsutsui and Mino 1990).

This restriction requires that the integrand in equation (15) is independent of the state variable. This independence arises if, for example,  $U'$  grows at a constant rate,  $\chi'$  is a constant, and if  $F(S) - \chi(S)$  is linear (so that  $\frac{\partial g(\tau; S_t)}{\partial S_t}$  is independent of the state).

### 3.2.2 Non-uniqueness

We simplify notation by writing  $z$  instead of  $z(S_\infty)$ ; this function is defined in equation (13). We also write  $r_0 = r(0)$ , the short run discount rate. The following proposition characterizes the set of candidate equilibria for  $T \leq \infty$ .

**Proposition 2** *Suppose that  $U_S \equiv 0$  and  $f(S, x) = F(S) - x$ . (i) Any value  $S_\infty$  that satisfies*

$$F'(S_\infty) = \bar{r} + \chi'(S_\infty) \int_0^T \theta(\tau) (r(\tau) - \bar{r}) e^{z\tau} d\tau. \quad (16)$$

*and the stability condition, equation (13) is a candidate (interior) steady state. (ii) There exists either no candidate interior steady states, or there exists a continuum of candidate steady state; in the latter case, there exists a continuum of candidate equilibrium control rules  $\chi(S; S_\infty)$ . (iii) For  $T = \infty$ , the steady state condition (16) simplifies to*

$$1 = \chi'(S_\infty) \left( \int_0^\infty \theta(\tau) e^{z\tau} d\tau \right). \quad (17)$$

*At every candidate state  $\chi'(S_\infty) > 0$  and*

$$\bar{r} < F'(S_\infty) < r_0. \quad (18)$$

*In the neighborhood of the steady state, the “effective discount rate” is between the short run discount rate  $r_0$  and the long run rate  $\bar{r}$ .*

With constant discounting, the second term on the right side of (16) vanishes, leaving the standard steady state condition  $F'(S_\infty) = \bar{r}$ . This equation determines a unique  $S_\infty$  if  $F$  is concave with  $F'(0) > \bar{r}$ . For concave  $U$ , the unique solution to the control problem is the consumption rule that satisfies the Euler equation and drives the state to  $S_\infty$ . The transversality condition provides the boundary condition to the Euler equation.

In contrast, with non-constant discounting, the steady state condition (16) does not identify a unique steady state. The right side of this equation contains the undetermined function  $\chi'(S_\infty)$ . We do not have “outside” information that identifies the value of  $\chi'(S_\infty)$ . Consequently, there exist a continuum of candidate equilibria.

The multiplicity of equilibria under non-constant discounting arises for essentially the same reason as in the differential game studied by Tsutsui and Mino (1990): an “incomplete transversality condition”, i.e., the lack of a “natural boundary condition”. The equilibrium problem with non-constant discounting is equivalent to a game amongst a succession of planners, each of whom chooses the control variable for a short (or infinitesimal) amount of time. The equilibrium conditions, together with the assumptions of Markov perfection and differentiability of the control rule, do not provide enough information to identify agents’ beliefs regarding the response of other agents to a departure from the steady state. That is, the equilibrium conditions do not identify  $\chi'(S_\infty)$ . Consequently, these conditions do not identify a unique steady state. However, the stability requirement puts bounds on  $\chi'(S_\infty)$  and on the range of candidate steady states, as Proposition 2.iii, equation (18), shows.

The bounds in equation (18) are necessary, not sufficient for candidate equilibria; the example below shows how the exact bounds implied by equation (17) and  $z < 0$  can be calculated. However, the general case (for  $T = \infty$ ) provides insight into the relation between this model and the model studied by Barro (1999). Barro uses a general equilibrium model in which the growth of the representative agent’s wealth depends on prices that are determined by the level of economy-wide aggregate capital. The agent takes these prices as given and solves a non-stationary problem with a linear equation of motion. In contrast, in our model the decision-maker solves a stationary problem taking the growth function in equation (2) as given. Thus, the two models are not identical, although they are similar.

Barro’s linear Markov equilibrium (under logarithmic utility) is observationally equivalent to a model with the constant discount rate  $\lambda \equiv \left(\int_0^\infty \theta(\tau) d\tau\right)^{-1}$ . (See Barro’s equation (13); his discount factor is  $e^{-\rho\tau + \phi(t)} = \theta(\tau)$  in our notation.) Any *steady state* (as distinct from the control rule that generates that steady state) in our model can obviously be supported as a steady state to a model with constant discounting. Inequality (18) shows that this constant discount rate is between  $\bar{r}$  and  $r_0$ . Barro’s equation (15) shows that  $\lambda$  satisfies  $\bar{r} < \lambda < r_0$ , so the two models do not appear contradictory. However, as noted above, the bounds in (18) are necessary, not sufficient for candidate equilibria, so we do not actually know that  $\lambda$  satisfies the bounds implied by stability. For the example below, we show that  $\lambda$  does not satisfy these bounds.

In order to discuss the general case, define  $y = F'(S_\infty)$  and write equation (17) as

$$Q(y, z) \equiv 1 - (y - z) \left( \int_0^\infty \theta(\tau) e^{z\tau} d\tau \right) = 0. \quad (19)$$

Equation (19) gives  $y$  (and therefore gives  $S_\infty$ ) as an implicit function of  $z$ . We re-write this implicit function as the explicit function  $y = q(z)$ ; stability requires  $z < 0$ . Note that  $q(0) = \lambda$ . This fact implies

**Corollary 1** *If the function  $q(z)$ , implicitly defined by equation (19), is monotonically decreasing for  $z \leq 0$ , then any interior stable steady state satisfies  $F'(S_\infty) > \lambda$ . In this case, any differentiable interior Markov perfect stable steady state can be supported by a model with constant discounting using a discount rate that is strictly greater than  $\lambda$ .*

In contrast, in Barro's formulation of the problem, the linear MPE is observationally equivalent to a constant discount rate of  $\lambda$ .

The monotonicity condition in the Corollary depends only on the discount function  $\theta$ . For general functional forms we cannot guarantee that the condition holds. Therefore, we turn to an example.

**An example** We consider the following example, in which  $0 \leq \gamma < \delta$  and  $0 \leq \beta \leq 1$ :

$$\theta(t) = \beta e^{-\gamma t} + (1 - \beta) e^{-\delta t},$$

which implies

$$\bar{r} \equiv \gamma < \beta\gamma + (1 - \beta)\delta \equiv r_0 \quad \lambda = \frac{\delta\gamma}{(1 - \beta)\gamma + \beta\delta}.$$

For this example (where  $T = \infty$ ), using  $y \equiv F'(S_\infty)$ , we can write equation (19) as the explicit function

$$\begin{aligned} y &= q(z) \equiv \frac{((1 - \beta)\delta + \beta\gamma)z - \delta\gamma}{z - (1 - \beta)\gamma - \beta\delta} \implies \\ \frac{dq}{dz} &= \frac{-\beta(\delta - \gamma)^2(1 - \beta)}{(-z + \gamma - \beta\gamma + \beta\delta)^2} < 0, \end{aligned}$$

For this example, the monotonicity condition of Corollary 1 holds; in addition  $q(0) = \lambda$  and  $q(z) \rightarrow r_0$  as  $z \rightarrow -\infty$ . Therefore, any stable steady state MPE can be supported as a steady state for a problem with a constant discount rate strictly between  $\lambda$  and  $r_0$ .

These results have several implications. If the regulator at an arbitrary time were able to make commitments, a positive steady state to her problem satisfies  $F'(S_\infty) = \gamma$ . The steady state stock in a MPE is always lower than this value. A positive steady state stock for the planner who can make commitments requires  $F'(0) > \gamma$ , whereas a positive steady state in the

MPE requires  $F'(0) > \lambda$ . The inability to make commitments might result in exhaustion of the resource stock. If  $r_0 > F'(0) > \lambda$  the resource might be exhausted or preserved in the steady state.

If  $F'(0) < \lambda$  and  $F(0) = 0$ , the unique steady state is  $S_\infty = 0$ . In this case, the existence of a unique steady state implies a unique solution to the Euler equation, i.e. a unique MPE. In this situation, the problem has a complete transversality condition (i.e. it has a “natural boundary condition”).

### 3.2.3 Selecting a particular equilibrium

This subsection considers the problem of selecting a particular steady state, thereby choosing a particular equilibrium. We first discuss several technical approaches to equilibrium selection that either fail or have serious limitations. We then show how the equilibria can be Pareto ranked.

**Technical approaches to equilibrium selection** Here we assume that there exists an interior stable steady state; by Proposition 2, this steady state is not unique. The selection of a particular MPE requires the selection of a particular steady state. It might seem that this selection could be achieved (and the indeterminacy could thus be resolved) by using the fact that the Ramsey Rule, equation (14), must hold for all  $t$ . This requirement means that it is legitimate to differentiate both sides of the Ramsey Rule with respect to  $t$  and to evaluate the result in the steady state, thereby obtaining an additional algebraic equation. It might appear that this additional algebraic equation provides the additional information needed to determine  $\chi'(S_\infty)$ , thereby enabling us to determine  $S_\infty$  and the equilibrium path.

To explain why this procedure does not resolve the indeterminacy, we proceed as follows. Define

$$M(S) \equiv \eta(\chi(S)) \frac{\chi'(S)(F(S) - \chi(S))}{\chi(S)} = \eta(x_t) \frac{\dot{x}}{x}$$

and for simplicity set  $\bar{r} = 0$ , so we can write the time derivative of the Ramsey Rule as

$$M'(S)\dot{S} = F''(S)\dot{S} - \frac{d}{dS} \left( \frac{K'(S)}{U'(\chi(S))} \right) \dot{S}. \quad (20)$$

By dividing both sides of equation (20) by  $\dot{S}$  we obtain

$$M'(S) = F''(S) - \frac{d}{dS} \left( \frac{K'(S)}{U'(\chi(S))} \right). \quad (21)$$

Equation (21) must hold everywhere, including at the steady state. However, the derivative on the right side of (21) involves  $\chi''(S)$ , the second derivative of the control rule. Equation (21), evaluated at the steady state, indeed provides another equation, but it also introduces another unknown, the value of  $\chi''(S)$  evaluated at the steady state. Obviously, taking higher order derivatives of the Ramsey Rule will not help matters.

One resolution is to *assume* that the control rule is linear in the neighborhood of the steady state, i.e. to set  $\chi''(S_\infty) = 0$ . This assumption and equations (16) and (21) (evaluated at the steady state) provide three equations that can be used to determine a locally unique steady state, and thereby determine the equilibrium path. A similar assumption was sometimes used in the literature on “consistent conjectural variations”; the assumption that conjectures are linear in the neighborhood of a Nash equilibrium can identify a unique consistent conjectural variation equilibrium. The “local linearity” assumption provides a practical way to identify an equilibrium, but it may be difficult to justify in conjectural variation, differential game, or hyperbolic discounting models.

A second way to select a unique equilibrium takes the limit of the equilibria of a sequence of finite horizon games, as the horizon extends to infinity. Economists who think that the finite horizon model of the prisoners’ dilemma provides a poor guide to the corresponding infinite horizon model might not find this approach attractive.

A third alternative identifies an equilibrium control rule that is defined for “all” values of the state variable. For example, in the one-state variable symmetric-player linear-quadratic differential game, only the linear equilibrium control rule exists over the entire real line; non-linear equilibria are defined only for part of the real line. (See footnote(1).) A similar criterion might be used for the hyperbolic discounting model. A limitation of this selection criterion is that the relevant definition of state space is often problem-specific; for example, it may include only a portion of the real line, over which many equilibrium rules might be defined.<sup>2</sup>

**Pareto ranking the steady states** We can interpret the equilibrium problem with non-constant discounting as a game amongst a succession of agents, each of whom wants to maximize the

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<sup>2</sup>Although the differential game model provides a useful analogy to the model of hyperbolic discounting, the two problems have important differences. For example, in the linear-quadratic differential game, a phase portrait can be used to show that the domain of any non-linear control rule is a subset of the real line. We do not know of analogous techniques for the model of hyperbolic discounting; phase portrait analysis does not appear useful in this context.

present discounted value of current and future welfare. An agent is indexed by the time at which she makes the decision. The agent at time  $t$  discounts utility at time  $t + \tau$ ,  $\tau \geq 0$ , using the discount factor  $\theta(\tau)$ . An equilibrium is Pareto Efficient if no other decision rule gives all (current and future) agents a higher payoff. In our setting, an equilibrium is Constrained Pareto Efficient if no other differentiable Markov Perfect decision rule gives all agents a higher payoff.

Given a (candidate) equilibrium control rule  $\chi(S; S_\infty)$ , the equilibrium payoff of an agent when the current state is  $S$  is the value function  $W(S; \chi(S; S_\infty))$ , the solution to equation (11). Here we show explicitly the dependence of this value function on the equilibrium decision rule, indexed by the steady state toward which this decision rule drives the state. We simplify notation using the following definitions:

$$\omega(S; S_\infty) \equiv W(S; \chi(S; S_\infty)), \quad \omega_2(S; S_\infty) = \frac{\partial \omega(S; S_\infty)}{\partial S_\infty}.$$

Define  $\Psi$  as the open set of candidate steady states. If the monotonicity condition in Corollary 1 holds, the supremum of  $\Psi$  is  $\max\{0, F'^{-1}(\lambda)\}$ . For the example where  $\theta(t) = \beta e^{-\gamma t} + (1 - \beta) e^{-\delta t}$ , we have  $\Psi = (F'^{-1}(r_0), F'^{-1}(\lambda))$ , provided that  $F'^{-1}(r_0) > 0$ .

An equilibrium rule  $\chi(S; S_\infty^*)$  is Constrained Pareto Superior to another rule  $\chi(S; S_\infty^\#)$  if and only if both  $S_\infty^*$  and  $S_\infty^\#$  are elements of  $\Psi$  and

$$\omega(S; S_\infty^*) \geq \omega(S; S_\infty^\#) \tag{22}$$

for all initial conditions  $S$  and strict inequality holds for some  $S$ .

Inequality (22) is difficult to check because it must hold for all  $S$ . A local condition that is simple to check provides a necessary condition for Constrained Pareto Efficiency. The local condition checks whether the current agent and all her successors would be willing to switch from a reference decision rule  $\chi(S; S_\infty)$  to a neighboring rule, given that the current state is  $S_\infty$ . If all agents (current and future decision-makers) would be willing to make the switch, the reference decision rule is not Constrained Pareto Efficient

The state trajectory is monotonic in the neighborhood of the steady state ( $z < 0$ ); if we switch from one decision rule to a neighboring rule, the state adjusts monotonically to the new steady state. Therefore, the local condition for Pareto ranking requires checking whether inequality (22) holds for values of  $S$  between the steady states of the reference and the neighboring decision rules, rather than for all possible values of  $S$ .

Clearly, a rule that is *less conservative* (i.e., that involves higher consumption for a given value of the state variable) than the reference rule always harms agents sufficiently far in the



future; those future agents consume at a lower steady state level (relative to consumption under the reference rule). Therefore, a rule that is less conservative than an arbitrary candidate cannot Pareto dominate that candidate.

A perturbation to a *more conservative* rule benefits agents sufficiently far in the future, since they consume at a higher level, as a result of the higher steady state. The current agent and her nearby successors have lower consumption but they appreciate the higher welfare of their successors, so it is not clear whether they benefit from the switch to the more conservative rule. We say that the decision rule  $\chi(S; S_\infty)$  is “locally Constrained Pareto dominated” by a more conservative neighboring rule if and only if  $\omega_2(S_\infty; S_\infty) > 0$ . Using this definition, we have the following

**Proposition 3** For  $T = \infty$

$$\text{sign}(\omega_2(S_\infty; S_\infty)) = \text{sign}(F'(S_\infty) - \lambda). \quad (23)$$

*Any equilibrium with a steady state that satisfies  $F'(S_\infty) > \lambda$  is locally Constrained Pareto dominated by an equilibrium with a higher steady state. If the monotonicity condition in Corollary 1 holds, the Pareto ranking of the MPE is identical to the ranking of the steady states of these equilibria: more conservative rules are always locally Pareto superior to less conservative rules.*

Pareto dominance may provide a reasonable basis for equilibrium selection. Corollary 1 and Proposition 3 provide conditions under which Pareto dominance selects a conservative rule. We cannot speak of the “most conservative” MPE since  $\Psi$  is an open set. It is worth noting that Pareto dominant rules imply slower asymptotic convergence to the steady state. That is, a lower value of  $|z|$  implies a lower value of the effective steady state discount rate (under the monotonicity property of Corollary 1) and this implies a higher steady state.

## 4 Conclusion

We obtained a DPE that must be satisfied by a differentiable MPE in a problem where a decision maker has declining discount rates and cannot commit to future actions. This DPE can be used as a basis for numerical solutions or to derive an Euler equation.

We emphasized the special case in which the state variable is not an explicit argument of the flow of utility, and where the equation of motion is linear in the control. With this model,

the Euler equation leads to an easily interpreted Ramsey Rule; this equation generalizes the familiar Ramsey Rule. It includes a term that arises from non-constant discounting.

The optimality and stability conditions are (in general) satisfied by a continuum of values of the state variable. The lack of a unique stable steady state means that we do not have a boundary condition that leads to a unique solution of the Euler equation; therefore, there is not a unique candidate equilibrium. We identified the conditions under which models of constant and decreasing discount rates are observationally equivalent. Even if the conditions for observational equivalence are satisfied, there are other MPE that are not observationally equivalent to the equilibrium under constant discounting. We found conditions under which more conservative Markov decision rules Pareto dominate less conservative neighboring rules.

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# Appendix: Proof of Propositions and Discussion of Numerical Algorithm

## Proof of Proposition 1

We begin by finding the DPE for the discrete time model. We then take the formal limit to obtain the DPE that corresponds to the continuous time model.

For the discrete stage problem we use the payoff and the equation of motion defined in equation (4). Define  $\theta_i = \theta(\varepsilon i)$  and define  $H_i = H(S_{t+i\varepsilon}) = U(S_{t+i\varepsilon}, \chi(S_{t+i\varepsilon}))$ , where  $\chi(\cdot)$  is a function to be determined. Each period lasts for  $\varepsilon$  units of time, so if the current calendar time is  $t$ , the calendar time  $j$  periods in the future is  $t + j\varepsilon$ . The discount rate decreases for the first  $n$  periods and is thereafter constant. For  $i = 1, 2, \dots, n - 1$ , define

$$\theta_{n-i} V_i(S_{t+(n-i)\varepsilon}) = \theta_{n-i} H_{n-i} \varepsilon + \theta_{n-i+1} V_{i-1}(S_{t+(n-i+1)\varepsilon}) \quad (24)$$

and

$$\theta_n V_0(S_{t+n\varepsilon}) = \theta_n H_n \varepsilon + \theta_{n+1} V_0(S_{t+(n+1)\varepsilon}). \quad (25)$$

The function  $\theta_{n-i} V_i(\cdot)$  is the present value discounted back to time  $t$  (the ‘‘current time’’) of the equilibrium continuation payoff from time  $t + (n - i)\varepsilon$  onwards. For periods  $n, n + 1, \dots$  the discount rate is constant, so the value function in equation (25) is stationary; the index 0 on the value function denotes that there are 0 periods to go before the value function becomes stationary. At each of the stages  $1, 2, \dots, n - 1$  periods in the future, the subsequent discount rate changes, so the value functions in these periods also change. The subscript on the functions  $V_i$  in equation (24) denote the number of subsequent periods during which the discount rate will be non-constant.

Using these two equation to ‘‘solve backwards’’ we obtain the relation

$$\theta_1 V_{n-1}(S_{t+\varepsilon}) = \sum_{i=1}^n \theta_i H_i \varepsilon + \theta_{n+1} V_0(S_{t+(n+1)\varepsilon}). \quad (26)$$

The DPE at period  $t$  is

$$W(S_t) = \max_x \{U(S_t, x) \varepsilon + \theta_1 V_{n-1}(S_{t+\varepsilon})\} \quad (27)$$

and the maximized DPE is

$$W(S_t) = H_i \varepsilon + \theta_1 V_{n-1}(S_{t+\varepsilon}). \quad (28)$$

(It is understood that  $S_{t+\varepsilon}$  is the value that results from the maximization in equation (27).) Equation (28) holds at all periods, including period  $t + \varepsilon$ . Advancing this equation by one period and using equation (26) (also advanced by one period) gives

$$\begin{aligned} W(S_{t+\varepsilon}) &= \sum_{i=0}^n \theta_i H_{i+1} \varepsilon + \theta_{n+1} V_0(S_{t+(n+2)\varepsilon}) \implies \\ \frac{W(S_{t+\varepsilon}) - \sum_{i=0}^n \theta_i H_{i+1} \varepsilon}{\theta_{n+1}} &= V_0(S_{t+(n+2)\varepsilon}). \end{aligned} \quad (29)$$

Equation (25) involves a stationary function; it holds at time  $t + (n + 1)\varepsilon$ , which implies

$$\theta_{n+1} V_0(S_{t+(n+1)\varepsilon}) = \theta_{n+1} H_{n+1} \varepsilon + \theta_{n+2} V_0(S_{t+(n+2)\varepsilon}). \quad (30)$$

Substituting equation (29) into (30) implies

$$\theta_{n+1} V_0(S_{t+(n+1)\varepsilon}) = \theta_{n+1} H_{n+1} \varepsilon + \frac{\theta_{n+2}}{\theta_{n+1}} \left( W(S_{t+\varepsilon}) - \sum_{i=0}^n \theta_i H_{i+1} \varepsilon \right). \quad (31)$$

Substituting equation (26) into equation (27) and then using equation (31) leads to the DPE for the discrete stage problem:

$$\begin{aligned} W(S_t) &= \max_x \left\{ U(S_t, x) \varepsilon + \sum_{i=1}^n \theta_i H_i \varepsilon + \theta_{n+1} V_0(S_{t+(n+1)\varepsilon}) \right\} \\ &= \max_x \left\{ U(S_t, x) \varepsilon + \sum_{i=1}^n (\theta_i - e^{-\bar{r}\varepsilon} \theta_{i-1}) H_i \varepsilon + e^{-\bar{r}\varepsilon} W(S_{t+\varepsilon}) \right\}. \end{aligned} \quad (32)$$

The last equality uses the following facts

$$\theta_{n+1} - \frac{\theta_{n+2}}{\theta_{n+1}} \theta_n = 0, \quad \frac{\theta_{n+2}}{\theta_{n+1}} = e^{-\bar{r}\varepsilon}$$

because the discount rate is constant from period  $n$  onward.

We now take the limit of the discrete time DPE as  $\varepsilon \rightarrow 0$ . This limit requires the Taylor expansion of the right side of the DPE. This expansion has the same form as the standard DPE with constant discounting, except for the presence of the term

$$J(\varepsilon, S_{t+\varepsilon}) \equiv \sum_{i=1}^n (\theta_i - e^{-\bar{r}\varepsilon} \theta_{i-1}) H_i \varepsilon = \sum_{i=1}^n e^{-\int_0^{\tau_i - \varepsilon} r(s) ds} \left( e^{-\int_{\tau_i - \varepsilon}^{\tau_i} r(s) ds} - e^{-\bar{r}\varepsilon} \right) H_i \varepsilon. \quad (33)$$

There are  $n = \frac{T}{\varepsilon}$  terms in  $J$ ; the  $i$ 'th term involves

$$\left( e^{-\int_{\tau_i - \varepsilon}^{\tau_i} r(s) ds} - e^{-\bar{r}\varepsilon} \right) \varepsilon.$$

Each of these terms is  $o(\varepsilon)$ , so  $J(\varepsilon, S_{t+\varepsilon})$  is asymptotic to  $\frac{o(\varepsilon)}{\varepsilon}T = 0$  as  $\varepsilon \rightarrow 0$ . Consequently, the first order Taylor approximation of  $J(\varepsilon, S_{t+\varepsilon})$  is

$$J(\varepsilon, S_{t+\varepsilon}) = 0 + J_\varepsilon(0, S_t)\varepsilon + J_S(0, S_t)(S_{t+\varepsilon} - S_t) + o(\varepsilon).$$

The argument used above to show that  $J(\cdot) \sim o(\varepsilon)$  also establishes that  $J_S(0, S_t) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . A straightforward calculation establishes

$$-J_\varepsilon(0, S_t) = \int_0^T \theta(\tau)(r(\tau) - \bar{r})H(S_{t+\tau})d\tau \equiv K(S_t; T). \quad (34)$$

We use equation (34) to write the first order Taylor approximation of the discrete DPE. We divide by  $\varepsilon$  and let  $\varepsilon \rightarrow 0$  to obtain the continuous time DPE, equation (11).

### Proof of Proposition 2

(i) Evaluating equation (14) at a steady state (where  $\dot{x} = 0$ ) gives the steady state condition

$$\begin{aligned} F'(S_\infty) &= \bar{r} + \frac{\int_0^T \theta(\tau)(r(\tau) - \bar{r})U'(g(\tau; S_t))\chi'(S_\infty)\frac{\partial g(\tau; S_t)}{\partial S_t}d\tau}{U'(u_\infty)} \\ &= \bar{r} + \chi'(S_\infty) \int_0^T \theta(\tau)(r(\tau) - \bar{r})\frac{\partial g(\tau; S_t)}{\partial S_t}d\tau. \end{aligned} \quad (35)$$

In the neighborhood of the steady state we have

$$\begin{aligned} \dot{S} &\approx z(S_t - S_\infty) \Rightarrow \\ g(\tau; S_t) &\approx e^{z\tau}S_t + S_\infty(1 - e^{z\tau}) \Rightarrow \\ \frac{\partial g(\tau; S_t)}{\partial S_t} &\approx e^{z\tau}, \end{aligned} \quad (36)$$

where  $z = F'(S_\infty) - \chi'(S_\infty) < 0$  because of stability. Substituting equation (36) into (35) results in the steady state condition (16).

(ii) Equation (16) is a single equation involving two unknowns and therefore generally has no solutions or a continuum of solutions. In the latter case, there are a continuum of candidate steady states. Associated with each steady state  $S_\infty$  is a function that satisfies the Euler equation,  $\chi(S; S_\infty)$ , i.e. a candidate equilibrium.

(iii) For  $T = \infty$ , we have

$$\int_0^\infty \theta(\tau)r(\tau)e^{z\tau}d\tau = -\int_0^\infty \theta'(\tau)e^{z\tau}d\tau = 1 + z \int_0^\infty \theta(\tau)e^{z\tau}d\tau \quad (37)$$

Using the definition of  $z$  and equation (37) we can rewrite equation (16) as

$$\begin{aligned} z &= F' - \chi' = \bar{r} + \chi' \left[ - \int_0^\infty \theta'(\tau) e^{z\tau} d\tau - \bar{r} \int_0^\infty \theta(\tau) e^{z\tau} d\tau - 1 \right] \\ z - \bar{r} &= \chi' \left[ (z - \bar{r}) \int_0^\infty \theta(\tau) e^{z\tau} d\tau \right] \end{aligned} \quad (38)$$

Equation (38) implies equation (17). The fact that

$$\frac{1}{r_0 - z} < \int_0^\infty \theta(\tau) e^{z\tau} d\tau < \frac{1}{\bar{r} - z},$$

implies equation (18).

### Proof of Proposition 3

Define  $z = F'(S_\infty) - \chi_1(S; S_\infty)$ . (Since here we show the steady state as an argument of  $\chi$  we write  $\chi_1$  instead of  $\chi'$ .) Consider a perturbation of the control rule  $\chi(S; S_\infty)$  to  $\chi(S; S_\infty) - \varepsilon$  for small  $\varepsilon > 0$ . This perturbation increases the steady state from  $S_\infty$  to  $S_\infty - \frac{\varepsilon}{z}$ . The function  $g(\tau; S)$  is the solution to the equation of motion for  $S$  under a particular control rule (equation (7).) With some abuse of notation, here we use  $g(\tau; S, \varepsilon)$  to denote the solution to the state equation under the perturbation. Beginning at the steady state  $S_\infty$  at time 0, the value of  $S$  at time  $\tau \geq 0$  under this perturbation is<sup>3</sup>

$$S(\tau) = g(\tau; S_\infty, \varepsilon) \approx S_\infty - \frac{\varepsilon}{z} + \frac{\varepsilon}{z} e^{z\tau}. \quad (39)$$

For arbitrary initial condition  $S$ , the value of the program under the control rule  $\chi(S; S_\infty)$  is

$$\begin{aligned} \omega(S; S_\infty) &= \int_0^\infty \theta(\tau) U(\chi(\tau)) d\tau, \\ \chi(\tau) &\equiv \chi(g(\tau; S); S_\infty). \end{aligned}$$

At  $S = S_\infty$  the value of the program under the perturbation (i.e., replacing  $\chi(S; S_\infty)$  by  $\chi(S; S_\infty) - \varepsilon$ ) is

$$\begin{aligned} \omega(S_\infty; S_\infty - \frac{\varepsilon}{z}) &= \int_0^\infty \theta(\tau) U(\chi(\tau, \varepsilon) - \varepsilon) d\tau \\ \chi(\tau, \varepsilon) &\equiv \chi(g(\tau; S_\infty, \varepsilon); S_\infty). \end{aligned} \quad (40)$$

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<sup>3</sup>It is more precise to replace the function  $z$  in equation (39) by  $F'(S_\infty - \frac{\varepsilon}{z}) - \chi_1(S; S_\infty - \frac{\varepsilon}{z})$  since we are writing the approximation of the trajectory under the perturbed decision rule, not the approximation of the trajectory under the original decision rule. This correction complicates the notation but does not change anything of substance. When we evaluate derivatives at  $\varepsilon = 0$  the formulation in equation (39) and the correction in this footnote lead to the same result.

Taking derivatives (with respect to  $\varepsilon$ ) of equation (40), and evaluating the result at  $\varepsilon = 0$ , gives

$$\begin{aligned}
\omega_2(S_\infty; S_\infty) &= -z \int_0^\infty \theta(\tau) U'(\chi_\infty) \left( \frac{\partial \chi}{\partial S} \frac{\partial g(\tau; S_\infty, \varepsilon)}{\partial \varepsilon} - 1 \right) d\tau \\
&= -z U'(\chi_\infty) \int_0^\infty \theta(\tau) \left( \frac{\partial \chi}{\partial S} \frac{1}{z} (e^{z\tau} - 1) - 1 \right) d\tau \\
&= -z U'(\chi_\infty) \left( \frac{1}{z} - \left( \frac{\partial \chi}{\partial S} \frac{1}{z} + 1 \right) \int_0^\infty \theta(\tau) d\tau \right) \\
&= -U'(\chi_\infty) \left( 1 - F'(S_\infty) \int_0^\infty \theta(\tau) d\tau \right)
\end{aligned} \tag{41}$$

The second equality uses the approximation in equation (39); the third equality uses equation (17), and the fourth equality used the definition of  $z$ . Since  $U'(x) > 0$  by assumption, the last line of equation (41) implies equation (23).

### Comments on numerical solution

The DPE (11) is standard, except for the presence of the unknown function  $K(\cdot)$ . If this function were known, standard methods (e.g. the collocation method using Chebyshev polynomials) could be used to solve the problem. The presence of the function  $K(\cdot)$  complicates the algorithm but it does not introduce any conceptual difficulties. We merely need to add an additional nested procedure to approximate this function. We discuss the necessary modification to the standard value function iteration algorithm.

At iteration  $i$  of a value function algorithm, we have an approximation of the decision rule,  $\chi^i(S)$ . We want to use this approximation to construct an approximation of  $K(\cdot)$  that can be used for the next iteration of the value function iteration. We discuss the case where  $T < \infty$ .

Using equation (6) define

$$\kappa^i(S, t) = \int_t^T \theta(\tau) (r(\tau) - \bar{r}) H^i(S_{t+\tau}) d\tau, \tag{42}$$

where  $H^i(S) \equiv U(S, \chi^i(S))$ , our approximation of  $H$  at iteration  $i$ . By construction, our approximation of  $K$  at iteration  $i$  is  $K^i(S) = \kappa^i(S, 0)$ . The function  $\kappa^i(S, t)$  is the solution to the partial differential equation (PDE)

$$0 = \theta(t) (r(t) - \bar{r}) H^i(S_t) + \kappa_t^i(S, t) + \kappa_S^i(S, t) \chi^i(S) \tag{43}$$

with the boundary condition  $\kappa^i(S, T) = 0$ . This PDE, unlike the DPE, does not involve optimization. We can approximate the solution to the PDE and the boundary condition, again



by using Chebyshev polynomials and the collocation method. This approximation requires a procedure that is nested in the standard value function algorithm.

The output of this procedure is our approximation  $K^i(S) = \kappa^i(S, 0)$ . We use this approximation, together with our  $i$ 'th iteration approximations of  $W$  and  $\chi$  to obtain the  $(i + 1)$ 'th approximation of the value function:

$$\bar{r}W^{i+1}(S) = U(S, \chi^i(S)) - K^i(S) + W^{ii}(S_t)f(S, \chi^i(S)).$$

This value function is used in equation (11) to obtain the  $(i + t)$ 'th approximation,  $\chi^{i+1}(S)$ .

In order to select a particular equilibrium decision rule we can impose an additional condition on the policy function. Using the methods described in the text, we can identify the set of candidate steady states. To select the policy rule that drives the state to a particular steady state,  $S_\infty$ , at each iteration  $i$  we impose the conditions

$$\begin{aligned} f(S_\infty, \chi^i(S_\infty)) &= 0 \\ \kappa^i(S_\infty, t) &= H^i(S_\infty) \int_t^T \theta(\tau) (r(\tau) - \bar{r}) d\tau \\ \bar{r}W^{i+1}(S_\infty) &= U(S, \chi^i(S_\infty)) - K^i(S_\infty) + W^{ii}(S_\infty)f(S_\infty, \chi^i(S_\infty)) \end{aligned}$$

As noted in the text, we cannot be sure that the resulting policy function exists for all of state space. We can experiment by looking for a policy rule that is defined over  $(S_\infty - \epsilon, S_\infty + \epsilon)$ , and then gradually increasing  $\epsilon$ .

An alternative numerical approach uses a Taylor approximation evaluated at the steady state. A particular value of  $S_\infty$  implies values of  $\chi(S_\infty)$  and  $\chi'(S_\infty)$ . Using the procedure described in Section 3.2.2 (see equation (21)) we can obtain the value  $\chi''(S_\infty)$ . Taking further derivatives of the Euler equation, evaluated at the steady state, provides higher order derivatives of  $\chi(S_\infty)$  which can be used to construct a Taylor expansion in the neighborhood of the steady state. Since this approach uses only information at the steady state, it is likely to be less accurate than the modification of the value function iteration described above.