Food Scares and Demand Recovery Patterns: An Econometric Investigation

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Abstract

This paper aims to propose a flexible stochastic approach to measure the time pattern of a food scare, which does not require the inclusion of additional explanatory variables such as a media coverage indices and easily accommodates the reoccurrence of the same or different scares. We show the results of an application to Italian demand for beef and chicken, which has been affected by the BSE and dioxin scares over the last decade.

Keywords: demand analysis, food scare, BSE, Almost Ideal Demand System, Kalman filter

Introduction

The measurement of consumer response to food scares has been the subject of many empirical investigations. It is a policy relevant task, as it provides the basis for calibrating countermeasures and establishing potential compensations. This paper aims to propose a flexible stochastic approach to measure the time pattern of a food scare, which does not require the inclusion of additional explanatory variables such as a media index and easily accommodates the reoccurrence of the same or different scares.

Sociological studies acknowledge that food scares exhibit a fairly standard pattern. Beardsworth and Keil (1996) classify public reaction in five steps: (i) initial equilibrium characterised by unawareness or lack of concern about the potential food risk factor; (ii) news about a novel potential risk factor and public sensitisation; (iii) public concern is raised as the risk factor becomes a major element of interest and concern in public debate and media; (iv) public response begins, usually with avoidance of the suspect food item; (v) public concern gradually decreases as attention switches from the issue, leading to the establishment of a new equilibrium. The same study highlights that public response in stage (iv) is often exaggerated and unrelated to the objective risk and even after the new equilibrium is reached in stage (v) a "chronic low-level anxiety may persist and can give rise to a resurgence of the issue at a later date".

Despite this general framework can be applied to most of food scare events, the duration of the single steps and the potential reoccurrence of the same scare remain a relevant econometric issue. Previous studies have followed different approaches to measure demand response. One direction is based on the assumption that consumer reaction is directly related to the amount of news released. Smith et al. (1988) and Liu et al. (2001) estimated the impact of the eptachlor contamination of milk in the Hawaiian island of Oahu in 1983 by including a variable related to media coverage in a demand function. On the same case study, Foster and Just (1989) discard the media variable and substitute it with a nonlinear shift on the intercept which allows for an exponential decrease in the food scare effects and also some long-term persistence. Burton and Young (1996), Verbeke and Ward (2001) and Piggot and Marsh (2004) extend Deaton and Muellbauer Almost Ideal Demand System (AIDS) to account for a media index specifically built for distinguishing the impact on meat demand of positive and negative news about Bovine Spongiform Encelopathy (BSE).

Even though the empirical performance of the above models is generally acceptable, we argue that they have some key limitations that reduce their reliability in many situations, not least the one of scare resurgence. Our objection is founded on three main considerations.

The first is that discrimination between positive and negative information is a highly subjective operation. For example, news about the incubation period of the Creutzfeldt-Jakob disease (CJD), which has been linked to BSE, informed the public about a possible latency period of up to 20 years. While this could be a source of anxiety for younger consumer, the same information could lead to a lower hazard perception for the elderly one. Furthermore, Smith et al. (1988) noted the extremely high correlation between news classified as positive and negative, as their amount is related to the media interest rather than scientific evidence, which usually takes too long to be advertised and rarely influence behaviour in the short term.

A second consideration concerns the way information is discounted over time in consumer perception, as it is recognised that within the same food scare event the marginal effect of additional information is decreasing. Also, the acute phase of a scare is characterised by the social amplification phenomenon (Beardsworth and Keil, 1996) which is generated by the initial 'news spiral', but is recognised as a self-limiting process. Some researchers (Smith et al., 1998) address this issue by including lags of the media variable, others (Verbeke and Ward, 2001) correct their index in order to account for decreasing lagged impacts, but both approaches require some subjective and undesirable assumptions.

The third argument against the modelling of consumer reaction through a media index or the nonlinear shift by Foster and Just is related to the crisis reoccurrence. It is clear that the marginal effect of novel or confirmatory news about a food risk factor already known to the public is likely to be different than in the period of the first occurrence. This is evident when one considers the effects of the various scares that affected European meat consumption over the last few years, beginning with the first BSE wave in the UK (March 1996), to continue with E-coli in Scotland (December 1996), dioxin in Belgium (May 1999), a second wave of the BSE scare throughout in late 2000 after a surge in the number of cases identified in France. In early 2001, the introduction of compulsory tests on bovines in the whole European Union led to the detection of BSE in Italy and other countries, which generated a dramatic reduction in beef consumption, even bigger than the one experienced within the first crisis. This outcome which is consistent with the persisting low-level anxiety discussed by Beardsworth and Keil.

The approach proposed in this paper is based on the inclusion of a stochastic shift within the AIDS framework. The model, which is estimated using Harvey's (1989) structural time series techniques allows a direct estimate of the time-varying pattern of consumer response based on actual data. Thus, the subjective and often difficult and expensive operation of retrieving media coverage data becomes unnecessary. The application is based on Italian aggregate household demand of beef and chicken, which has been greatly affected by the meat-related scares that have occurred in Europe over the last decade.

The model

A flexible stochastic framework for modelling the time-varying impact of food scares is provided by the structural time series Almost Ideal Demand System. This model has been recently employed to model time-varying tastes and seasonality in food demand (Fraser and Moosa, 2002, Mazzocchi, 2003) and it basically consists in allowing some or all of the model parameters to follow a pre-determined stochastic specification[1]. In this paper we adopt a dynamic version of the complete linearly-approximated aggregate AIDS based on the partial-adjustment form suggested by Alessie and Kapteyn (1991):

$$w_{it} = \alpha_{it}^{*} + \sum_{j=1}^{n} \gamma_{ij} \ln p_{jt} + \beta_i \ln\left(\frac{Y_t}{k_t P_t^{*}}\right) + u_{it} \qquad i{:}1,2,\dots,n \tag{1}$$

[1] In principle, all the AIDS coefficient could be specified as time-varying, but it has been shown (Mazzocchi, 2003) that the model is unable to distribute the effects across the exogenous variables in an economically sensible way.

where w_{it} is the expenditure share for the i-th good at time t, p_{jt} is the price of the j-th good, Y_t is the total expenditure, P_t^* is the Stone index, k_t is an aggregation index computed as in Deaton and Muellbauer (1980) to account for household heterogeneity and u_t is a white-noise normally distributed error.

The intercept in (1) is a function of the vector of lagged shares to account for habits, of a linear trend to account for gradually changing tastes and of (monthly) seasonal factors:

$$\alpha_{it}^* = \alpha_i + \sum_{j=1}^{n} \rho_{ij} w_{i,t-1} + \sum_{s=1}^{12} \phi_{is}\delta_{ts} + \lambda_i t \tag{2}$$

where δ_{ts} is a dummy variable equal to 1 when the time period t falls in month s and 0 elsewhere, and the sum of the seasonal factors over 12 consecutive months is constrained to be 0, $\sum_{s=1}^{12} \phi_{is} = 0$.

System (1) fulfils the demand theory requirements when the following conditions are satisfied:

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_i = 1 \quad \sum_{i=1}^{n} \rho_{ij} = 0 \quad \sum_{i=1}^{n} \lambda_i = 0 \\ &\sum_{i=1}^{n} \gamma_{ij} = 0 \quad \sum_{i=1}^{n} \beta_i = 0 \quad \sum_{i=1}^{n} \phi_{is} = 0 \quad \forall j \end{aligned} \tag{3a}$$

$$\sum_{j=1}^{n} \gamma_{ij} = 0 \qquad \forall i \tag{3b}$$

$$\gamma_{ij} = \gamma_{ji} \qquad \forall i, j \tag{3c}$$

The above constraint ensure respectively adding-up (3a), homogeneity (3b) and symmetry (3c). An additional constraint is necessary to ensure identification of the dynamic system (Edgerton, 1996):

$$\sum_{j=1}^{n} \rho_{ij} = 0 \quad \forall i \tag{3d}$$

In order to measure the effect of one or more food scares occurring after time period t_0, we augment the intercept of (1) with a dummy shift whose coefficient is allowed to vary according to a random walk. The intercept allowing for a response to the food scare(s) is augmented as follows:

$$\alpha_{it}^* = \alpha_i + \sum_{j=1}^{n} \rho_{ij} w_{i,t-1} + \sum_{s=1}^{12} \phi_{is}\delta_{ts} + \lambda_i t + \psi_t h_t \tag{4}$$

where $h_t=1$ for all time periods after the occurrence of the first food scare and is 0 elsewhere and the stochastic coefficient Ψ_t is assumed to follow a random walk with a normal white-noise error to capture the evolving pattern of the food scare:

$$\psi_{it} = \psi_{i,t-1} + e_{it} \tag{5}$$

Estimation

The system of equations described by (1) , (4) and (5) and subject to the constraints in (3) is a structural time series model (Harvey, 1989) and the estimation of the time-varying parameter can be accomplished by rewriting the model in the state-space form and applying a maximum-likelihood algorithm such as the expectation-maximisation (EM) algorithm by Dempster et al. (1977). The state-space form of the system is given by defining a *measurement equation* and a *transition equation* as follows:

$$w_t = Z_t a_t + e_t^M \tag{6a}$$

$$a_t = T a_{t-1} + e_t^T \tag{6b}$$

where the $n \times 1$ vector w_t contains the expenditure shares, the $m \times 1$ state vector a_t includes the m unknown parameters of system and the $n \times m$ matrix Z_t contains the exogenous variables and other fixed values, so that (1) is equivalent to (6a), apart from the stochastic specification of the time-varying shift. The stochastic transition patterns for the random-walk coefficient is defined in the transition equation (6b), which represents the relationship between the state vector a_t and its lagged values, through the $m \times m$ transition matrix T_t, whose values are known. The stochastic specification of the model is completed by the disturbance vectors e_{it}^M and e_{it}^T, each with mean zero and with covariance matrices equal to H_t and Q_t respectively. A detailed discussion about the state-space specification of a time-varying AIDS model allowing for cross-equation theoretical constraint is provided in Mazzocchi (2003).

Some further assumptions can considerably reduce the computational burden. Thus, we set H and Q to be time-independent and adopt a diagonal structure for Q, which implies that the errors of the transition equation are independent. Once a model is expressed in the state-space form, the Kalman filter (KF) can be applied. The KF a recursive procedure for computing the optimal estimates of the state vector at time t using all available information at time t, once some acceptable priors for the initial state vector and covariance matrix have been defined. The other procedure necessary for estimating (6) is the Kalman smoother (KS). The KS is a backward procedure, which starts from the state vectors computed through the KF and produces 'smoothed' estimates. Furthermore, the KF allows to derive the log-likelihood function as a function of the unknown parameters in the system and the other parameters appearing in the state-space form, namely the error covariance matrices H and Q. The representations of the KF and KS, and the log-likelihood function are reported in the Appendix.

Maximum likelihood estimates can now be obtained using the EM algorithm, whose application to the estimation of stochastic coefficient models is illustrated by Shumway and Stoffer (1982) and Watson and Engle (1983). The EM algorithm is an iterative maximisation procedure that starts with the definition of the initial values for the state vector, for its covariance matrix and for H and Q.

The following steps are then repeated iteratively: (1) get estimates of the state vector and its covariance matrix through the KF; (2) feed the filtered estimates into the KS to obtain smoothed estimates; (3) maximise the log-likelihood function conditional to the smoothed values to estimate the error covariance matrices H and Q; (4) use the smoothed estimates of H, Q and the initial state vector to restart the algorithm from step 1 and repeat step 1-3 until convergence is achieved.

The EM algorithm has the desirable property that each step always increases the likelihood and convergence is guaranteed (Wu, 1983). On the other hand, the limitation of the EM algorithm is that it may stop at some local maximum, so that the appropriate starting values are provided by the SUR estimates of the constant coefficient AIDS (Mazzocchi, 2003).

Application

An ideal setting for testing the performance of the AIDS model allowing for a time-varying shock is given by aggregate Italian meat demand. Over the last decade, the Italian meat market has been subject to several food scares and in all occasions consumer response has been quite strong, with a sharp and sudden fall in consumption and a slow recovery pattern, while it is still debated whether there has been a permanent impact. The first informational shock to Italian household was the news about a potential link between BSE and CJD from the UK in March 1996. Despite BSE cases in Italy were negligible and linked to imported cattle, the fall in consumer trust towards beef was made evident by the drop in both the quantity consumed and prices, while substitute meats showed a rather stable consumption despite a significant rise in prices. In April 1996, household real expenditure in beef fell by 18.0% with respect to April 1995 and real beef prices went down by 2.8%, while real expenditure in chicken raised by 1.7% despite a 7.2% price increase. By the end of 1998 and accounting for the structural decline that characterised the market well before the BSE crisis, beef consumption had returned to the pre-BSE level, while prices were still clearly below their expected level. At the end of May 1999 the very short, but European-wide dioxin crisis also affected the meat sector, especially chicken. In June, Italian households' real expenditure in chicken decreased by 13.9% with respect to the same month in 1998 and real chicken prices fell by 1.8%. After the summer consumption had already gone back to previous levels and this crisis was not comparable to the BSE one in terms of economic impact, but it still added anxiety to the consumer and affected the slow process of trust restoration. In November 2000, a significant increase in the number of BSE cases was registered in France after the adoption of sample tests on bovines and several countries including Italy suspended French beef imports. This lead to a sudden and huge shock on Italian household beef consumption (-32.2% in terms of beef real expenditure and –0.7% in terms of prices with respect to November 1999), which was exacerbated by the detection of the first BSE case in Italy in January 2001. Beef consumption was almost halved (-49.2% with respect to January 2000), while real beef price went down by 1.2%. A slow recovery started in late Spring 2000, but was still far from being completed at the end the year. Real expenditure in chicken showed a sharp growth in the first months after the crisis (up to +32.0% in January 2001) and prices again reacted significantly (still +18.0% in March 2001). It is clear that the specification of a traditional demand system would fail with such data and the various fluctuations over a long period (1996-2001) would prevent a simple dummy variable specification to account for the shocks. Also, it would be very problematic and subjective to build a media index able to account for several scares on different products.

Three versions of the homogeneity and symmetry-restricted dynamic Almost Ideal Demand System were estimated: (a) with no shift accounting for the food scares; (b) with a fixed dummy shift from March 1996; and (c) with a random walk shift from March 1996. The data series were obtained from the ISTAT Household Expenditure Survey. Monthly observations from January 1986 to December 2001 were used to estimate a 4-equations system for beef, chicken, other foods and a residual equation for all remaining goods. Systems (a) and (b) where estimated through an iterated Seemingly Unrelated Regression estimator, while system (c), augmented with the stochastic shift defined in (4) and (5), was estimated through the EM algorithm as discussed in previous section. The residual equation was dropped from estimation in order to avoid singularity of the covariance matrix (see Barten, 1969 or Bewley, 1986). Systems

Stability tests on system (a) show the relevance of the multiple structural breaks implied by the food scares, while tests on system (b) are aimed to assess whether a simple dummy shift on the first scare date might be able to accommodate also the subsequent shocks. For both models, table 1 reports the Chow test (Fisher, 1970) and the Nyblom test of the null of constant coefficients against the alternative of at least one coefficient following a random walk (Nyblom, 1989). This latter test does not require any assumption on the break date.

Table 1. Stability tests on the dynamic AIDS model without intervention (a) and with dummy intervention (b)

	March 1996 (bse)	May 1999 (dioxin)	October 2000 (bse2)	
	Model (a) - No shift			
	Chow Breakpoint test		Chow Forecast test(a)	Nyblom test(b)
Beef	1.72 *	1.73 *	5.95 **	5.30 **
Chicken	2.42 **	3.07 **	3.20 **	4.05
Other foods	2.43 **	1.15	0.53	5.83 **
	Model (b) - Dummy shift on March 1996			
Beef		1.76 *	6.66 **	4.72 *
Chicken		3.40 **	4.00 **	3.96
Other foods		1.10	0.51	5.55 **

Notes:

(a) Chow Breakpoint test not applicable due to the lack of degrees of freedom

(b) Critical values at 95% (99%) confidence level are 4.43 (4.88) for the model without shift and 4.62 (5.09) for the model with a dummy shift

The stability tests show the inadequacy of model (a) which does not account for structural break and structural break diagnostics worsen as the dioxin crisis and the latest BSE crisis are considered. If no break date is assumed (as in the Nyblom test), evidence for at least one random walk coefficient emerges for beef and other foods, while there is no clear sign of structural shock for chicken. If a single and constant shift on the intercept accounting for the first BSE scare is included (b), there is no sign of improvement in Chow diagnostics and also the Nyblom test still captures the instability of at least one parameter.

We now focus on model (c), where a random walk intervention is considered after March 1996. Estimates from the dynamic AIDS with constant shift were used as starting values for the EM algorithm. Parameters estimates and some model diagnostics are reported in Table 2.

Table 2. Estimates from the dynamic AIDS model with a random walk shift from March 1996

	Beef	Chicken	Other foods
α	-0.0525	0.0671 **	-0.0823
λ	-0.0001 **	-0.0001 **	-0.0001
ρ_1	0.5170 **	-0.0872 **	-0.3671
ρ_2	-0.6544 **	0.2143 **	-0.3877
ρ_3	0.0526	-0.0709 **	0.5296
ρ_4	0.0847	-0.0560	0.2241
γ_1	0.0097	0.0002	-0.0168
γ_2	0.0002	0.0070 **	-0.0076 *
γ_3	-0.0168 *	-0.0076 *	-0.0004
β	-0.0107 **	-0.0002 **	-0.0478 **
ϕ_1	-0.0005	-0.0003	0.0036
ϕ_2	0.0001	-0.0001	-0.0005
ϕ_3	-0.0007	-0.0001 **	0.0061 **
ϕ_4	0.0004 *	0.0000	0.0002
ϕ_5	0.0001	0.0001 **	0.0028
ϕ_6	-0.0002	0.0000 **	-0.0011
ϕ_7	0.0001	0.0001	-0.0051
ϕ_8	0.0001	0.0003	-0.0014 **
ϕ_9	0.0000	0.0003	0.0005
ϕ_{10}	0.0004 *	0.0000	0.0023
ϕ_{11}	0.0001	0.0002	0.0016
ϕ_{12}	0.0002	-0.0007	0.0050 **
Adj. R^2	0.96	0.82	0.51
Ljung-Box	7.78 *	0.82	9.79 **
R^2_S	0.252	0.004	-0.391
R^2_D	0.335	0.138	-0.114

A plot of the time-varying interventions for chicken and poultry is shown in Figure 1. The first observation in the graph corresponds to the March 1996 BSE scare and highlights the expected negative effect for beef and a positive one for chicken. Such impact is reabsorbed in few months and the model captures a positive trend for beef and a negative one for chicken. The dioxin crisis in itself has little relevance, even if the chicken shift registers a negative peak. The impact of the 2000 crisis is by far the widest one. The negative shift in beef reaches its peak in January 2001, then there is a recovery pattern which is completed by mid-2001. Similarly, there is a very strong positive effect on chicken demand, which is still present by the end of 2001.

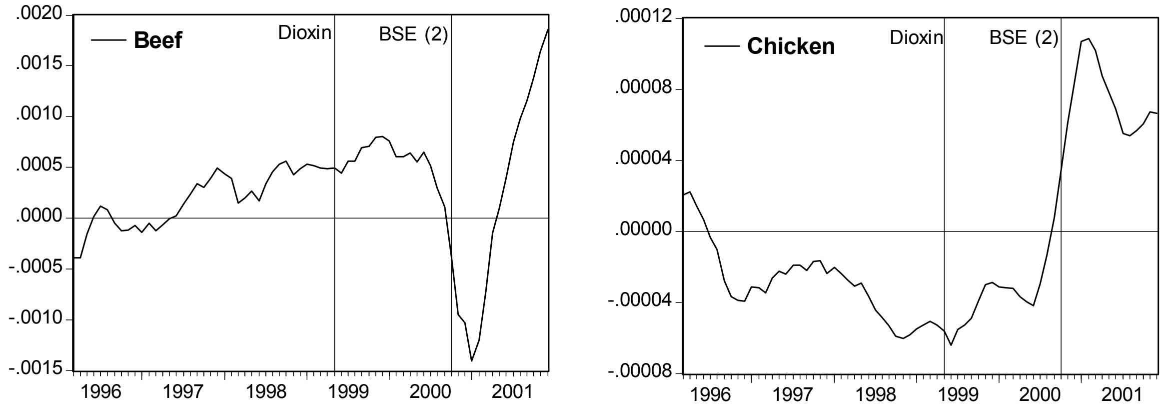

Figure 1. Plot of the time-varying shifts for beef and chicken

The interventions plotted in figure 1 are meant to capture the shifts in preference due to the food scares, i.e. excluding any effect due to changes in prices. For all considered crisis, there is clear evidence of the social amplification process in the first month, then the negative psychological effect is recovered relatively quickly. This does not necessarily mean that beef demand has fully absorbed the effects of the scare, since consumption is increased due to lover prices and vice versa for chicken.

Conclusion

We suggest that a stochastic approach to model the impact of a food scare over time should be preferred to the methods based on simple dummy shifts or media coverage indicators, especially in the cases where the same scare or different scares involving the same product reoccur over time. This method, based on a random walk specification of the intervention variable, avoids the need for subjective assumptions on the cumulated impact of information and the difficult distinction between positive and negative information. A dynamic Almost Ideal Demand System with a stochastic shift on the intercept after the onset of the first scare is expected to model the evolving pattern of consumer anxiety, maintaining the capability to capture subsequent events affecting the consumption of the same foods. This model allows to isolate the effect on consumer preferences other than the impact on demand due to the change in prices. Estimation is achieved through the Kalman-filter based EM algorithm.

The application of the dynamic AIDS model with stochastic shift is shown on Italian data, to assess the time-varying impact of two waves of the BSE crisis (1996 and 2000) and the dioxin crisis in between. Empirical results show the scarce relevance of the dioxin crisis in terms of preference shift, while not excluding for a more relevant effect through prices. The impact of the first BSE crisis on preferences seems to be reabsorbed in few months, but the second wave of the scare at the end of 2000 had a much stronger effect on preferences than the first one and the positive shift in chicken demand was still persisting 14 months after the onset of the crisis.

The model could be further improved to overcome some of its limitation. First, different stochastic structures such as an AR(1) shift could be tested and compared to the random walk assumption. Another issue is the stability of the price and expenditure coefficients, as consumer response to food safety information is likely to affect the behavioural response of the consumer.

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Appendix: Kalman filter, smoother and the log-likelihood function

The Kalman filter is a recursive procedure producing the optimal estimates of the state vector at time t conditional upon the available information in the same time period. The optimal filtered estimator at time t is defined as

$$a_{t|t-1} = Ta_{t-1} \tag{A1}$$

and its covariance matrix is

$$P_{t|t-1} = TP_{t-1}T' + Q \tag{A2}$$

where $Var(a_t) = P_t$ is the covariance matrix for the state vector. Equations (A1) and (A2) are the prediction equations of the Kalman filter. Once the actual observation w_t becomes available, the optimal estimator is updated according to the previous prediction error. This happens through the following updating equations:

$$a_t = a_{t|t-1} + P_{t|t-1} Z_t' F_t^{-1} \left(y_t - Z_t' a_{t|t-1} \right) \qquad \text{(A3)}$$

$$P_t = P_{t|t-1} - P_{t|t-1} Z_t' F_t^{-1} Z_t P_{t|t-1} \quad \text{where } F_t = Z_t P_{t|t-1} Z_t' + H \qquad \text{(A4)}$$

The equations described in (A1 – A4) constitute the Kalman filter.

Once the full set of filtered estimates $a_{t|t-1}$ and a_t are computed through the Kalman filter, it becomes possible to smooth the estimates of the state vector by exploiting all the information available in the data set. In other words, the Kalman smoother allows the computation of the least square estimates of the state vector at time *t*, conditional to the whole set of τ observations, i.e. $a_{t|\tau} = E\left(\alpha_t \middle| \Im_\tau \right)$. The fixed interval smoothing algorithm (alternative algorithms are discussed in Harvey, 1989, p.150) is a backward recursive procedure, described by the following equations:

$$a_{t|\tau} = a_t + P_t^* \left(a_{t+1|\tau} - T a_t \right) \qquad \text{(A5)}$$

$$P_{t|\tau} = P_t + P_t^* \left(P_{t+1|\tau} - P_{t+1|\tau} \right) P_t^{*\prime} \qquad \text{(A6)}$$

where

$$P_t^* = P_t T' P_{t+1|t}^{-1} \qquad \text{(A7)}$$

The smoother runs from $t=\tau-1$ to $t=1$, with $a_{\tau|\tau} = a_\tau$ and $P_{\tau|\tau} = P_\tau$ as starting values. Estimates obtained through the Kalman smoother show mean square error inferior or equal to those obtained through the Kalman filter, as they are based on a larger set of observations.

Given the assumption of a normal distribution for the disturbances in the model and the initial state vector, the distribution of the vector of observation w_t conditional on the set of observation up to time t-1 is itself normal, where the mean and covariance for such distribution can be derived through the Kalman filter. Hence, it becomes possible to write explicitly the log-likelihood function for a multivariate normal model:

$$\log L(w, \Psi) = -\frac{\tau g}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^{\tau} \log \left| F_t \right| - \frac{1}{2} \sum_{t=1}^{\tau} (w_t - Z_t a_{t|t-1})' F_t^{-1} (w_t - Z_t a_{t|t-1}) \qquad \text{(A8)}$$

where Ψ represents all unknown parameters of the model.