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ON THE CHOICE OF DESIGNS FOR THE ESTIMATION OF PRODUCTION FUNCTIONS

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I. Introduction

A problem facing economists and agronomists in estimating crop production functions from experimental data is the choice of a design[1/] for the experiment. Two of the primary uses of these estimated production functions are to determine economically optimum input levels and to predict crop response within the range of the experimental points. Thus the design must allow for precise estimation of the portion of the production surface with positive diminishing marginal product and in the region of the physical maximum. Both the theoretical precision of the production function and the effects of the spacing of the design points on the production surface should be considered in selecting a design. Variables such as weather and initial soil fertility level which influence yield are uncontrolled in many field studies of crop response, and unless several experiments are conducted simultaneously the experimenter must wait until the following year to conduct an experiment utilizing results of his present data. Therefore, the economist would like to obtain reliable information concerning the region to the left of and including maximum crop response during each year of experimentation.

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1/ In this paper design refers to the specific treatment combinations included in the experiment.

During 1961 an experiment was conducted to compare five basic designs used in agronomic-economic studies. The purpose of the work reported in this paper is to (1) compare production surfaces and economic optima estimated from the several designs and (2) evaluate the relative precisions of the functions estimated from these various designs in predicting the observed production surface.

II. Design and Procedure

In recent years in agronomic-economic studies, many of the field fertility experiments which have been conducted by the colleges have used a simple composite design2/ or some modification of it as alternatives to a complete factorial design. Five basic designs which have been used in agronomic-economic studies were investigated in the experiment reported here: (1) the complete factorial, (2) the interlaced factorial,3/ (3) a simple central composite, (4) a double cube, and (5) a "triple cube".4/

2/ Davies, O. L., Design and Analysis of Industrial Experiments, Oliver and Boyd, London, 1956, pp. 532-537.

3/ The interlaced factorial design was developed by Clifford G. Hildreth, Department of Agricultural Economics, Michigan State University, for the Mississippi-TVA cooperative agronomic-economic project at Mississippi.

4/ The "triple cube" design was developed by Thomas E. Tramel, Department of Agricultural Economics, Mississippi State University, for the Mississippi-TVA cooperative agronomic-economic project, and is reported in Economics and Technical Analysis of Fertilizer Innovations and Resource Use, edited by Baum, E. L., Heady, Earl O., Pesek, John T., and Hildreth, Clifford G., Iowa State University Press, Ames, Iowa, 1957, pp. 168-175.

The over-all design was a completely randomized incomplete factorial consisting of two observations on each of the 141 treatment combinations appearing in the designs under study.[5/] Three complete factorials were included: the 5^3 factorial (125 treatment combinations), a 3^3 factorial (27 treatment combinations) consisting of the transformed levels $0, \pm 1$ (defined later), and a 3^3 factorial consisting of the transformed levels $0, \pm 2$. The double (23 treatment combinations) and triple cube (31 treatment combinations) designs are modifications of the simple central composite design (15 treatment combinations). The double cube design is formed by adding a 2^3 factorial to the simple composite either inside or outside the cube of the simple composite within the limits of $\pm\alpha$.[6/] In this study two double cubes were included, one with the additional cube of the transformed levels ± 0.5, and the other with the additional cube of the transformed levels ± 1.5. The cube of the simple composite was of the transformed levels ± 1. The triple cube is formed by adding two 2^3 factorials to the simple composite within the limits of $\pm\alpha$ one inside and one outside the cube of the simple composite. The additional cubes for the triple cube in this study were of the transformed levels ± 0.5 and ± 1.5. The transformed level of α was 2 for all of the composite designs. The interlaced factorial (53 treatment combinations) is formed from two 3^3 factorials which overlap each other and a 2^3 factorial of the extreme levels of the independent variables. In the study the transformed levels for the two 3^3 factorials were $0,-1,-2$ and $-1,0,1$ and for the 2^3 factorial were ± 2.

5/ Portions of this paper were taken from Havlicek, Betty P., "A Statistical comparison of Experimental Results from Six Basic Designs Used in Agronomic-Economic Studies," Unpublished M.S. Thesis, University of Tennessee, August 1961.

6/ α is the extreme level of the variables included in the design. See description of the simple composite design in Davies, O.L., op. cit., pp. 532-537.

The experiment was conducted in the TVA greenhouse at Wilson Dam, Alabama. This was done in an attempt to hold constant variables not under study and thus obtain a smaller experimental error than would be obtained in a field experiment. The production surface under study was the response of corn to N (nitrogen), P (phosphorus), and K (potassium). There are numerous functional forms which may be utilized in productivity studies. In this study different functional forms were not investigated. Theoretically crop response to fertilizer nutrients is curvilinear and if the estimated production on surface is to be amenable to economic analysis it must account for the region of response reflecting diminishing positive marginal products. Initial analyses indicated that a quadratic function fit the data well and would allow an investigation of the linear, quadratic, and interaction components of the response. The quadratic function fitted to the data[7/] by standard least squares techniques was of the form

$$\hat{Y} = b_0 + b_1N + b_2P + b_3K + b_{11}N^2 + b_{22}P^2 + b_{33}K^2 + b_{12}NP + b_{13}NK + b_{23}PK$$

where the b's are estimates of the population regression coefficients, $\hat{Y}$ is an estimate of yield for a particular nutrient combination, and N, P, and K are the application levels of nitrogen, phosphorus, and potassium, respectively. If the quadratic function is estimating the production surface in the economically relevant region of production, i.e., where the marginal product is positive but decreasing, the coefficients of the squared terms will be negative and all other coefficients will be positive. A function with these characteristics is referred to as logical in this paper. Positive coefficients on the squared terms mean that the production function is estimating the response surface to the left of the region mentioned above and the function represents the region where the marginal

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7/ The mean yield of the two observations were analyzed, N, P, and K were applied in milligrams per pot (mg./pot) and yield was measured in grams of dry matter per pot. Although yields of corn in grams per pot obtained under greenhouse conditions are not directly comparable to grain yields obtained under field conditions when the corn is grown to maturity, the greenhouse results should give approximations of what can be expected under field conditions for a particular design relative to another design.

physical product of fertilizer is still increasing. An estimated production function with positive coefficients on the squared terms will be considered illogical for studying the production surface in the economically relevant region.

A transformation of the independent variables was employed to take advantage of the semi-orthogonality property of the composite designs and the orthogonality of the complete factorial designs. As a result of the transformation the linear and interaction variables were orthogonal. This transformation did not render the interlaced factorial design semi-orthogonal; however, the degree of inter-correlation between the coefficients of the variables is small. The transformed independent variables and original mean yield values were employed to solve for the estimated production functions. The transormed and actual levels of the independent variables are given in Table 1.

Least squares regression solutions were obtained for all of the designs and then the function estimated from the 5^3 factorial was employed as a basis against which theoretical and experimental comparisons of the other designs were made. There were two reasons for using the 5^3 factorial as a basis against which the other designs were compared:

(1) The 5^3 factorial was the largest orthogonal design and the observations were spread over the entire observed surface.

(2) The composite and interlaced factorial designs have been used in crop response studies as alternatives to the complete factorial designs. Because of reason (1) above it was felt that the function estimated from the 5^3 factorial design should be the best estimate of the true production function in the entire experimental region observed. Thus an underlying question was how well do the composite and interlaced factorial designs (each with less than half the number

Table 1

Transformed and Actual Levels of N, P, and K Used
in the Various Experimental Designs
Greenhouse Corn Experiment, TVA, 1961[a]

N		P		K	
Actual mg./pot[b]	Transformed	Actual mg./pot[b]	Transformed	Actual mg./pot[b]	Transformed
200	-2.0	100	-2.0	100	-2.0
325	-1.5	200	-1.5	200	-1.5
450	-1.0	300	-1.0	300	-1.0
575	-0.5	400	-0.5	400	-0.5
700	0.0	500	0.0	500	0.0
825	0.5	600	0.5	600	0.5
950	1.0	700	1.0	700	1.0
1075	1.5	800	1.5	800	1.5
1200	2.0	900	2.0	900	2.0

a/ The greenhouse experiment was conducted jointly during the spring of 1961 by personnel from Soils and Fertilizer Research Branch and Agricultural Economics Branch, Division of Agricultural Relations, Tennessee Valley Authority.

b/ Milligrams of elemental nutrient per pot.

of observations included in the 5^3 factorial design) approximate the results obtained from the 5^3 factorial design.

Comparisons are made of the theoretical precisions of the functions (Section III), of the estimated regression coefficients (section V), of the ability of the estimated functions to predict the observed surface (section VI), and of the estimated maxima and economic optima for an assumed price ratio (section VII). The standard regression results are presented in section IV.

III. Theoretical Precision

Standardized design matrices were used to compute the theoretical precisions of the functions estimated from all of the designs except the interlaced factorial and the incomplete factorial which utilized all of the observations. For each design

$$\sum x_i = 0$$
$$\sum x_i^2 = n$$

where x_i is the i^{th} level of the standardized level of N, P, and K, as suggested by Box and Hunter[8/] and n is the number of treatment combinations in the design The theoretical precision of the ii^{th} regression coefficient is given by nc_{ii},[9/] where n is the number of observations included in the design, and c_{ii} is the ii^{th} element of $(X'X)^{-1}$, the inverse of the standardized raw sums of squares and cross products matrix. These precisions are given in Table 2. Since $\sigma^2(X'X)^{-1}$ is the variance-covariance matrix of the regression coefficients, the smaller the value of nc_{ii} the more precisely the coefficient is estimated. The precisions of the coefficients of the respective designs relative to the 5^3 factorial are also given in Table 2.

8/ Box, G.E.P., and Hunter, J.S. "Multi-factor Experimental Designs for Exploring Response Surfaces," The Annals of Mathematical Statistics, 28:196, March 1957.

9/ Ibid, p. 199.

Each design estimates the linear regression coefficient as precisely as does the 5^3 factorial design. The triple cube estimates the constant and interaction and terms most precisely (158 and 117 per cent, respectively, as precisely as the 5^3 factorial), the double cube design (± 0.5)[10] estimates the quadratic term most precisely (237 per cent as precisely as the 5^3 factorial). Thus a decision between these designs is not clear but depends on which coefficient or coefficients the experimenter is interested in determining most precisely Alternatively, the variance function $V(x)$, for a design enables one to consider the accuracy of the coefficients jointly. The variance function $V(x) = nV(\hat{Y})/\sigma^2$ provides a standardized measure of the precision with which a design allows the response to be estimated at any point x. The variance function for each standardized design in Table 2 is given in Table 3. The values of $V(x)$ for the center of each design and for the extreme levels of each design are also given in Table 3.

The triple and double (± 0.5) cubes estimate the response at the center of the respective designs most precisely, $[V(x) = 3.3$ and 3.8, respectively$]$. If one evaluates the variance function at $x_1 = x_2 = x_3 =$ the extreme levels of each variable in the design, the 3^3 and 5^3 factorials estimate response most precisely $[V(x) = 13.8$ and 23.3, respectively$]$. Thus if the experimenter has a good approximation of the maximum and centers his design at this point, he will estimate response in the area of the maximum with the smallest variance by using the double (± 0.5) or tiple cube designs. If the experimenter does not have a good approximation of the maximum but feels it may lie some distance above or below his estimate, the variance functions indicate he would be better off with a 3^3 factorial than one of the composite designs under study here.

10/ Levels of independent variables in parentheses distinguish two variations of the same basic design.

Table 2. Theoretical Precision of Standardized Experimental Designs Relative to the 5^3 Factorial Design.

Element	5^3 Factorial nc_{ii}	3^3 Factorial nc_{ii}	3^3 Factorial Relative Precision(%)
nc_{oo}	5.285714	7.000000	75
nc_{11}, nc_{22}, nc_{33}	1.000000	1.000000	100
$nc_{11,11}$, $nc_{22,22}$, $nc_{33,33}$	1.428571	2.000000	71
$nc_{12,12}$, $nc_{13,13}$, $nc_{23,23}$	1.000000	1.000000	100

Element	Double Cube (±1.5) nc_{ii}	Double Cube (±1.5) Relative Precision(%)	Triple Cube nc_{ii}	Triple Cube Relative Precision(%)
nc_{oo}	6.643613	80	3.340759	158
nc_{11}, nc_{22}, nc_{33}	1.000000	100	1.000000	100
$nc_{11,11}$, $nc_{22,22}$, $nc_{33,33}$	1.674170	85	1.131052	126
$nc_{12,12}$, $nc_{13,13}$, $nc_{23,23}$	1.036307	96	0.853193	117

Element	Simple Composite nc_{ii}	Simple Composite Relative Precision(%)	Double Cube (±0.5) nc_{ii}	Double Cube (±0.5) Relative Precision(%)
nc_{oo}	11.666667	45	3.773181	140
nc_{11}, nc_{22}, nc_{33}	1.000000	100	1.000000	100
$nc_{11,11}$, $nc_{22,22}$, $nc_{33,33}$	1.540741	93	0.601610	237
$nc_{12,12}$, $nc_{13,13}$, $nc_{23,23}$	2.133333	47	1.657289	60

Table 3. Variance Functions For Standardized Experimental Designs

Experimental Design	Variance Function
5^3 Factorial	$V(x)=5.285714-1.857142(x_1^2+x_2^2+x_3^2)+1.428571(x_1^4+x_2^4+x_3^4)+x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2$
3^3 Factorial	$V(x)=7.000000-3.000000(x_1^2+x_2^2+x_3^2)+2.000000(x_1^4+x_2^4+x_3^4)+x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2$
Simple Composite	$V(x)-11.666667-6.111110(x_1^2+x_2^2+x_3^2)+1.540741(x_1^4+x_2^4+x_3^4)+4.148147(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)$
Double Cube (± 0.5)	$V(x)=3.773181-0.848788(x_1^2+x_2^2+x_3^2)+0.601610(x_1^4+x_2^4+x_3^4)+1.980073(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)$
Double Cube (± 1.5)	$V(x)=6.643612-2.762408(x_1^2+x_2^2+x_3^2)+1.674170(x_1^4+x_2^4+x_3^4)+1.243341(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)$
Triple Cube	$V(x)=3.340759-0.560506(x_1^2+x_2^2+x_3^2)+1.131052(x_1^4+x_2^4+x_3^4)+0.502393(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)$

	Center of Design		Extremes 1/ of Design	
	x_1, x_2, x_3	$V(x)$	x_1, x_2, x_3	$V(x)$
5^3 Factorial	0	5.285714	$\pm(2\frac{1}{2})$	23.285714
3^3 Factorial	0	7.000000	$\pm\left[(3/2)^{\frac{1}{2}}\right]$	13.750000
Simple Composite	0	11.666667	$\pm\left[(15/4)^{\frac{1}{2}}\right]$	182.916642
Double Cube (± 0.5)	0	3.773181	$\pm\left[(46/9)^{\frac{1}{2}}\right]$	193.085885
Double Cube (± 1.5)	0	6.643612	$\pm\left[(46/17)^{\frac{1}{2}}\right]$	48.303649
Triple Cube	0	3.340759	$\pm\left[(31/9)^{\frac{1}{2}}\right]$	55.687406

1/ This treatment combination appears only in the 5^3 and 3^3 factorials; for the other designs only one variable at a time is equal to the extreme value.

IV. Standard Regression Results

Thus far, the theoretical precisions of the designs have been considered; the remainder of the paper will be concerned with the experimental results of the study reported here. Had the design been centered at or near the maximum, each estimated production surface would have been an unbiased estimate of the true production surface in the region of the maximum. However, the design was actually centered far to the left of the maximum.

The observed mean yield values for the treatment combinations of the 5^3 factorial are plotted in Figures 1 through 5. Diminishing marginal productivity to each fertilizer nutrient is indicated on each graph by the slope of the curves drawn through the mean yield values. Interactions between the fertilizer nutrients are indicated by observing simultaneously the surfaces plotted in each figure. The surface is still rising in Figure 5, indicating that the maximum has not been reached in this experiment. The observed surface does indicate that a plateau is being reached.

The transformed production functions estimated from each design, including the incomplete factorial design with all the sample points in the experiment, are given in Table 4. The regression coefficients, standard errors of the coefficients, coefficient of determination, error mean square, and F values are also given.

Each equation fitted resulted in a significant reduction in the "sum of squares due to regression", but the R^2 values differed markedly from one design to another. If one were to rely solely on the R^2 values, one would conclude that the 3^3 complete factorial with the levels (0, ±1) provided the "best" estimate of the desired production function. However, all the

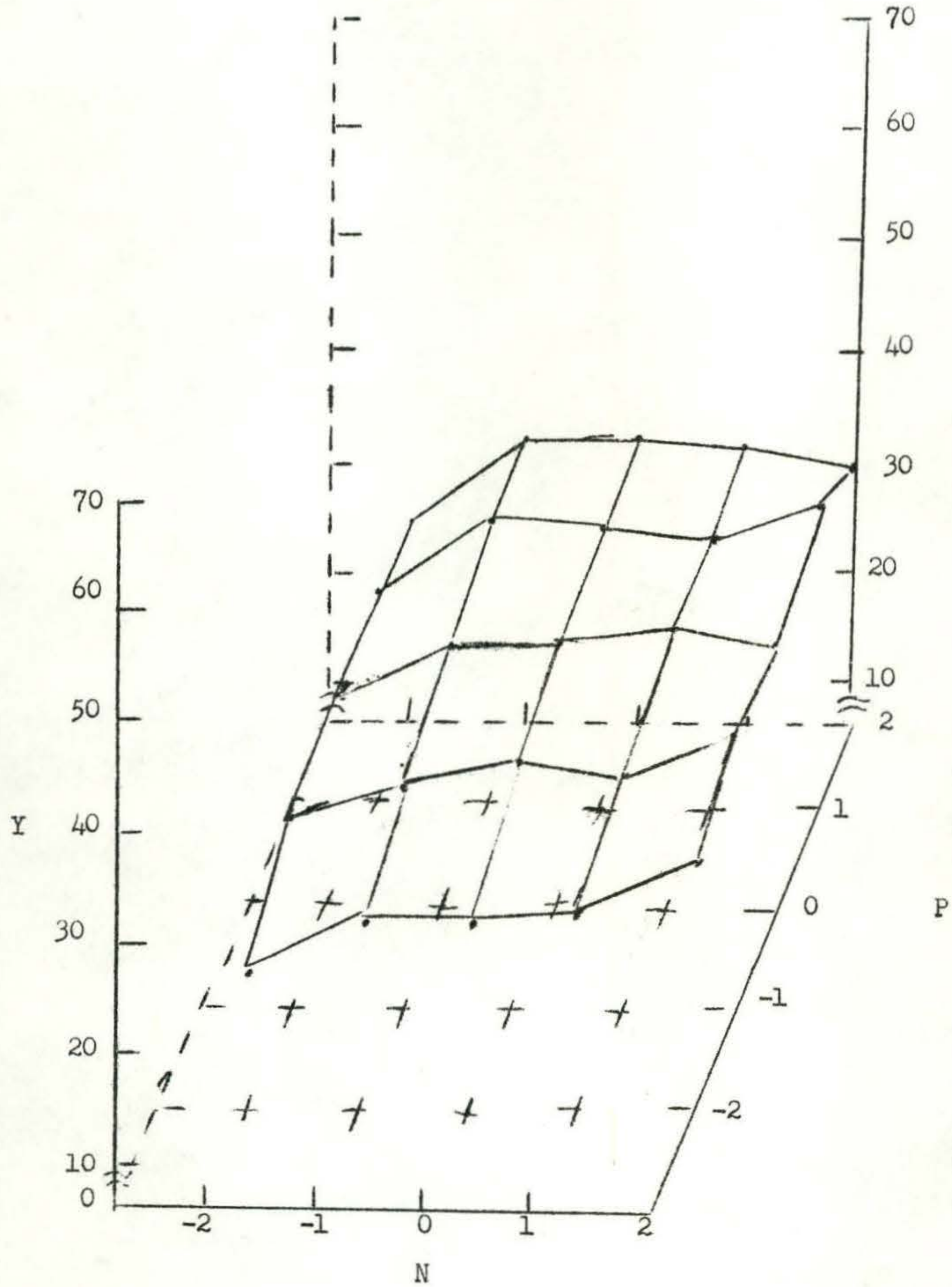

Figure 1. Response to N and P at K = -2

For the 5^3 Factorial Design

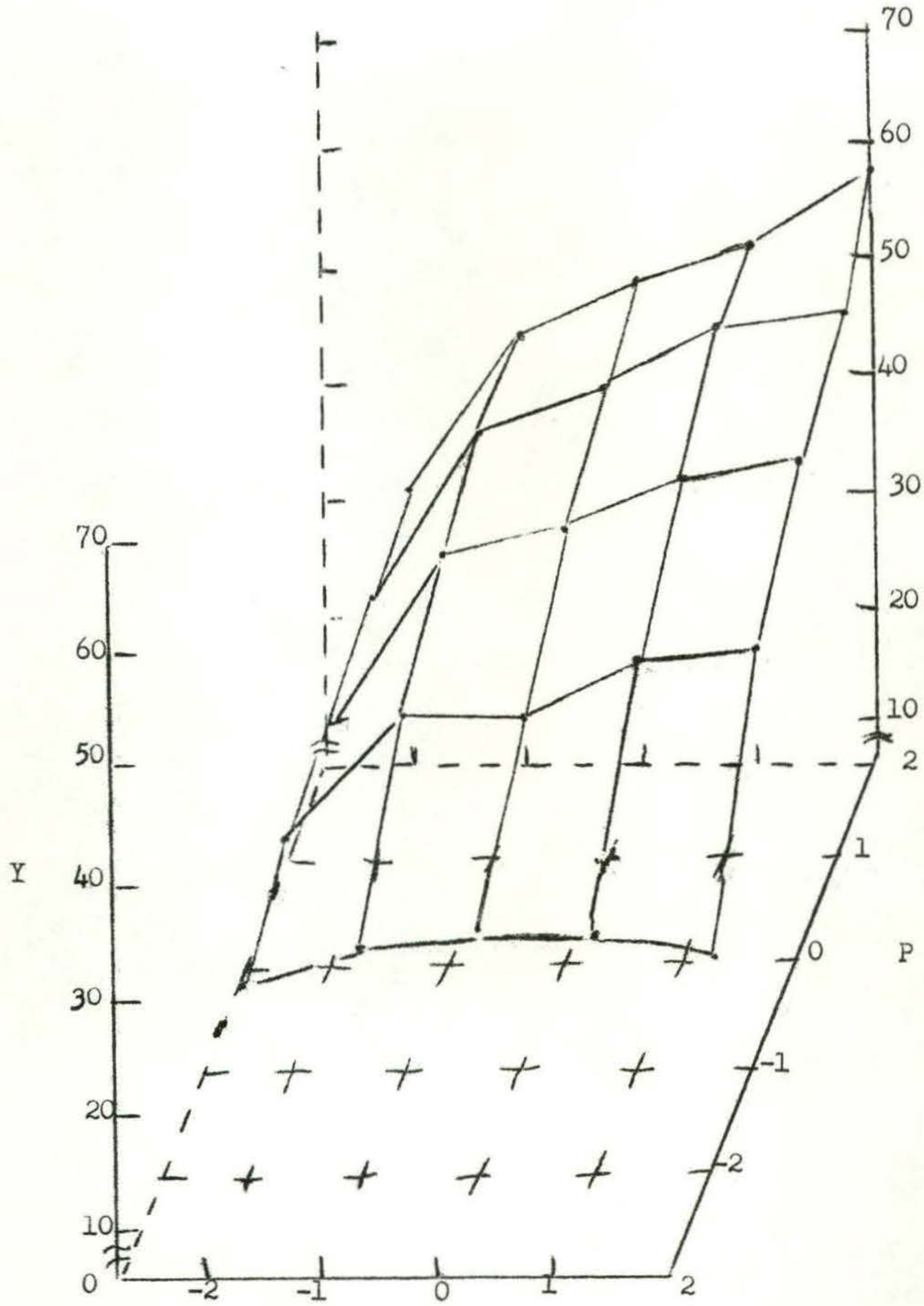

Figure 2. Response to N and P at K = -1

For the 5^3 Factorial Design

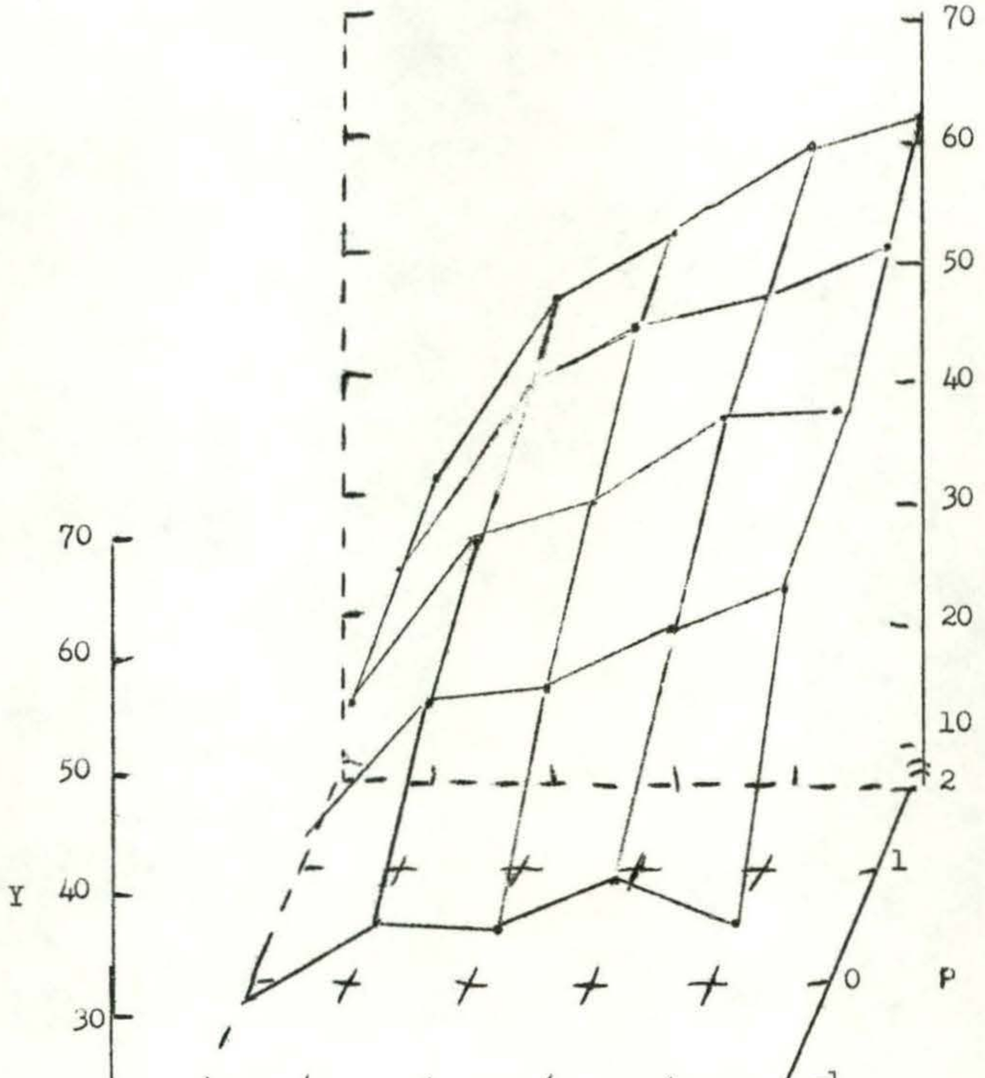

Figure 3. Response to N and P at K = 0

For the 5^3 Factorial Design

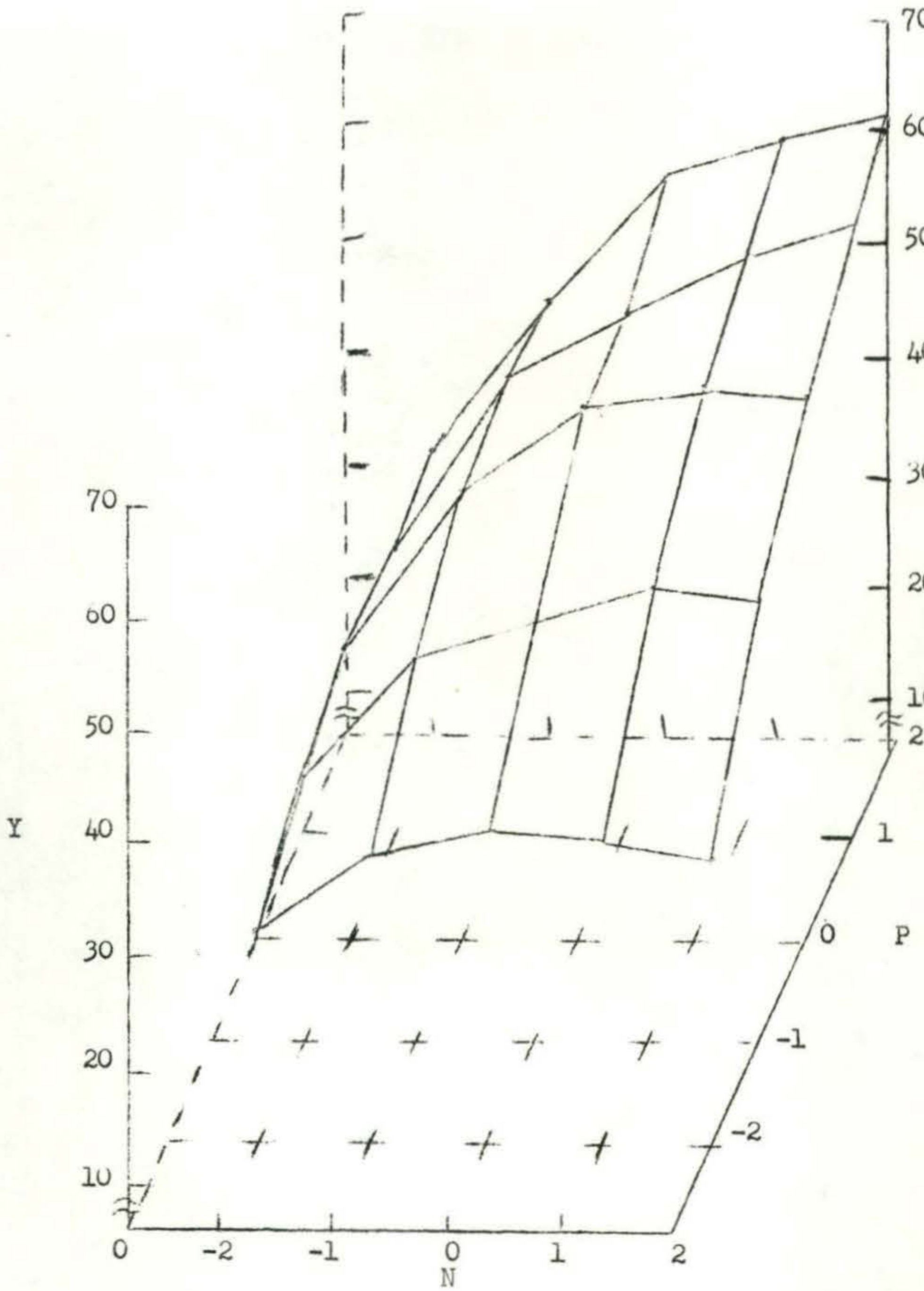

Figure 4. Response to N and P at K = 1 for the

5^3 Factorial Design.

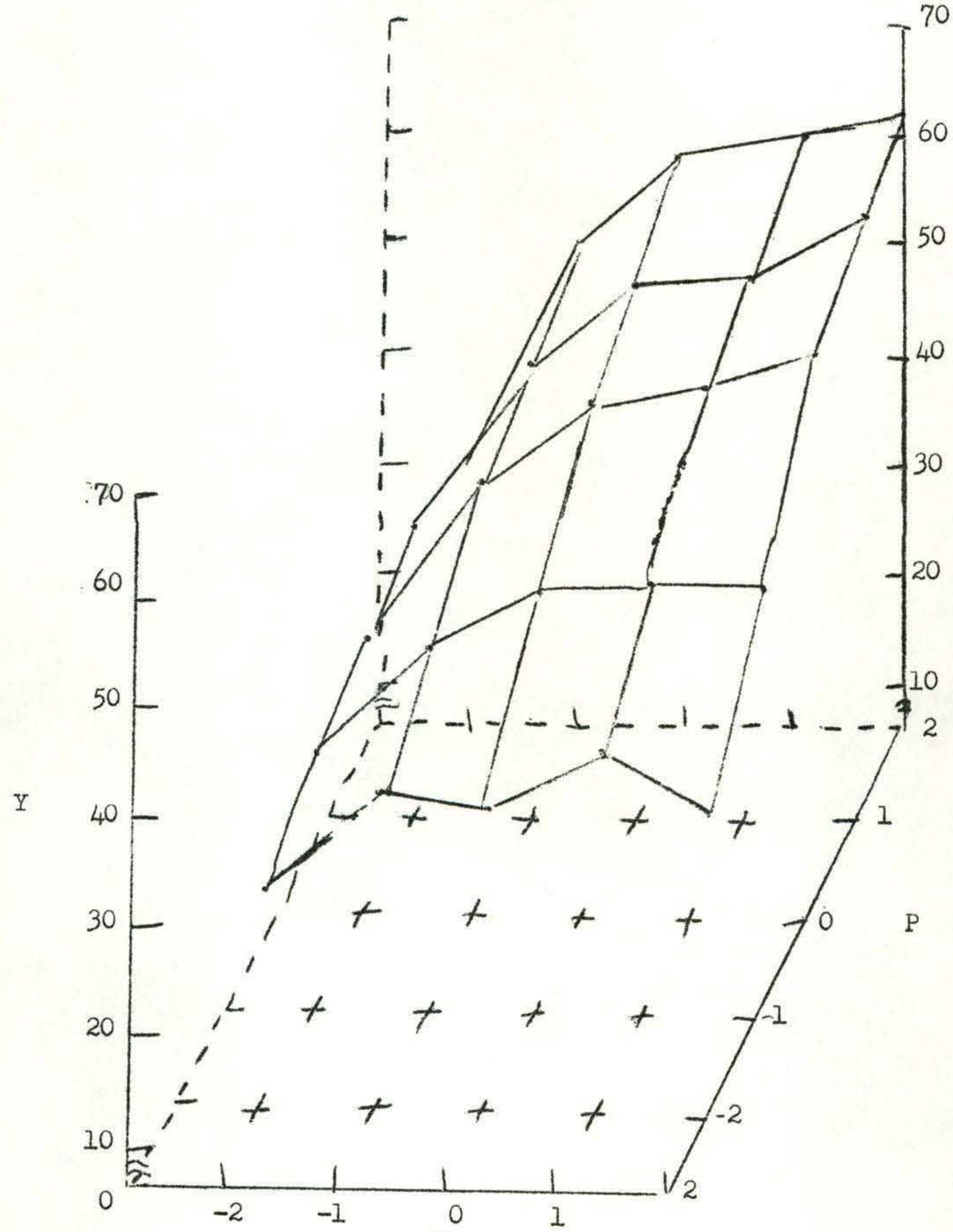

Figure 5. Response to N and P at K = 2
For the 5^3 Factorial Design.

Table 4. Estimated Response Equations with Standard Errors of the Coefficients for Greenhouse Corn Experiment, TVA, 1961, t-Test, R^2, S_e^2, N, and F for Various Transformed Design Matrices.

Experimental Design	b_o	N	P	K	N^2	P^2	K^2	NP	NK	PK
Incomplete Factorial										
b	47.2528	4.0989	3.9072	3.9379	-1.2899	-1.0775	-1.3111	1.0537	0.8647	0.4738
s_b	0.9207	0.3077	0.3077	0.3077	0.2625	0.2625	0.2625	0.2174	0.2174	0.2174
t	51.323***	13.321***	12.698***	12.798***	4.914***	4.105***	4.995***	4.847***	3.976***	2.179***
n = 141	R^2=0.8259	S_e^2=25.56	F=69.06***							
Interlaced Factorial										
b	48.3740	3.9451	3.8056	3.7468	-1.5697	-1.5650	-1.4191	0.8203	0.9002	0.5258
s_b	0.6678	0.3770	0.3770	0.3770	0.3255	0.3255	0.2276	0.2276	0.2276	0.2276
t	72.438***	10.464***	10.094***	9.938***	4,822***	4.808***	4.360***	3.604***	3.955***	2.310***
n = 53	R^2=0.9469	S_e^2=8.46	F=85.14***							
Simple Composite										
b	48.8700	4.7494	4.9344	4.1156	-1.5862	-1.7712	-1.3212	1.0262	0.2888	0.1388
s_b	4.1845	1.1862	1.1862	1.1862	1.4256	1.4256	1.4256	1.6775	1.6775	1.6775
t	11.679***	4.004***	4.160***	3.470***	1.113 N.S.	1.242 N.S.	<1 N.S.	<1 N.S.	<1 N.S.	<1 N.S.
n = 15	R^2=0.9047	S_e^2=22.51	F=5.28**							
Double Cube (±0.5)										
b	42.2440	5.7433	4.2539	4.8411	0.2585	0.0735	0.5235	0.9741	0.8076	-0.9765
s_b	2.9731	1.7302	1.7302	1.7302	1.5169	1.5169	1.5169	2.5177	2.5177	2.5177
t	14.209***	3.319***	2.459**	2.798***	<1 N.S.	<1 N.S.	<1 N.S.	<1 N.S.	<1 N. S.	<1. N.S.
n = 23	R^2=0.6617	S_e^2=53.88	F=2.83**							
Double Cube (±1.5)										
b	41.7874	3.5391	2.7544	2.9056	0.8314	0.6464	1.0964	0.3493	-0.0433	-0.2981
s_b	3.2283	1.0301	1.0301	1.0301	1.0963	1.0963	1.0963	0.8625	0.8625	0.8625
t	12.944***	3.436***	2.674***	2.821***	<1 N.S.	<1 N.S.	1 N.S.	<1 N.S.	<1 N.S.	<1 N.S.
n = 23	R^2=0.6899	S_e^2=36.08	F=3.21							

Table 4 (Con'd)

Experimental Design	b_o	N	P	K	N^2	P^2	K^2	NP	NK	PK
Triple Cube										
b	40.8149	4.1033	2.5353	3.3356	1.0121	0.8271	1.2771	0.3471	0.0501	-0.4871
s_b	2.3195	1.1776	1.1776	1.1776	1.1622	1.1622	1.1622	1.0094	1.0094	1.0094
t	17.596***	3.484***	2.153**	2.832***	<1 N.S.	<1 N.S.	1.099 N.S.	<1 N.S.	<1 N.S.	<1 N.S.
n = 31	R^2=0.5774, S_e^2=49.92, F=3.19**									
3^3 Factorial (0,+1)										
b	47.9130	3.9578	4.4256	2.6250	0.2544	-2.3856	-0.3472	0.8992	0.5375	0.0117
s_b	0.6105	0.2826	0.2826	0.2826	0.4895	0.4895	0.4895	0.3461	0.3461	0.3461
t	78.4816***	14.0050***	15.6603***	9.2887***	<1 N.S.	4.8735***	<1 N.S.	2.5981***	1.5530*	<1 N.S.
n = 27	R^2=0.9706, S_e^2=1.44, F=62.36***									
3^3 Factorial (0, +2)										
b	46.5756	3.9800	3.9164	4.2336	-1.4175	-1.2887	-1.2229	1.1027	1.1185	0.7004
s_b	2.0868	0.4830	0.4830	0.4830	0.4183	0.4183	0.4183	0.2958	0.2958	0.2958
t	22.319***	8.240***	8.108***	8.765***	3.389***	3.081***	2.924***	3.728***	3.781***	2.368**
n=27	R^2=0.9415, S_e^2=16.80, F=30.42***									
5^3 Factorial										
b	48.5508	4.1399	4.1705	4.0360	-1.5553	-1.3429	-1.5765	1.1225	0.9353	0.5626
s_b	0.6806	0.2093	0.2093	0.2093	0.1769	0.1769	0.1769	0.1480	0.1480	0.1480
t	71.335***	19.780***	19.926***	19.283***	8.792***	7.591***	8.912***	7.584***	6.320***	3.801***
n=125	R^2=0.9282, S_e^2=10.96, F=165.11***									

*** Denotes significance at 1 per cent level.
** Denotes significance at 5 per cent level.
* Denotes significance at 10 per cent level.
N.S. denotes not significant.

regression coefficients in this equation are not significantly different from zero nor is the sign of the coefficient for N^2 logical in the sense that logical was defined in section II. At the same time the coefficient for N^2 was not significantly different from zero.

The difficulty with the response functions obtained using the simple composite, double cube, and triple cube designs in this experiment is either that (1) the signs on the quadratic and/or interaction terms are illogical for the region of the maximum of the surface or (2) the quadratic and interaction terms are nonsignificant. The coefficients which have illogical signs are nonsignificant and the R^2 values for the functions are low. It would appear that the real difficulty with the functions for the simple composite, double cube, and triple cube designs lies in the fact that the experiment was miscentered. The miscentering and spacing of the design points have caused the triple and double cubes and 3^3 factorial $(0, \pm 1)$ to be biased estimates of the true production surface in the region of the physical maximum.

Figures 6, 7, and 8 show graphically the observed average response to N, P, and K based on the 5^3 factorial and triple cube designs. The figures support the positive signs on the squared terms observed in the triple and double cube functions. One reason for the discrepancy between the observed average yield curves of the two designs is the difference in the degree of hidden replication in the 5^3 factorial. The variability in yield for any particular level of N, P, and K was quite large.

All the coefficients estimated by the functions for the incomplete factorial, interlaced factorial, 5^3 factorial, and 3^3 factorial $(0, \pm 2)$ designs are significant and all signs are logical. The R^2 value is highest for the interlaced factorial (0.95) and lowest for the incomplete factorial (0.83).

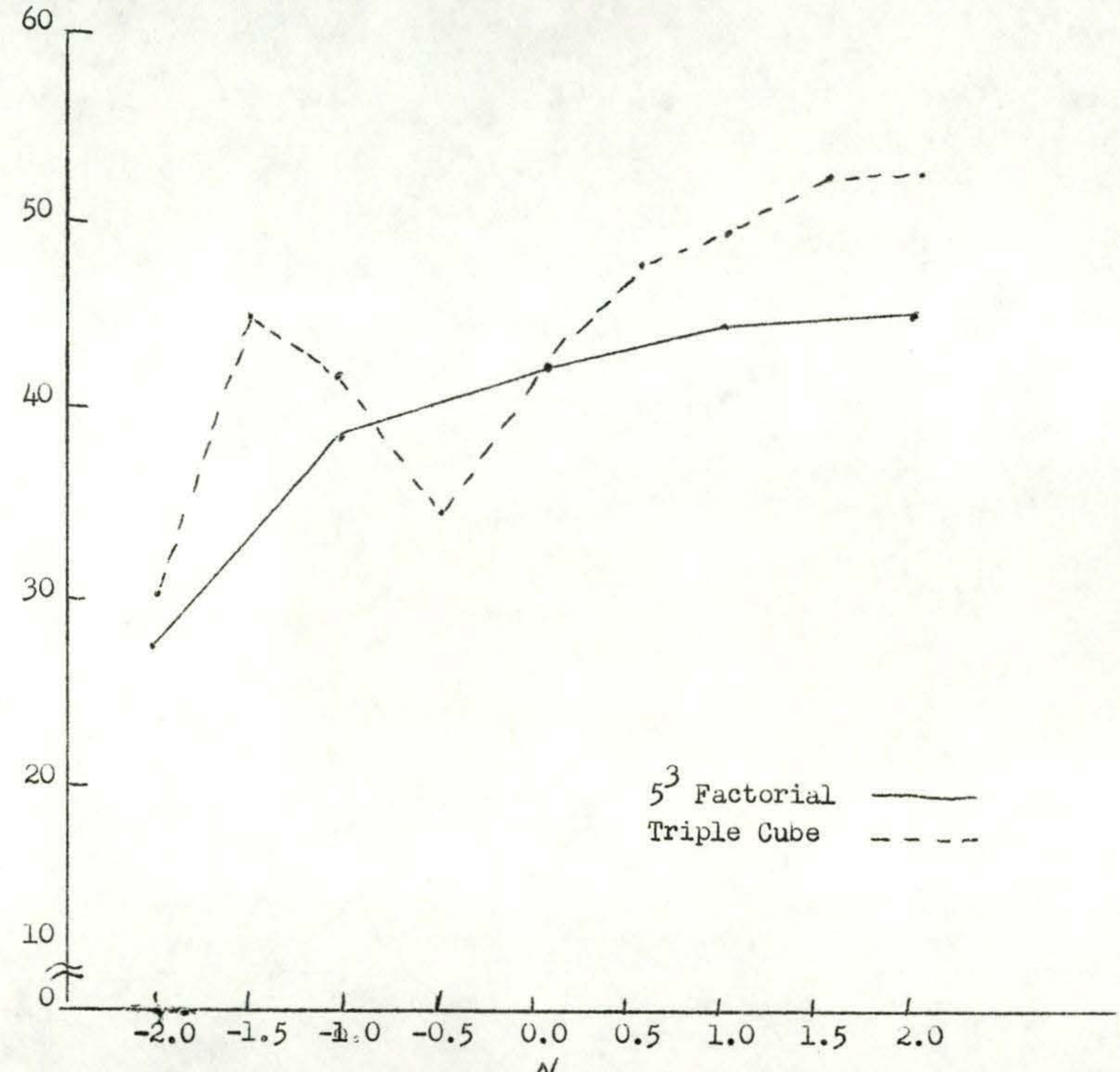

Figure 6. Average Response to Nitrogen for the 5^3 Factorial and Triple Cube Designs

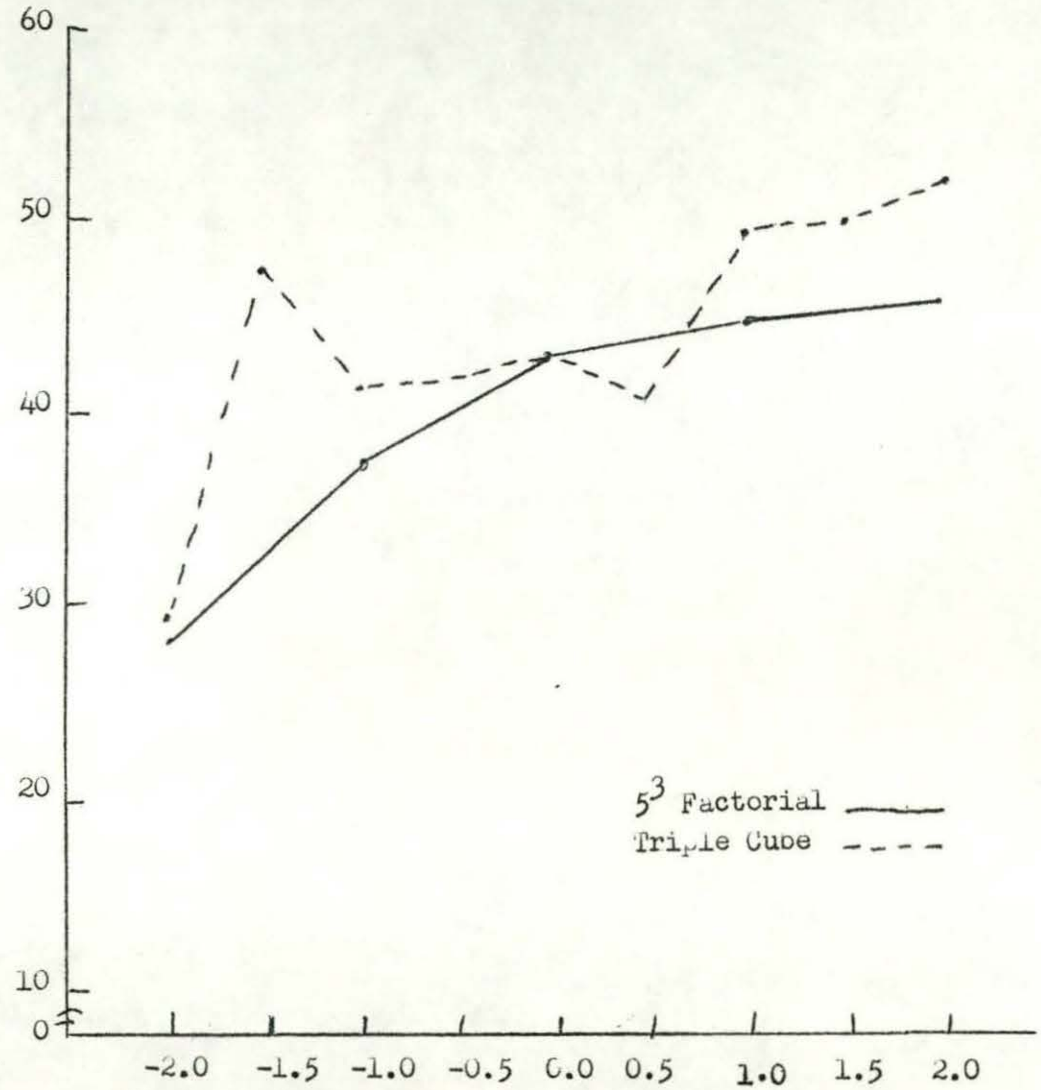

Figure 7. Average Response to Phosphorus For the 5^3 Factorial and Triple Cube Designs.

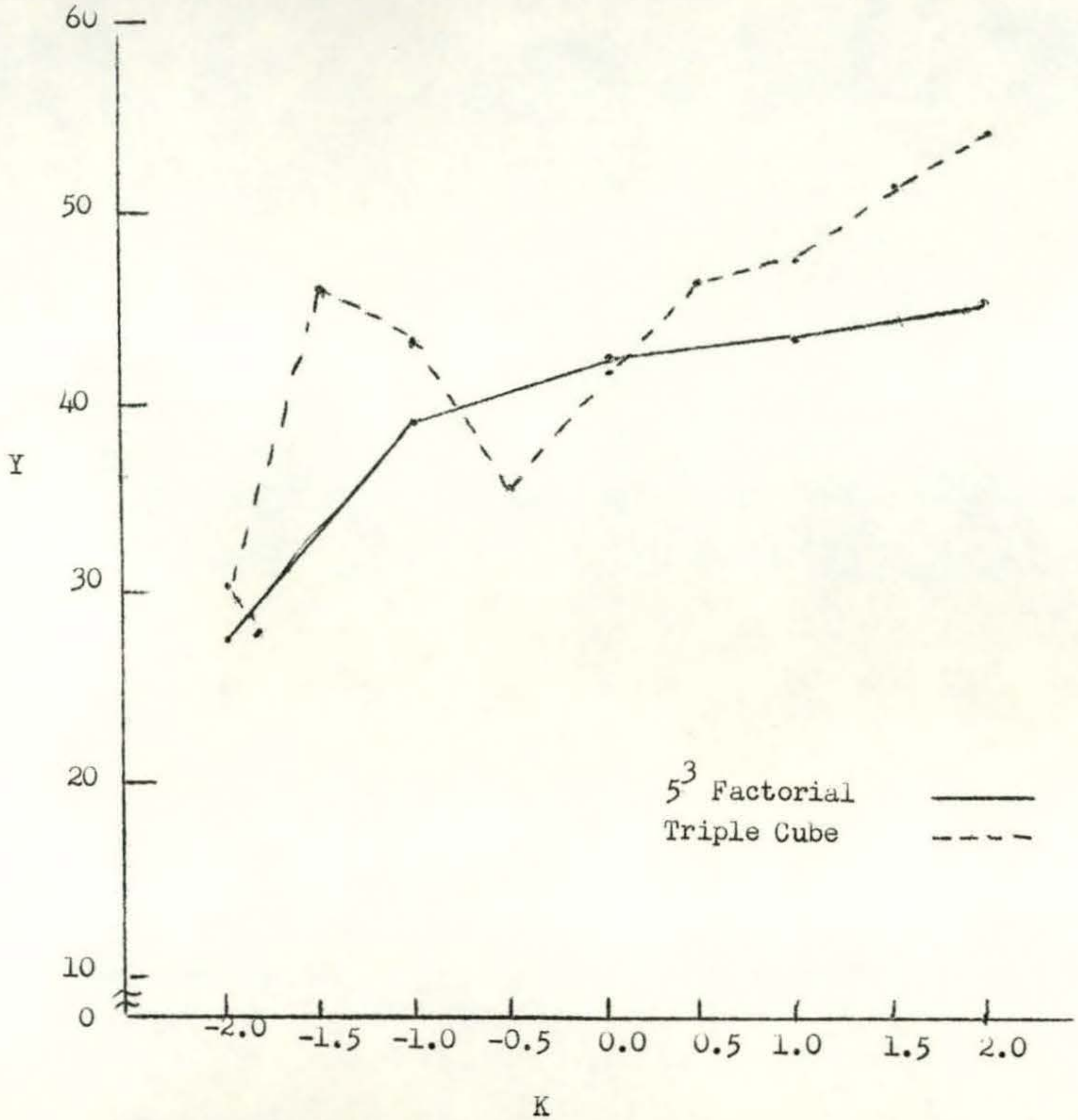

Figure 8. Average Response to Potassium for the 5^3 Factorial and Triple Cube Designs.

V. Comparison of Regression Coefficients

Each of the production functions given in Table 4 is an estimate of the same true production function within the sample space of the respective designs. The spacing of the points in each design will determine which area of the surface is receiving the most emphasis in the estimated production function.

As one means of comparing the various functions to the function computed from the 5^3 factorial design, the 95 per cent confidence interval for each of the coefficients of the 5^3 factorial was computed (Table 5). Corresponding coefficients of other functions which lay outside this confidence interval were assumed to be significantly different from the coefficient estimated in the function from the 5^3 factorial design[11/] and are denoted by an asterisk. This procedure was felt to be applicable because of the small degree of intercorrelation between the estimated coefficients which were not independent (Table 6).

The 3^3 factorial $(0,\pm2)$, interlaced factorial, incomplete factorial, and 5^3 factorial appear to be estimating essentially the same area of the production surface. None of the coefficients of the incomplete factorial, one coefficient of the interlaced factorial, and two coefficients of the 3^3 factorial $(0,\pm2)$ functions lie outside the confidence intervals for the coefficients of the 5^3 factorial. More than half of the regression coefficients estimated by the 3^3 $(0,\pm1)$, double cube, and triple cube designs lay outside the 95 per cent conficence intervals, and thus it is assumed in this experiment that these functions are estimates of a different area of the production surface because of the spacing of the points in these designs.

11/ Ezekiel, Mordecai, and Fox, Karl A., Methods of Correlation and Regression Analysis, 3rd edition, John Wiley and Sons, Inc., New York, 1959, p. 424.

parison of b_i's for Various Transformed Designs with

N^2	P^2	K^2	NP	NK	PK
-1.2086	-0.9962	-1.2298	1.4126	1.2254	0.8527
-1.5553	-1.3429	-1.5765	1.1225	0.9353	0.5626
-1.9020	-1.6896	-1.9232	0.8324	0.6452	0.2725
-1.2899	-1.0775	-1.3111	1.0537	0.8647	0.4738
0.2544*	-2.3856*	-0.3472*	0.8992	0.5375*	0.0117*
-1.4175	-1.2887	-1.2229*	1.1027	1.1185	0.7004
-1.5862	-1.7712*	-1.3212	1.0262	0.3888*	0.1388*
0.2585*	0.0735*	0.5235	0.9741	0.8076	-0.9765*
0.8314*	0.6464*	1.0964*	0.3493*	-0.0433*	-0.2981*
1.0121*	0.8271*	1.2771*	0.3471*	0.0501*	-0.4871*
-1.5697	-1.5650	-1.4191	0.8203*	0.9002	0.5258

gns.

b_1 and b_4 b_2 and b_5 b_3 and b_6	b_2 and b_6 b_1 and b_6 b_3 and b_5 b_2 and b_4 b_3 and b_4 b_1 and b_5	b_4 and b_5 b_4 and b_6 b_5 and b_6	b_3 and b_9 b_2 and b_9 b_3 and b_8 b_1 and b_7 b_2 and b_7 b_1 and b_8	b_3 and b_7 b_2 and b_8 b_1 and b_9	b_5 and b_9 b_5 and b_9 b_6 and b_8 b_4 and b_7 b_5 and b_7 b_4 and b_8	b_6 and b_7 b_5 and b_8 b_4 and b_9	b_7 and b_8 b_7 and b_9 b_8 and b_9
0	0	-0.060254	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0.653846	0	0	0	0	0
0	0	0.268267	0	0	0	0	0
0	0	-0.155076	0	0	0	0	0
0.386646	-0.155594	-0.324778	0.109672	0.078961	-0.126211	0.175802	-0.061317

VI. Predictive Ability of Functions

The predictive ability of a response function may be used to experimentally evaluate jointly the estimated coefficients of the function and provide another means of evaluating the response functions obtained from the various designs. All of the designs except the 3^3 (0, $\pm$1) cover the same range of levels of the independent variables and the uncontrolled variability is the same for each design. The designs differ in the number of points observed on the surface and the spacing of these points. Therefore, differences in the ability of the designs to predict all of the observed mean yields reflect differences in the spacing of the sample points on the surface and the sample size.

Predicted mean yield values for all of the treatment combinations included in the experiment were computed from each estimated production function. The sum of the squares of the deviations of these predicted yield values from the actual mean yield values divided by the degrees of freedom for the incomplete factorial are given in Table 7. The precision[12/] relative to the incomplete factorial of each of the functions in estimating yield is also given in Table 7. The 5^3 factorial function was 96 per cent as precise as the incomplete factorial function in estimating all of the observed mean yields, the interlaced factorial, 94 per cent and the 3^3 factorial (0,$\pm$2), 92 per cent as precise as the incomplete factorial function. As indicated previously the incomplete factorial had all the points included in the solution of the function, the 5^3 factorial had 125 of the points, the interlaced factorial had 53 of the points and the 3^3 factorial (0,$\pm$2) had 27 of the points included in the solution of their respective functions. The 5^3 factorial, interlaced factorial, 3^3 factorial

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12/ In this paper the experimental precision of a function in estimating all the observed mean yields in the experiment is defined as the sum of the squares of the deviations of the predicted yields from the observed mean yields divided by the degrees of freedom of the incomplete factorial. The smaller this figure the greater the precision of the function.

Table 7. Precision of Designs in Predicting All Observed Values Relative to the Incomplete Factorial and F-Tests of Variances, Greenhouse Corn Experiment, TVA, 1961.

Design	$\frac{\Sigma(Y-\hat{Y})^2}{131}$	Relative Precision (%)	F (To 5^3 Factorial)
Incomplete Factorial	25.56	--	
5^3 Factorial	26.53	96	
Interlaced Factorial	27.28	94	1.03,N.S.1/
3^3 Factorial $(0,\pm2)$	27.90	92	1.05,N.S.1/
Simple Composite	32.30	79	1.22,N.S.2/
3^3 Factorial $(0,\pm1)$	54.25	47	2.04*
Double Cube (±0.5)	79.01	32	
Double Cube (±1.5)	127.24	20	
Triple Cube	135.51	19	

1/ Not significant at 0.30

2/ Not significant at 0.10, $F_{.30} = 1.20$

* Significant at 0.0005.

(0,±2), and simple composite functions fall in one group based on predictive ability, while the 3^3 factorial (0, ±1), double cubes, and triple cube functions fall in groups with a much lower predictive ability.

The deviations of all functions except the incomplete factorial function were also tested to the deviations of the 5^3 factorial function, and results of these tests are also given in Table 7. The deviations of the interlaced factorial, 3^3 factorial (0,±2), and simple composite functions are not significantly different from the deviations of the 5^3 factorial function at the 0.30, 0.30, and 0.10 level of significance, respectively. The deviations of the 3^3 factorial (0,±1) function are significantly different from the deviations of the 5^3 factorial function at the 0.0005 level of significance, and thus the deviations of the two double cube and triple cube functions are also significantly different. For purposes of predicting response to fertilizer within the range of the experiment, production functions estimated from the 5^3 factorial, interlaced factorial, 3^3 factorial (0,±2), and simple composite designs are not significantly different from each other.

VII. Estimated Maxima and Economic Optima

Production functions are used by the economist to determine maximum and optimum input levels and corresponding net revenues. If there is a cost associated with the input, the output resulting from the optimum input level yields maximum net revenue.[13] When the input has no cost the input level which yields maximum output also yields maximum revenue and thus defines the upper boundary of the area of rational production. It appears desirable to investigate these points for this data. Recognizing that it is impossible to determine maximum and economically optimum fertilization rates for greenhouse data of this nature, one may assume for the present that this data is

13/ Net revenue refers to the returns above the cost of the fertilizer inputs.

representative of what one would obtain under field conditions and carry out a limited economic analysis on the data by assigning assumed price ratios of yield output and fertilizer inputs.

The predicted input levels of N, P, and K necessary to obtain the maximum physical yield are given in Table 8 and differed widely for different designs. The amount of N, P, and K necessary to obtain the maximum physical yield was the highest for the 3^3 factorial (0, ± 2) and lowest for the simple composite design with estimated yields of 83.2 gms. and 63.8 gms., respectively. The varying maximum input and output values illustrate well the point that one should treat with caution estimates from a regression function which require extrapolation of the data. Had any one of these designs been employed alone, the maximum solution should not have been treated as a sound result but as an indication that the design should be re-run with the design centered at a much higher input level than in the study reported in this paper.

Finally, and probably most important from the economist's viewpoint, the optimum input levels of N, P, and K were estimated when the input to output price ratios of P_N/P_Y, P_P/P_Y, and P_K/P_Y were 1.8, 1.5, and 0.3, respectively. The optima for various designs and the 95 per cent confidence limits[14] on the optimum input levels estimated from the 5^3 factorial function are given in Table 8. The range between estimated optima was found to be 645, 496, and 589 mgs. of N, P, and K, respectively. The optimum input levels estimated from the production functions of the interlaced factorial, 3^3 factorial (0, ± 2), and simple composite designs lay either outside or barely inside the confidence limits on the optimum inputs estimated from the production function of the 5^3 factorial design. The optimum input levels for P and K estimated from the incomplete factorial design did lie within the 95

14/ Smith, W.G., and Havlicek, Joseph, Jr. "An Application of Setting Simultaneous Confidence Limits on the Optimum Levels of Three Variables," Proceedings of the Agricultural Economics and Rural Sociology Section, Association of Southern Agricultural Workers, 1962, Volume II.

Table 8. Maximum and Optimum Inputs for Various Designs with 95 Per Cent Confidence Limits on Optimum Inputs Estimated From 5^3 Factorial Design. Greenhouse Corn Experiment, TVA, 1961.

Design	N	P	K	$\hat{Y}$ (Gm./Pot)
			Maximum (Mg./Pot)	
5^3 Factorial	1588	1232	1097	69.6
Incomplete Factorial	1975	1543	1325	74.1
Interlaced Factorial	1351	966	1016	62.8
3^3 Factorial (0, ± 2)	2227	1663	1738	83.2
Simple Composite	1292	931	886	63.8
			Optimum (Mg./Pot)	
5^3 Factorial	1284	991	963	67.5
Incomplete Factorial	1533	1186	1121	72.9
Interlaced Factorial	1114	802	904	61.2
3^3 Factorial (0, ± 2)	1736	1294	1428	79.8
Simple Composite	1091	798	839	62.5
Assumed Price Ratio	1.8	1.5	0.3	
95% Confidence Limits For Optimum of 5^3 Factorial				
Upper	1519	1223	1163	69.2
Lower	1124	843	833	64.2

per cent confidence limits on the estimates from the 5^3 factorial design. Estimated optimum yields for the optimum input levels computed from each of the alternative functions considered here lay outside the optimum yields for the confidence limit inputs. While there is a wide range in the input levels, the range of the optimum yields is from 61.2 grams per pot to 79.8 grams per pot. The importance of the range of these optimum input estimates depends upon the difference in net revenue resulting from using the various estimates. This could be studied in the context of the reduction in net revenue from using one function when another function is a better estimate of the true production function.[15/]

The maximum and optimum levels of input were not estimated from the production functions of the double and triple cubes and 3^3 factorial (0, ± 1) designs because the functions did not indicate diminishing marginal productivity to all of the inputs.

VIII. Conclusions

In this paper the interlaced factorial, double cube, triple cube, simple composite, and 3^3 factorial designs were considered as alternatives to the 5^3 factorial design when an experimenter is studying the area of the maximum of a production surface. The choice of an alternative design to the 5^3 factorial design depends upon the experimenter's past knowledge of the location of the maximum. In agricultural crop studies the location of this maximum is known to vary with factors, such as moisture conditions and

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15/ Havlicek, Joseph Jr., and Seagraves, James A., "The 'Cost of the Wrong Decision' as a Guide in Production Research," Journal of Farm Economics, 44:157, February 1962.

initial soil fertility level, not subject to the control of or at this time measurable by the experimenter.[16/]

Assuming that the designs are approximately centered in the region of interest on the production surface, the variances of individual coefficients of the estimated production function do not give an answer to the decision problem because the variances and the precisions of the coefficients relative to the 5^3 factorial vary within one design. The variance function (which considers the precision of all coefficients) at the center of the design is smallest for the double (± 0.5) and triple cube designs. At the extreme levels of the designs the 3^3 factorial possesses the smallest variance function and this should be considered if the experimenter feels the design may lie very far to the left or right of the assumed maximum.

The region of interest in this study was the area near the maximum of the production surface; however, the estimated maximum actually lay to the right of the experimental region. On the basis of the observed surface and the canonical form of the production function estimated from the 5^3 factorial design, the area of this maximum is relatively flat. The remaining comments based on experimental results reported in this paper assume that the design is centered far to the left of the maximum and reflect results of the spacing of design points when miscentering of the design occurs.

All of the regression coefficients estimated from the 5^3 factorial, interlaced factorial, 3^3 factorial ($0, \pm 2$), and simple composite designs possessed logical signs for the region near the maximum. Half of the coefficients estimated from the simple composite design and most of the coefficients estimated from 3^3 factorial ($0, \pm 2$) and interlaced factorial

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16/ For example, in an NPK cotton fertility field experiment at one location in Mississippi during the years 1956-59 the following predicted maximum physical yields in pounds of seed cotton per acre were obtained: 1956, 1,791 lbs.; 1957, 2,032 lbs.; 1958, 1,688 lbs.; 1959, 1,701 lbs. (Source: Mississippi-TVA cooperative agronomic-economic project annual report.) Many other similar examples could be cited.

fell within 95 per cent confidence limits placed on the corresponding coefficients of the 5^3 factorial design. In order to better evaluate the effect on the estimated coefficients of the spacing of design points on the observed surface, each production function was used to predict all of the observed yields in the experiment. The deviations of these predicted yields from observed yields for the interlaced factorial, 3^3 factorial $(0,\pm2)$, and simple composite functions were not significantly different at the 0.30, 0.30, and 0.10 levels of significance, respectively, from the deviations of the corresponding predicted yield estimates from the 5^3 factorial function. Thus the spacing of points did not appear to have a significant over-all effect on the coefficients of these designs. The 3^3 factorial $(0,\pm2)$, interlaced factorial, and simple composite designs appear to approximate the 5^3 factorial function.

The double cubes, triple cube, and 3^3 factorial $(0,\pm1)$ estimated production functions contained coefficients with illogical signs for the region near the maximum, although the coefficients were not significantly different from zero. More than half of the coefficients in each of these functions lay outside the 95 per cent confidence limits placed on the coefficients estimated from the 5^3 factorial design. The predicted yields computed from the production functions of these designs for the complete area under study showed that the deviations of these predicted yields from the observed yields were significantly different at the 0.0001 level of significance. Therefore, the production functions from the double cubes, triple cube, and 3^3 factorial $(0,\pm1)$ do not accurately approximate the production function from the 5^3 factorial design and the functions do not accurately represent the production surface in the region of diminishing marginal productivity when the design points are spaced as they were in the experiment reported in this paper.

Thus, based on the theoretical variance functions and experimental results, the 3^3 factorial $(0,\pm 2)$ appears to be the best alternative to the 5^3 factorial design if the experimenter feels that the maximum may lie to the right of the anticipated maximum and in a relatively flat area of the surface. Based on predictive ability of the functions over the entire observed surface and estimated regression coefficients, the experimenter will do as well with the interlaced factorial design as with the 5^3 factorial design. As knowledge of the location of the maximum is improved, the experimenter may reduce the number of experimental points and use the simple composite design with no detrimental effects from the spacing of points.

Estimated maxima and economic optima for a specified price ratio differed widely from design to design indicating that if the maximum lies outside or near the edge of the experimental region, the investigator should use caution in evaluating these estimates.

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