SHADOW PRICES IN PMP AND CONSEQUENCES FOR CALIBRATION AND ESTIMATION OF PROGRAMMING MODELS

Thomas Heckelei
Institute for Agricultural Policy, Market Research, and Economic Sociology
University of Bonn, Germany; e-mail: heckelei@agp.uni-bonn.de

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1 Introduction

The main objective of this note is a review of PMP with a specific focus on the determination of dual values of resource constraints in this approach. Based on analytical derivations it is shown that the shadow prices implied by the first phase of PMP are not consistent with the assumption that the resulting calibrated model is the data generating process. Apart from creating reservations about the interpretability of the calculated shadow prices in general, this has more serious consequences for the estimation of parameters in the multiple observation context. An alternative approach simultaneously estimating parameters and shadow prices based on a complete set of optimality conditions is suggested based on Heckelei 2002 and Heckelei and Wolff 2003. A short outlook on more general applications of these bi-level estimation problems is given at the end.

2 Using dual values of calibration constraints: the PMP approach

As the word "positive" in PMP implies, the original motivation of PMP was to increase the reliability of a constrained optimisation model by using observed behaviour in the specification phase. Calibration constraints were introduced for observed levels of endogenous variables, and their dual values impacted on the specification of appropriate non-linear functions. The resulting model then reproduced these endogenous variables without the calibration constraints. The following paragraphs review that approach in detail.

The starting point of PMP is a profit maximising linear programming problem:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{Max}}\, Z = \mathbf{p}'\mathbf{x} - \mathbf{c}'\mathbf{x} \\ &\qquad \text{subject to} \\ &\mathbf{Ax} \le \mathbf{b} \qquad [\boldsymbol{\lambda}] \\ &\mathbf{x} \ge \mathbf{0} \end{aligned} \tag{1}$$

where

- Z = objective function value
- **p** = (N×1) vector of product prices
- **x** = (N×1) vector of production activity levels
- **c** = (N×1) vector of accounting cost per unit of activity
- **A** = (M×N) matrix of coefficients in resource constraints
- **b** = (M×1) vector of available resource quantities
- **λ** = (M×1) vector of dual variables associated with the resource constraints

The general idea of PMP is to use information contained in dual variables of calibration constraints, which bound the LP-problem to observed activity levels (Phase 1). These dual values are used to specify a non-linear objective function such that observed activity levels are reproduced by the optimal solution of the new programming problem without bounds (Phase 2).

Phase 1 of this procedure is formally described by extending model (1) in the following way:

$$\begin{aligned} &\max_{\mathbf{x}} Z = \mathbf{p}'\mathbf{x} - \mathbf{c}'\mathbf{x} \\ &\qquad \text{subject to} \\ &\mathbf{Ax} \le \mathbf{b} \qquad [\boldsymbol{\lambda}] \\ &\mathbf{x} \le (\mathbf{x}^{o} + e) \quad [\boldsymbol{\rho}] \\ &\mathbf{x} \ge [\mathbf{0}] \end{aligned} \tag{2}$$

where:

$\mathbf{x}^o$ = (N×1) vector of observed activity levels

$\boldsymbol{\varepsilon}$ = (N×1) vector of a small positive numbers

$\boldsymbol{\rho}$ = dual variables associated with the calibration constraints

The addition of the calibration constraints will force the optimal solution of the linear programming model (2) to exactly reproduce the observed base year activity levels $\mathbf{x}^o$, given that the specified resource constraints allow for this solution (which they should if the data are consistent, see Hazell and Norton 1986: 266f). 'Exactly' is accurately understood to mean within the range of the positive perturbations of the calibration constraints, $\boldsymbol{\varepsilon}$, which are included to guarantee that all binding resource constraints of model (1) remain binding here and thus avoid a degenerate dual solution.

We can partition the vector $\mathbf{x}$ into two subsets, an ((N-M)×1) vector of 'preferable' activities, $\mathbf{x}^p$, which are bounded by the calibration constraints, and a (M×1) vector of 'marginal' activities, $\mathbf{x}^m$, which are constrained solely by the resource constraints. To simplify notation, without loss of generality, we assume that all elements in $\mathbf{x}^o$ are nonzero and all resource constraints are binding. Then, the Kuhn-Tucker conditions imply that

$$\boldsymbol{\rho}^p = \mathbf{p}^p - \mathbf{c}^p - \mathbf{A}^{p}{}'\boldsymbol{\lambda} \tag{3}$$

$$\boldsymbol{\rho}^m = [\mathbf{0}] \tag{4}$$

$$\boldsymbol{\lambda} = \left(\mathbf{A}^{m}{}'\right)^{-1}\left(\mathbf{p}^m - \mathbf{c}^m\right) \tag{5}$$

where the superscripts p and m indicate subsets of original vectors and matrices corresponding to preferable and marginal activities, respectively. Whereas the dual values of the calibration constraints are zero for marginal activities ($\boldsymbol{\rho}^m$) as shown in (4), they are equal to the difference of price and marginal cost for preferable activities ($\boldsymbol{\rho}^p$) as seen in (5), latter being the sum of variable cost per activity unit ($\mathbf{c}$) and the marginal cost of using fixed resources ($\mathbf{A}^{p\prime}\boldsymbol{\lambda}$). It should be noted here, that *the dual values of the resource constraints ($\boldsymbol{\lambda}$) only depend on objective function entries and coefficients of marginal activities*.

In *Phase 2* of the procedure, the dual values of the calibration constraints $\boldsymbol{\rho}^p$ are employed to specify a non-linear objective function such that the marginal cost of the preferable activities are equal to their respective prices at the base year activity levels $\mathbf{x}^o$. Given that the implied variable cost function has the right curvature properties (convex in activity levels) the solution to the resulting programming problem will be a 'boundary point, which is the combination of binding constraints and first order conditions' (Howitt 1995a: 330).

Howitt (1995a) and Paris and Howitt (1998) interpret the dual variable vector $\boldsymbol{\rho}$ associated with the calibration constraints as capturing any type of model mis-specification, data errors, aggregation bias[1], risk behaviour and price expectations.

In principle, any type of non-linear function with the required properties qualifies for Phase 2. For reasons of computational simplicity and lacking strong arguments for other type of functions, a quadratic cost function is often employed (exceptions: Paris and Howitt 1998). The general version of this variable cost function to be specified is then

$$C^v = \mathbf{d}'\mathbf{x} + \frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} \tag{6}$$

with:

$\mathbf{d}$ = (N×1) vector of parameters associated with the linear term and

$\mathbf{Q}$ = (N×N) symmetric, positive (semi-) definite matrix of parameters associated with the quadratic term.

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[1] To deal with aggregation errors in regional or sector modelling, ÖNAL and MCCARL (1991) provide the theoretical basis of an exact aggregation procedure based on extreme point representation under the assumption of full information on every farm and suggest empirical approximation procedures using the available aggregate information on all farms.

The parameters are then specified such that the linear 'marginal variable cost' ($\mathbf{MC}^V$) functions fulfil

$$\mathbf{MC}^V = \frac{\partial C^v(x^o)}{\partial x} = \mathbf{d} + \mathbf{Q}\mathbf{x}^o = \mathbf{c} + \boldsymbol{\rho}\,. \tag{7}$$

Note, however, that the derivatives (7) of this *variable* cost function do not incorporate the opportunity cost of fixed resources ($\mathbf{A}^{p\prime}\boldsymbol{\lambda}$) which remain captured in the ultimate model by the dual values of the resource constraints.

Given that we have a set of parameters satisfying (7), we obtain the final non-linear programming problem that reproduces observed activity levels as

$$\begin{aligned} &\max_{\mathbf{x}} Z = \mathbf{p}'\mathbf{x} - \mathbf{d}'\mathbf{x} - \frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} \\ &\text{subject to} \\ &\mathbf{A}\mathbf{x} \le \mathbf{b} \qquad [\boldsymbol{\lambda}] \\ &\mathbf{x} \ge \mathbf{0} \end{aligned} \tag{8}$$

The dual values of the resource constraints in model (8) at $\mathbf{x}^o$ do not differ from the one in model (2). They are still determined by the marginal profitability of the marginal activities at their observed levels $\mathbf{x}^{om}$, $(\mathbf{A}^{m\prime})^{-1}$ $[\mathbf{p}^m - (\mathbf{d}^m + \mathbf{x}^{om}\mathbf{q}^m)]$, which remains equal to $(\mathbf{A}^{m\prime})^{-1}[\mathbf{p}^m\text{-}\mathbf{c}^m]$ in the specification step, because of (4) and (7). Consequently, the value of equation (5) remains unchanged. Apart from problematic consequences for the estimation of cost function parameters under certain assumption on the data generating process discussed below, these shadow values are unlikely to represent realistic land values in the context of agricultural programming models. They would only be valid if a marginal reduction of the land resource would lead to a reduction in marginal activities only. This however, is rarely realistic, as rotational effects will lead to a reduction also in preferable activities. More technically speaking, land values determined by marginal activities only neglect multi-output-multi-input technology effects.

3 Dual values and supply response of the quadratic model

The problem of condition (7) is that it implies an underdetermined specification problem as long as we consider a flexible functional form. In the case of the second order flexible quadratic function we have N+N(N+1)/2 parameters which we try to specify on the basis of N pieces of information (the marginal variable cost equations (7)). There are an infinite number of parameter sets which satisfy these conditions, i.e. lead to a perfectly calibrating model, but each set implies a different response behaviour to changing economic incentives.

In order to see the consequences of an arbitrary – apart from satisfying (7) – specification of non-linear cost terms, we derive the supply functions implied by the PMP calibrated model (8). If we start from the Lagrangian formulation

$$L(\mathbf{x}) = \mathbf{p}'\mathbf{x} - \mathbf{d}'\mathbf{x} - 0.5\mathbf{x}'\mathbf{Q}\mathbf{x} + \lambda[\mathbf{b} - \mathbf{A}\mathbf{x}] \tag{9}$$

and continue to assume that all optimal activity levels are positive we obtain the first order conditions in gradient format as

$$\frac{\partial \mathbf{L}}{\partial \mathbf{x}} = \mathbf{p} - \mathbf{d} - \mathbf{Q}\mathbf{x} - \mathbf{A}'\boldsymbol{\lambda} = \mathbf{0} \tag{10}$$

and

$$\frac{\partial \mathbf{L}}{\partial \boldsymbol{\lambda}} = \mathbf{b} - \mathbf{A}\mathbf{x} = \mathbf{0}\,. \tag{11}$$

Solving (10) for $\mathbf{x}$ results in

$$\mathbf{x} = \mathbf{Q}^{-1}\left(\mathbf{p} - \mathbf{d} - \mathbf{A}'\boldsymbol{\lambda}\right) \tag{12}$$

and substituting the right hand side of (12) into (11) allows to solve for

$$\boldsymbol{\lambda} = \left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\left(\mathbf{A}\mathbf{Q}^{-1}\left(\mathbf{p} - \mathbf{d}\right) - \mathbf{b}\right). \tag{13}$$

The vector of optimal activity levels as a function of exogenous model parameter can then be expressed as

$$\mathbf{x} = \mathbf{Q}^{-1}\left(\mathbf{p} - \mathbf{d}\right) - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\left(\mathbf{A}\mathbf{Q}^{-1}\left(\mathbf{p} - \mathbf{d}\right) - \mathbf{b}\right). \tag{14}$$

The gradient of (14) with respect to the price vector is proportional to the marginal supply response in this case (since product supply is constant per activity unit) and given by

$$\frac{\partial \mathbf{x}}{\partial \mathbf{p}} = \mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}'\left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}'\right)^{-1}\mathbf{A}\mathbf{Q}^{-1} \tag{15}$$

which finally reveals that the full **Q**-matrix is relevant for the supply response of each single product. This is even true when **Q** is diagonal (and consequently $\mathbf{Q}^{-1}$ as well), because the fixed allocable inputs (resource constraints) still link all production activities with each other. The second summand in (15) which is $-\mathbf{Q}^{-1}\mathbf{A}'$ times the gradient of $\boldsymbol{\lambda}$ with respect to **p** ensures that all elements of $\mathbf{Q}^{-1}$ enter each element of the supply gradient.

The different methods developed to choose among the infinite number of calibrating parameter sets increasingly recognised the need to introduce additional information in order to avoid arbitrary simulation behaviour. For an overview on different methods used see Umstätter (1999), Röhm (2001) or Heckelei and Britz (2005).

4 Calibration and estimation of optimization models without dual values of calibration constraints

The last section hinted at the danger of specifying models based on PMP that imply arbitrary simulation behaviour. One problem is the thin information base provided by just one year of observations on activity levels. In fact, the data in this case do not provide any information on second order properties (Hessian matrix) of the objective function. If a change in economic incentives and the resulting behaviour is not observed, then the information for parameter specification must come from other sources. Even if one would be able to specify the 'true' model with respect to behavioural assumptions and functional form, the parameters are still not identified. The only convincing use of PMP with just one observation is the use as a calibration method in combination with elasticities or other exogenous information on technology or behavioural response with respect to changes in activity levels.

The main focus of this section, however, shall be the inclusion of additional data looking for the bridge to typical econometric models. The question we need to address first is, whether the PMP procedure itself is designed to make best use of additional data information. We show that the marginal conditions derived from the first phase of PMP are inappropriate. They represent a mis-specified model in the sense that the inclusion of additional observations will never allow to recover the underlying model which is assumed to have generated the data.

In order to see this, we will use some of the elements already introduced in the previous sections, but look at the methodology from an econometrician's point of view. This includes the assumption that the ultimate model to be specified is the 'true' model structure, or at least one that is believed to be a good approximation of the true model: Apparently, many PMP modellers thought that the final model with a non-linear objective function to be optimised under linear resource constraints is a reasonable representation of the behaviour of agricultural producers, otherwise it would not have made any sense to use this structure as the ultimate specification. The PMP procedure, however, enforces shadow prices and marginal cost values that differ from the ones implied by the non-linear model.

Suppose the quadratic model (8) is the true data generating process. The derivations (9) to (13) have shown that the shadow prices of the resource constraints under the assumption that all activity levels are positive at the optimum can be calculated as $\boldsymbol{\lambda} = \left(\mathbf{AQ}^{-1}\mathbf{A}'\right)^{-1}\left(\mathbf{AQ}^{-1}\left(\mathbf{p}-\mathbf{d}\right)-\mathbf{b}\right)$. This is clearly different from the dual values of the resource constraints obtained in the first phase of PMP (see equation (5)) which only depend on quantities related to the marginal activities and were given by $\boldsymbol{\lambda} = \left(\mathbf{A}^{m}{}'\right)^{-1}\left(\mathbf{p}^{m}-\mathbf{c}^{m}\right)$. The second phase of PMP then uses these dual values at the observed activity levels through enforcement of the 'marginal cost' equations (7), thereby implicitly imposing wrong values for the marginal variable cost as well. Given this discrepancy, it is impossible to recover the true non-linear objective function no matter how many observations on activity levels are used. The use of the *biased* marginal cost equations as estimating equations in some econometric exercise with multiple observations generally leads to inconsistent estimates. The PMP approach *in this context* is fundamentally flawed in the sense that it imposes first order conditions which are incompatible with the non-linear model it ultimately tries to recover.

A remedy is to directly use the first order conditions of the quadratic programming problem (10) and (11) for parameter specification allowing simultaneously estimating parameters and shadow prices of limiting resources. After adding error terms, the first order conditions serve as estimating equations which are fitted by some econometric criterion to the observations on activity levels. A general formulation for the quadratic model is given by

$$\underset{\mathbf{d},\mathbf{Q}}{\text{Min}}\ \text{H}\left(\mathbf{e}\right) \text{ subject to} \tag{16}$$

$$\mathbf{p}-\mathbf{d}-\mathbf{Q}\left(\mathbf{x}^{o}-\mathbf{e}\right)-\mathbf{A}'\boldsymbol{\lambda}=\mathbf{0} \tag{17}$$

$$\mathbf{b}-\mathbf{A}\left(\mathbf{x}^{o}-\mathbf{e}\right)=\mathbf{0}. \tag{18}$$

The objective function H(**e**) could be for example generalized least squares or maximum entropy being minimized by adjusting error terms and parameter values. The shadow prices are direct implicit functions of the parameters as can be seen by equation (13) above. Although not illustrated here, this approach can also accommodate observations (and error terms) on shadow prices of resource constraints to make best use of all available information (see).
In the PMP approach, equation (17) – without error terms – also basically ensures the calibration of the model solution to one base year observation on activity levels.[2] The fundamental difference is that the shadow values $\boldsymbol{\lambda}$ are set *a-priori* in the first phase of PMP and are not determined simultaneously with the parameters.

The estimation problem characterized by equations (16)-(18) can be seen as a special case of bi-level programs designed to estimate parameters of (non-) linear programming models. We can distinguish between an "outer" optimization problem, the econometric criterion, and the "inner" problem, the optimality conditions of the economic model. The outer problem can only choose parameters and fitted values that constitute an optimal solution to the inner problem. Heckelei and Wolff (2003) go beyond the typical quadratic PMP model and use the same type of approach to estimate parameters of crop-specific production and profit functions in the context of an explicit land allocation model. They also show for a small illustrative model, that complementarity conditions, i.e. the potential non-binding status of inequality restrictions, can also be accommodated with this approach. For a more realistic model, Jansson and Heckelei (2004) estimate transport cost, prices and trade flows in the context of a transport cost minimisation problem, where observations on prices and transport costs are available. Trade flows, however, are not. The estimation not only recovers a set of transport cost and prices consistent with cost minimisation, it also provides corresponding estimates of trade flows satisfying regional market balances. Whether a trade flow is positive or zero cannot be determined a-priori.

One drawback of these bi-level problems is of numerical nature. There are different approaches available for solving them, all of which have in common that they do not work equally well for all types of problems in this class. The difficulties in numerically finding a solution are especially severe

[2] For application of this calibration approach without the first phase of PMP see JUDEZ et al. 2001; BUYSSE et al. 2004, HENRY DE FRAHAN et al. 2005

for gradient solvers if the optimality conditions of the inner problems comprise complementary slackness conditions, because gradients of these restrictions are not continuously differentiable. To mitigate this problem somewhat, one could decide a-priori, for example by data inspections, which resource and non-negativity constraints are binding and which are not. For the former, we can then formulate equality restrictions with a nonzero shadow price and for the latter we can simply leave out the restrictions as they do not matter for the data generating process. However, this approach is not always applicable, as we often do not know whether constraints are binding or not, because of lacking data or variables measured with errors.

Consequently, bi-level estimation problems will often require the development of case specific solution algorithms or at least the necessity to "play" with parameters of existing solvers for this type of problems. This will probably limit for a while the use of this general methodology for estimating complex programming models to the methodologically interested analyst. This and constraints on data availability will most likely lead to continuing use of more simple methods of programming model calibration, hopefully avoiding arbitrary specification of non-linear parameters and shadow prices as in early PMP approaches. However, Jansson (2005) shows that the investment in the direction of programming model estimation may pay off under certain conditions. In this case, the estimator's properties in the context of a transportation models are superior to the performance of previous methods calibrating these models.

5 Conclusions

Using non-linear terms in the objective function, the original PMP approach calibrated programming models to observed base year activity levels and guaranteed a smooth supply response behaviour relative to previously used linear programming models. This feature was especially important for aggregate programming models at regional or sectoral level. Later, the issues of supply response behaviour of the resulting model and the determination of the shadow prices of limiting resources based on dual values of calibration constraint in the PMP approach came to the centre of attention.

In the context of using multiple observations, it could be shown that phase 1 of the original PMP approach with calibration constraints leads to inconsistent parameter estimates. Fortunately, there exists a conceptually simple alternative to use first order conditions of the programming problem as estimating or calibrating equations directly. This approach allows simultaneously estimating parameters of the cost function with dual values of resource constraints while treating the decision variables as stochastic. It is certainly a straightforward idea to estimate parameters in the framework later used for simulation, and this has been successfully applied in many standard applications of profit, cost, or indirect utility functions. However, the combination of, for example, dual profit functions and a model with an explicit primal representation of parts of the technology in form of inequalities leads to a more challenging model class. Such a situation defines a so-called bi-level program, where an outer objective function is optimized under constraints representing first order conditions of an inner optimization problem. These approaches require specific algorithms to be solved for large scale models. It is therefore not astonishing that so far, few consistent parameter estimations in a primal/dual framework are documented for real world application. Indeed, application to larger models just started and the development of algorithm suitable for bi-level problems for agricultural economic model promises to be a fruitful exercise.

Generally, we conclude that there is no need for the PMP approach with calibration constraints in phase 1 anymore. It has served its purpose of providing shadow prices of resource constraints as long as the method and its implication for supply response behaviour had not been fully understood. Now, calibration and estimation of programming models proceeds and should proceed using simultaneously explicit prior or data information on shadow prices and decision variables.

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