PMP, EXTENSIONS AND ALTERNATIVE METHODS: INTRODUCTORY REVIEW OF THE STATE OF THE ART

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Positive Mathematical Programming (PMP) as formalised in Howitt (1995a) to calibrate exactly constrained optimisation models has renewed the interest in Mathematical Programming (MP) modelling for analysing agricultural and environmental policies. It has generated numerous applications and extensions (e.g., Paris and Howitt, 1998; Paris, 2001) but also criticisms and alternative methods (Britz *et al*., 2003; Heckelei and Wolff, 2003). Issues of exact calibration and aggregation have also been addressed differently with a procedure based on extreme point representation (Önal and McCarl, 1989 and 1991).

This organised session aims to compare alternative methods of exact calibration in MP modelling, update on development in these methods and discuss some recent applications. After a short introduction by Henry de Frahan who recently reviewed these methods (Henry de Frahan *et al.,* 2005), this session assembles contributions of three key protagonists in the development of these methods. First, Heckelei explains the inconsistency in parameter estimation for the case of multiple observations if we rely on the first PMP calibration step, presents an alternative method to PMP for avoiding this problem and assesses merits and problems with this alternative method. Second, Howitt responds to the Heckelei and Wolff's (2003) critique, shows the benefits of relying on PMP-GME based micro flexible production function models for policy analysis and provides measures of information gain from the disaggregation of production models. Third, Schneider explains the Önal and McCarl approach as an alternative for exact aggregation and calibration and illustrates its use in the context of the FAOSM model. These three contributions are followed by a round table discussion.

The renewed interest in MP in the last fifteen years to model economic behaviour and, hence, help analyse agricultural and environmental policy originates from a combination of factors among which PMP plays a distinctive role. First, the emergence in the late 1980's of the positive mathematical programming (PMP) has brought an appealing breath of positivism in the determination of the optimising function parameters in replacement of various unsatisfactory *ad hoc* calibration techniques. This method formalised later by Howitt (1995a) makes it indeed possible to calibrate MP models exactly exploiting the observed behaviours of economic agents at either the disaggregated or aggregated level. Second, as a result of the former, PMP has provided a more flexible and realistic simulation behaviour of MP models avoiding unlikely abrupt discontinuities in the simulation solutions. Third, the increasing need to model and simulate behavioural functions under numerous technical, economic, policy and, more recently, environmental conditions has strengthened the recourse to MP models. Fourth, in an environment of often-limited amount of adequate information and data to treat complex decisions, MP models are nevertheless able to handle decision problems which econometrics cannot. This renewed interest in MP modelling for analysing agricultural and environmental policies has generated numerous applications as well as extensions at different investigation levels of which several are reported in Heckelei and Britz (2005).

Alternatively to PMP, another but less popular approach to calibrate programming models has been proposed exploiting the Dantzig-Wolfe (1961) decomposition. To correct aggregation errors so common in regional and sector models Önal and McCarl (1991) provide an aggregation procedure that has the potential to be exact under full information on all disaggregated entities irrespective to their heterogeneity. Like PMP, this aggregation procedure also is positive in the sense that its empirical applications exploit the observed behaviours of economic agents.

This introduction briefly reviews the PMP and the exact aggregation procedures, outlines their shortcomings and extensions and guides the participants to this organised session to ongoing work to correct these shortcomings.

1. The Standard PMP Procedure

PMP is a method to calibrate mathematical programming models to observed behaviours during a reference period by using the information provided by the dual variables of the calibration constraints (Howitt, 1995a; Paris and Howitt, 1998). The dual information is used to calibrate a non-linear objective function such that the observed activity levels are reproduced for the reference period but without the calibration constraints. The term "positive" that qualifies this method implies that, like in econometrics, the parameters of the non-linear objective function are derived from an economic behaviour assumed to be rational given all the observed and non-observed conditions that generates the observed activity levels. The main difference with econometrics is that PMP does not require a series of observations to reveal the economic behaviour, which as a drawback deprives PMP from inference and validation tests.

Formalised by Howitt (1995a), PMP follows a procedure in three steps. The first step consists in writing a MP model as usual but adding to the set of limiting resource constraints a set of calibration constraints that bound the activities to the observed levels of the reference period. Taking the case of maximising gross margins with upper bounded calibration constraints, we write the initial model as in Paris and Howitt (1998):

$$\text{Maximise } Z = \mathbf{p}'\,\mathbf{x} - \mathbf{c}'\,\mathbf{x} \tag{1}$$

$$\text{subject to: } \mathbf{A}\,\mathbf{x} \leq \mathbf{b} \qquad [\boldsymbol{\lambda}] \tag{1a}$$

$$\mathbf{x} \leq \mathbf{x_o} + \boldsymbol{\varepsilon} \qquad [\boldsymbol{\rho}] \tag{1b}$$

$$\mathbf{x} \geq 0 \tag{1c}$$

where:

- Z scalar of the objective function value,
- $\mathbf{p}$ (n x 1) vector of product prices,
- $\mathbf{x}$ (n x 1) non-negative vector of production activity levels,
- $\mathbf{c}$ (n x 1) vector of accounting costs per unit of activity,
- $\mathbf{A}$ (m x n) matrix of coefficients in resource constraints,
- $\mathbf{b}$ (m x 1) vector of available resource levels,
- $\mathbf{x_o}$ (n x 1) non-negative vector of observed activity levels,
- $\boldsymbol{\varepsilon}$ (n x 1) vector of small positive numbers for preventing linear dependency between the structural constraints (1a) and the calibration constraints (1b),
- $\boldsymbol{\lambda}$ (m x 1) vector of duals associated with the allocable resource constraints,
- $\boldsymbol{\rho}$ (n x 1) vector of duals associated with the calibration constraints.

Howitt (1995a) and Paris and Howitt (1998) interpret the dual variable vector $\boldsymbol{\rho}$ associated with the calibration constraints as capturing any type of model mis-specification, data errors, aggregate bias, risk behaviour and price expectations. In the perspective of calibrating a non-linear decreasing yield function as in Howitt (1995a), this dual vector $\boldsymbol{\rho}$ represents the difference between the activity average and marginal value products. In the alternative perspective of calibrating a non-linear increasing cost function as in Paris and Howitt (1998), this dual vector $\boldsymbol{\rho}$ is interpreted as a differential marginal cost vector that together with the activity accounting cost vector $\mathbf{c}$ reveals the actual variable marginal cost of supplying the observed activity vector $\mathbf{x_o}$.

To account for greater competitiveness among closed competitive activities that can be viewed as variant activities from a generic activity, Rohm and Dabbert (2003) add within this first step calibration constraints for these variant activities that are less restrictive than the calibration constraints for the generic activities.

The second step of PMP consists in using these duals to calibrate the parameters of the non-linear objective function. A usual case considers calibrating the parameters of a variable cost function C^v

that has the typical multi-output quadratic functional form, however, holding constant variable input prices at the observed market level as follows:[1]

$$C^v(\mathbf{x}) = \mathbf{d}'\,\mathbf{x} + \mathbf{x}'\,\mathbf{Q}\,\mathbf{x} / 2 \qquad (2)$$

where:
d (n x 1) vector of parameters of the cost function,
Q (n x n) symmetric, positive (semi-) definite matrix with typical element $q_{ii'}$ for activities i and i'.

The variable marginal cost vector $\mathbf{MC^v}$ of this typical cost function is set equal to the sum of the accounting cost vector **c** and the differential marginal cost vector $\boldsymbol{\rho}$ as follows:

$$\mathbf{MC^v} = \nabla C^v(\mathbf{x})_{\mathbf{xo}}' = \mathbf{d} + \mathbf{Q}\,\mathbf{x_o} = \mathbf{c} + \boldsymbol{\rho} \qquad (3)$$

where $\nabla C^v(\mathbf{x})$ is a (1 x n) gradient vector of first derivatives of $C^v(\mathbf{x})$ for $\mathbf{x} = \mathbf{x_o}$.

To solve this system of *n* equations for $[n + n(n + 1)/2]$ parameters and, thus, overcome the under-determination of the system, PMP modellers rely on various solutions. An earlier *ad hoc* solution consists in assuming that the symmetric matrix **Q** is diagonal, implying that the change in the actual marginal cost of activity *i* with respect to the level of activity *i'* ($i \neq i'$) is null and, then, in relying on additional assumptions. Among them, the average cost approach equates the accounting cost vector **c** to the average cost vector of the quadratic cost function, which leads to:

$$q_{ii} = 2\,\rho_i / x_{io} \text{ and } d_i = c_i - \rho_i \text{ for all } i = 1, \ldots n. \qquad (4)$$

In this case, however, the variable marginal costs of these so-called marginal activities that are constrained by the allocable constraints (1a) but not by the calibration constraints (1b) are independent of their levels while those of the so-called preferable activities that are constrained by the calibration constraints are. As a result, an exogenous shock on one preferable activity would uniquely modify the levels of this activity and the level of the marginal activities, not those of the other preferable activities (Gohin and Chantreuil 1999). One *ad hoc* solution to obtain an increasing marginal cost function for these marginal activities consists in retrieving some share of one limiting resource dual value λ and adding it to the calibration dual vector $\boldsymbol{\rho}$ to obtain a modified calibration dual vector $\boldsymbol{\rho}_{\mathbf{M}}$ (Rohm and Dabbert 2003).

Exogenous supply elasticities ε_{ii} are also used to derive the parameters of the quadratic cost function as in Helming *et al*. (2001):

$$q_{ii} = p_{io} / \varepsilon_{ii}\, x_{io} \text{ and } d_i = c_i + \rho_i - q_{ii}\, x_{io} \text{ for all } i = 1, \ldots n. \qquad (5)$$

All these *ad hoc* specifications would exactly calibrate the initial model as long as equations (2) are verified, but lead to different simulation responses to external changes.

A more severe solution consists in skipping the first step of PMP altogether. Judez *et al.* (2001) directly derive the unknown parameters of the final non-linear model from the Kuhn-Tucker conditions of such final model considering exclusively the activities whose observed levels are different from zero and the opportunity costs of the limiting resources as given exogenously to the model. To the extent that the opportunity costs of the limiting resources are lower than the dual values $\boldsymbol{\lambda}$ obtained from the initial linear programme, all activities are described with an increasing marginal cost. They use this approach to represent the economic behaviours of different farm types

[1] Other functional forms are possible. The generalized Leontief and the weighted-entropy variable cost function (Paris and Howitt, 1998) and the constant elasticity of substitution (CES) production function (Howitt, 1995b) in addition to the constant elasticity of transformation production function (Graindorge *et al*., 2001) have also been used. A von Neumann-Morgenstern expected utility approach has been used to account for a constant absolute risk aversion to price volatility (Paris 1997).

based on farm accounting data from the Spanish part of the European Farm Accountancy Data Network (FADN).

A subsequent development from Paris and Howitt (1998) to calibrate the marginal cost function is to exploit the maximum entropy estimator to determine all the $[n + n(n + 1)/2]$ elements of the vector **d** and matrix **Q** using the Cholesky factorisation of this matrix **Q** to guarantee that the calibrated matrix **Q** is actually symmetric positive semi-definite.[2] This estimator in combination with PMP enables to calibrate a quadratic variable cost function accommodating complementarity and competitiveness among activities still based on a single observation but using *a priori* information on support bounds. Nevertheless, as argued in Heckelei and Britz (2000), the simulation behaviours of the resulting calibrated model would be still arbitrary because heavily dominated by the supports.

The third step of PMP uses the calibrated non-linear objective function in a non-linear programming problem similar to the original one except for the calibration constraints. This calibrated non-linear model is consistent with the choice of the non-linear activity yield or cost function derived in the preceding step and exactly reproduces observed activity levels and original duals of the limiting resource constraints. The following PMP model is ready for simulation.

$$\text{Maximise } Z = \mathbf{p}'\,\mathbf{x} - \hat{\mathbf{d}}'\,\mathbf{x} - \mathbf{x}'\,\hat{\mathbf{Q}}\,\mathbf{x} / 2 \tag{6}$$

$$\text{subject to: } \mathbf{A}\,\mathbf{x} \leq \mathbf{b} \qquad [\boldsymbol{\lambda}] \tag{6a}$$

$$\mathbf{x} \geq \mathbf{0} \tag{6b}$$

where the vector $\hat{\mathbf{d}}$ and matrix $\hat{\mathbf{Q}}$ are the calibrated parameters of the non-linear objective function.

This calibration approach can be applied at the farm, regional and sector levels. When accounting data of a sample of *F* farms are available such as from the FADN, *F* PMP models can be defined for each farm of the sample. Simulations can then be performed on these individual PMP models and simulation results may be aggregated as in Buysse *et al.* (2004) and Henry de Frahan *et al.* (2005).

To represent the economic behaviours with regard to activities of farms whose initial observed supply level is zero during the reference period, Paris and Arfini (2000) add to the *F* PMP models a supplementary PMP model for the whole farm sample and calibrate a frontier cost function for all the activities included in the whole farm sample. Such solution to the self-selection problem provides a representation of economic behaviours even with regard to those activities of farms whose initial observed supply level is zero.

2. The Exact Aggregation Procedure

Alternatively to PMP, another but less popular approach to calibrate programming models has been proposed exploiting the Dantzig-Wolfe (1961) decomposition. To correct aggregation errors so common in regional and sector models Önal and McCarl (1991) provide an aggregation procedure that has the potential to be exact under full information on all disaggregated entities irrespective to their heterogeneity. Like PMP, this aggregation procedure also is positive in the sense that its empirical applications exploit the observed behaviours of economic agents.

The aggregation procedure advocated by McCarl (1982) and Önal and McCarl (1989 and 1991) that can be also used to calibrate programming models rests on the Dantzig-Wolfe (1961) decomposition. According to this decomposition any feasible solution of the production possibility set, i.e., the bounded set defined by the resource constraints, can be expressed as a convex combination of the extreme points.

By exploiting the extreme point representation of a linear system, the problem:

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[2] In short, the maximum entropy approach consists in estimating parameters regarded as expected values of associated probability distributions defined over a set of a priori discrete supports (Golan *et al*., 1996).

$$\text{Maximise } Z = \mathbf{p}'\,\mathbf{x} - \mathbf{c}'\,\mathbf{x} \tag{7}$$
$$\text{subject to: } \mathbf{A}\,\mathbf{x} \leq \mathbf{b}$$
$$x \geq 0$$

using the same notation as in model (1), can be equivalently stated as (Önal and McCarl 1989):

$$\text{Maximise } Z = \mathbf{p}'\,\mathbf{x} - \mathbf{c}'\,\mathbf{x} \tag{8}$$
$$\text{subject to: } \boldsymbol{\phi}\,\hat{\mathbf{X}} = \mathbf{x}$$
$$\sum_{\mathbf{i}}^{\mathbf{w}} \boldsymbol{\phi}_{\mathbf{i}} = 1$$
$$\mathbf{x} \geq \mathbf{0} \text{ and } \phi_i \geq 0 \text{ for all i}$$

where the (w x 1) vector contains the convex combination weights ϕ_i and the (n x w) matrix $\hat{\mathbf{X}}$ contains the extreme points of the linear system of the initial model constraints. Schneider will elaborate further on this exact aggregation procedure in his presentation.

3. PMP Shortcomings and Alternatives

The under-estimation of the marginal cost system and the consequently ambiguous treatment of the marginal versus the preferable activities are shortcomings of the PMP that have been already reported above. Other shortcomings are reviewed in the following.

To overcome other criticisms that have been raised against the use of a linear technology in limiting resources and the zero-marginal product for one of the calibrating constraints, Paris (2001) and Paris and Howitt (2001) generalise the PMP framework into a Symmetric Positive Equilibrium Problem (SPEP) and extend it to a full sample of farms sharing the same technology. These authors express the first step of this new structure as an equilibrium problem consisting of symmetric primal and dual constraints and the third step as an equilibrium problem between demand and supply functions of inputs, on the one hand, and between marginal cost and marginal revenue of the output activities, on the other hand. For these authors, the key novelty of this new framework is rendering the availability of limiting inputs responsive to output levels and input price changes. Britz *et al.* (2003), however, address several conceptual concerns with respect to the SPEP methodology and question the economic interpretation of the final model ready for simulations.

As Heckelei will explain in his presentation, even if we rely on multiple cross-sectional or chronological observations to overcome the problem of under-determination of the marginal cost system, then we face a fundamental inconsistency between the specification of the parameters of this system and the resulting quadratic optimisation model. His argument goes as follows (Heckelei and Wolff, 2003). On the one hand, the shadow values $\boldsymbol{\lambda}$ of the limiting resources implied by the ultimate model (6) are determined by the vectors **p**, **d** and **b** and the matrices **A** and **Q** of all the activities through its first-order condition. On the other hand, the various sets of shadow values $\boldsymbol{\lambda}$ of the limiting resources from the sample initial models (1) are solely determined by the vectors **p** and **c** and matrix **A** of only those marginal activities bounded by the resource constraints through their first-order conditions. As a result, the various sets of shadow values of the initial models are most generally different from the shadow values of the ultimate model. Since the first step simultaneously sets both the initial dual vectors $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ and the second step uses the initial dual vector $\boldsymbol{\rho}$ to estimate the vector $\mathbf{MC}^{v}$, this latter vector must generally be inconsistent with the ultimate model (6). The derived marginal conditions (3) are, therefore, most likely to be biased estimating equations yielding inconsistent parameter estimates.[3] This inconsistency makes PMP not well suited to the estimation of programming models that rely on multiple cross-sectional or chronological observations. Howitt will respond to that criticism in his presentation.

[3] In other words, the 'estimated' values of the shadow values λ cannot converge to the true shadow values λ as more observations are added because PMP always selects the highest possible values for shadow values λ.

To avoid inconsistency between steps 1 and 3 as further exposed in Heckelei and Britz (2005), Heckelei and Wolff (2003) suggest to skip the first step altogether and employ directly the optimality conditions of the desired programming model to estimate, not calibrate, simultaneously the shadow values of the limiting resources and the parameters of the marginal cost system. They illustrate this general alternative to the original PMP through three examples relying on the Generalised Maximum Entropy (GME) procedure for estimating the model parameters. Their examples deal with the estimation of the parameters of various optimisation models that (1) incorporate a quadratic cost function and only one constraint on land availability, (2) allocate variable and fixed inputs to production activities represented by activity-specific production functions or (3) allocate fixed inputs to production activities represented by activity-specific profit functions.

As stated by their authors, this alternative approach to PMP has some theoretical advantage over the original PMP for the estimation of programming models. It also has some empirical advantage over standard econometric procedures of duality-based behavioural functions for the estimation of more complex models characterized by more flexible functional forms and more constraints as well as the incorporation of additional constraints relevant for simulation purpose.

To exploit fully the richness of the farm-level data from the standard FADN, Polomé *et al.* (2005) rely on a panel data estimation to estimate the parameters of a marginal cost and input demand system avoiding the calibration step of PMP. The cost function has a quadratic form similar to the Paris and Howitt's (2001) quadratic-Leontieff cost function but modified to fully satisfy the regularity properties of a cost function. The estimation of the marginal cost and input demand system derived from this cost function is restricted to a sample of homogeneous crop farms of the Belgian part of the FADN that hypothetically share the same technology. This sample restriction allows limiting the number of parameters to estimate while still maintaining the farm fixed effect of the panel data model. Following Heckelei and Wolff (2003), the estimation of this system directly uses the optimality conditions of the desired programming model equating the marginal cost system to the observed output prices and the input demand system to the observed input uses. Restrictions to account for theoretical regularity properties but also for some specific policy aspects or resource constraints can be incorporated into the estimation. Once estimated, the marginal cost and input demand system is ready to be used in a profit maximisation program to simulate the farmers' economic behaviours when some exogenous conditions change. Results can then be aggregated and analysed according to farm type, size or location.

4. Preliminary Conclusions

PMP has renewed the interest in mathematical modelling for agricultural and environmental policies for several reasons. The main advantages of the PMP approach are the simplicity of the modelling of bio-economic constraints or policy instruments, the smoothness of the model responses to policy changes and the possibility to make use of very few data to simulate agricultural and environmental policies.

PMP is a method that has, however, been developed for situations in which the researcher has either very few information or faces a situation with a high heterogeneity in farms, but is willing to impose strong hypotheses on the functional form of the cost function. Without additional data, there is probably little improvement that can be achieved. As large samples such as the FADN become available, it becomes more and more useful to extend PMP and to prefer econometric estimation approaches to calibration approaches, as they are less demanding in terms of hypotheses and more robust.

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