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The Competitive Firm Under Uncertainty: The Case of Two Random Variables

by

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INTRODUCTION

Deterministic models of the theory of the firm are not adequate in describing investment behavior or factor demand under conditions of uncertainty and risk aversion. Recent attempts to construct models that take into consideration uncertainties and decision maker's attitudes toward risk have identified output price as a random variable and then sought to show how factor demand and output are affected by the random nature of the variable under the assumption of risk aversion. The level of output and factor demand for the risk averse case is then compared to the risk neutral case. In the risk neutral case random variables are replaced by their expected value. Thus, neutrality to risk is equivalent to certainty if one views the certain value of the variable as being its expected value. Nevertheless, it should kept in mind that whenever a variable quantity is introduced as random then the decision maker is in the realm of uncertainty about the exact value of that variable irrespective of whether he is risk neutral or risk averse. The purpose of this paper is to extend the recent work by introducing a random component into the production relationship simultaneously with output price as random variables. This model is altered in the last section to an n period model with both input and output prices random while assuming the production relationship to be non-random.

The approach taken in this paper is similar to that taken by Baron (1970) and to a lesser extent that taken by Sandmo (1971) and Batra and Ullah (1974). Baron, utilizing a cost function approach, showed that in the case where output price is defined as a random variable and under the assumption that the firm is risk averse, then the optimal level of output occurs such that the marginal cost is less than the expected price. Under similar assumptions, Sandmo showed the same result. Using a where w is the price of the input L and both are known with certainty. There are no fixed costs in this model. The production function $q(L)$ for a single output is an increasing concave function with continuous first and second derivatives, i.e., $q'(L) > 0$ and $q''(L) \leq 0$, that is deterministic in nature for the single input. The variable α, independent of the variable p and taking values $0 \leq \alpha \leq 1$ with density function $g(\alpha)$, is introduced to represent the random nature of production. No storage or inventory will be permitted so as to not deter from the generality of the model.

Let the firm be controlled by a single decision maker whose utility for profits is identified by an increasing concave utility function, U, i.e. $U'(\pi) > 0$ and $U''(\pi) \leq 0$, where U is a utitlity function in the von Neumann-Morgenstern sense. Letting $E(\pi)$ be the expected value of π, then

$$E(\pi) = \int_0^\infty \int_0^1 (p\alpha q(L) - wL)g(\alpha)f(p)d\alpha dp.$$

Differentiating with respect to L, we have

$$\frac{dE(\pi)}{dL} = \bar{p}\bar{\alpha}q'(L) - w$$

where $\bar{\alpha}, \bar{p}$ are the expected values of α and p respectively. Let L_0 be the value of L that maximizes $E(\pi)$. Then

$$q'(L_0) = \frac{w}{\bar{p}\bar{\alpha}}. \tag{1}$$

The above equation is the well known equilibrium relationship of profit maximization under certainty. In this context certainty refers to replacing every random quantity by its expected value in the profit function. Taking the expected value of the utility of profit and computing its derivative with respect to L we get

$$\frac{dE(U(\pi))}{dL} = E((p\alpha q'(L)-w)U'(\pi))$$
$$= q'(L)E(p\alpha(U'(\pi)) - wE(U'(\pi)).$$

If L_1 is the value of L that maximizes $E(U(\pi))$, then

$$q'(L_1) = \frac{wE(U'(\pi))}{E(p\alpha U'(\pi))} . \tag{2}$$

Rewrite $E(p\alpha U'(\pi))$ in the following equivalent form

$$E(p\alpha U'(\pi)) = \bar{p}\bar{\alpha}E(U'(\pi)) + E(p(\alpha-\bar{\alpha})U'(\pi)) + \bar{\alpha}E(p-\bar{p})U'(\pi)).$$

Substituting the above expression in (2) we get after factoring $\frac{w}{\bar{p}\bar{\alpha}}$

$$q'(L_1) = \frac{w}{\bar{p}\bar{\alpha}}\left[\frac{1}{1+W(U')+M(U')}\right] \tag{3}$$

where

$$W(U') = \frac{E(p(\alpha-\bar{\alpha})U'(\pi))}{\bar{p}\bar{\alpha}E(U'(\pi))}$$

and

$$M(U') = \frac{E((p-\bar{p})U'(\pi))}{\bar{p}E(U'(\pi))} .$$

Combining (3) and (1) we have

$$q'(L_1) = q'(L_0)\left[\frac{1}{1+W(U')+M(U')}\right].$$

The above equation can be rewritten in the following form

(4) $$q'(L_1) - q'(L_0) = -q'(L_1)[W(U')+M(U')].$$

Using the mean value theorem we can assert

(5) $$q'(L_1) - q'(L_0) = q''(L^*)(L_1-L_0),$$

and

(6) $$q(L_1) - q(L_0) = q'(\hat{L})(L_1-L_0),$$

where L^* and $\hat{L}$ are numbers between L_0 and L_1. Eliminating L_1-L_0 from (4) and (5) we get

(7) $$q(L_1) - q(L_0) = \frac{q'(\hat{L})}{q''(L^*)}[q'(L_1)-q'(L_0)].$$

Combining (7) and (4) we obtain

(8) $$q(L_1) - q(L_0) = -\frac{q'(L_1)q'(\hat{L})}{q''(L^*)}[W(U')+M(U')].$$

Since U is concave, $M(U')$ and $W(U')$ are negative numbers [see appendix]. Moreover $q'(L_1)$ and $q'(\hat{L})$ are positive while $q''(L^*)$ is a negative number because of our assumptions on $q(L)$. Hence (8) implies that $q(L_1) - q(L_0)$ is a negative number. In other words, output under uncertainty for a risk-averse firm is less than output under uncertainty for a risk neutral firm. Note that a deterministic price p implies $M(U')$ is zero, and similarly a deterministic output implies $W(U')$ is zero, thus $q(L_1) - q(L_0) = 0$. If either $M(U')$ or $W(U')$ is zero and the other is negative, then $q(L_1) - q(L_0)$ is a negative number, but will be larger in the case when both $M(U')$ and $W(U')$ are simultaneously negative. This extends Baron's result which he proved under the assumption that only the price is random.

II. INCREASING RISK AVERSION AND INCREASING RISK

The level of output of a purely competitive firm can be related to the degree of risk aversion as measured by the Pratt-Arrow[1] index of risk aversion $r(x)$ given by

$$r(x) = -\frac{U''(x)}{U'(x)} = -\frac{d}{dx}\ln U'(x).$$

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[1] See Arrow (1971) and Pratt (1964).

Assume $U(x)$ is a given concave utility function and β a small positive number. Let $r_a(x)$ be

$$r_a(x) = r(x) + \beta.$$

Then the firm whose index of risk aversion is $r_a(x)$ is more risk averse than the firm whose risk aversion index is $r(x)$. Let $U_a(x)$ be the utitlity function whose risk aversion index is denoted by $r_a(x)$. Then

$$U_a'(x) = ke^{-\beta x}U'(x),$$

where

$$k = \frac{U_a'(0)}{U'(0)}.$$

Let $L_1 = L_1(\beta)$ be the value of L which maximizes $E(U_a(\pi))$. Then

$$E(U_a'(\pi_1)(p\alpha q'(L_1)-w)) = 0,$$

where

$$\pi_1 = p\alpha q(L_1) - wL_1. \tag{9}$$

Let

$$z = z(\beta, L) = E(U_a'(\pi)(p\alpha q'(L) - w))$$
$$= kE(U'(\pi)\bar{e}^{\beta\pi}(p\alpha q'(L) - w)).$$

Thus

$$z(\beta, L_1(\beta)) = 0.$$

Using the implicit function theorem

$$\frac{dL_1}{d\beta} = -\frac{\partial z/\partial\beta}{\partial z/\partial L_1}.$$

But

$$\frac{\partial z}{\partial\beta} = -kE(U'(\pi_1)\pi_1\bar{e}^{\beta\pi_1}(p\alpha q'(L_1) - w)),$$

and

$$\frac{\partial z}{\partial L_1} = kE((p\alpha q'(L_1) - w)^2\bar{e}^{\beta\pi_1}(U''(\pi_1) - \beta U'(\pi_1)))$$
$$+ kE(U'(\pi_1)\bar{e}^{\beta\pi_1}p\alpha q''(L_1)).$$

Taking $\beta = 0$

$$(10) \quad \left.\frac{dL_1}{d\beta}\right|_{\beta=0} = \frac{E(U'(\pi_1)\pi_1(p\alpha q'(L_1) - w))}{E((p\alpha q'(L_1) - w)^2U''(\pi_1)) + E(U'(\pi_1)p\alpha q''(L_1))}.$$

Recalling (9) we have

$$
\begin{aligned}
E(U'(\pi_1)\pi_1(p\alpha q'(L_1)-w)) &= E(U'(\pi_1)(p\alpha q(L_1)-wL_1)(p\alpha q'(L_1)-w)) \\
&= q(L_1)q'(L_1)E(U'(\pi_1)p^2\alpha^2) \\
&\quad - (wL(q'(L_1)+wq(L_1))E(p\ U'(\pi_1)) \\
&\quad + w^2 LE(U'(\pi_1)) \\
&= q(L_1)q'(L_1)E(U'(\pi_1)p^2\alpha^2) \\
&\quad + [w^2 L_1 - \frac{w}{q'(L_1)}(wL_1 q'(L_1)+wq(L_1))]\cdot \\
&\quad \cdot E(U'(\pi_1)) \\
&= q(L_1)q'(L_1)E(U'(\pi_1)p^2\alpha^2) \\
&\quad - w^2\frac{q(L_1)}{q'(L_1)}E(U'(\pi_1)).
\end{aligned}
$$

From (2) the expression above reduces to

(11) $E(U'(\pi_1)\pi_1(p\alpha q'(L_1)-w))$

$$
\begin{aligned}
&= q(L_1)[w\frac{E(U'(\pi_1))E(U'(\pi_1)p^2\alpha^2)}{E(p\alpha U'(\pi_1))} - wE(p\alpha U'(\pi_1))] \\
&= \frac{wq(L_1)}{E(p\alpha U'(\pi_1))}[E(U'(\pi_1))E(U(\pi_1)p^2\alpha^2)-E^2(p\alpha(U'(\pi_1))].
\end{aligned}
$$

By Schwarz inequality we know that

$$E^2(FG) \le E(F^2)E(G^2),$$

where F and G are functions of p and α. Applying Schwarz inequality to $F = p\alpha\sqrt{U'}$ and $G = \sqrt{U'}$ implies that the quantity in the square brackets in (11) is positive. Thus the numerator in (10) is positive, while the denominator is negative. Then

$$\left.\frac{dL_1}{d\beta}\right|_{\beta=0} < 0,$$

which implies that

$$L_1(\beta) < L_1(0),$$

for small β, and hence

$$q(L_1(\beta)) < q(L_1(0)).$$

The above inequality shows that the firm with the larger index of risk aversion produces a lower level of output than the firm with a smaller index of risk aversion. Baron obtained a similar result under the more general assumption that $r_a(x) \geq r(x)$. However, his proof relies heavily on the fact that the integral in the expectation operator is a single integral while in our case it is a double integral since we are considering two random variables.

We will now apply the above result to the case where $r(x)$ is identically zero, that is, to the firm which is risk neutral. Then $r_1(x)$ has the constant value β corresponding to a firm with a small, but constant index of risk aversion. If $r(x)$ is identically zero, then $U'(x)$ is constant and $U''(x)$ is identically zero. Thus (10) reduces to

$$\left.\frac{dL_1}{d\beta}\right|_{\beta=0} = \frac{wq(L_0)}{\bar{p}\bar{\alpha}} \times \frac{E(p^2\alpha^2)-\bar{p}^2\bar{\alpha}^2}{q''(L_0)\bar{p}\bar{\alpha}},$$

where $L_1(0) = L_0$.

Letting $C_\alpha = \frac{\sigma_\alpha}{\bar{\alpha}}$ and $C_p = \frac{\sigma_p}{\bar{p}}$ be the coefficients of variation of α and p respectively, then

$$\left.\frac{dL}{d\beta}\right|_{\beta=0} = \frac{wq(L_0)}{q''(L_0)} (C_\alpha^2 C_p^2 + C_\alpha^2 + C_p^2).$$

Thus for small β

$$(12) \qquad L_1(\beta) - L_0 = \frac{wq(L_0)}{q''(L_0)} (C_\alpha^2 C_p^2 + C_\alpha^2 + C_p^2)$$

Hence the firm with a constant risk aversion index whose value is β produces less out-put than the risk neutral firm by an amount Δq given <u>approximately</u> by

$$\Delta q = q(L_1(\beta)) - q(L_0)) = q'(L_0)(L_1(\beta) - L_0).$$

Combining the above equation with (12) we obtain the formula

$$(13) \qquad \Delta q = \frac{w\beta q'(L_0) q(L_0)}{q''(L_0)} (C_\alpha^2 C_p^2 + C_\alpha^2 + C_p^2).$$

Recall that $q'(L_0)$ can be computed from (1).

Baron showed that under the assumption of a normally distributed price p that an increase in risk (i.e. an increase in the variance of p) results in a decrease in the optimal level of output. D. V. Coes (1977) and Y. Ishii (1977) arrived at the same conclusion for the case of a "mean preserving spread" without any prior assumption on the distribution.[2] A more generalized result can be interpreted from (12). That is, without making any prior assumptions on the distribution of the two random variables p and α, the level of optimal output will decrease if both or either of the variances implicit in the coefficients of variation C_p and C_α increase. An increase in either variance while holding the other constant will result in a smaller decrease in the level of optimal output than the case when both variances increase simultaneously.

III. OTPIMAL OUTPUT WITH INPUT AND OUTPUT PRICE RANDOM

In the following section a model of a competitive firm with n production periods is considered. The assumption on output price p from the previous section is retained, while the level of output y is assumed to be deterministic. Let $C(y)$ be that portion of the variable costs known with certainty throughout the n production periods and $\phi_i(y)$ be the amount of input needed during the i^{th} period of production. The price of the input at the beginning of the i^{th} period of production, denoted by u_i, is assumed to be random from the point of view of the entrepreneur at the initial stage of production planning. The total variable cost of production will be

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[2]See Rothschild and Stiglitz (1970) for a discussion of the concept "mean preserving spread" and other definitions of an increase in ris

$C(y) + \sum_{i=2}^{n} u_i\phi_i(y)$.[3] Under the above assumptions the profit function $\pi(y)$ is given by

$$\pi(y) = py - C(y) - \sum_{i=2}^{n} u_i\phi_i(y).$$

Assume the random variables p, u_i, $i = 2,\ldots,n$ are independent. Let $f(p)$ be the density of p and $g_i(u_i)$ be the density function of u_i, $i = 2,\ldots,n$. Taking the expected value of the profit function we get

$$E(\pi) = \bar{p}y - C(y) - \sum_{i=2}^{n} \bar{u}_i\phi_i(y),$$

where, as before, the bar above the symbol of a random variable denotes its expected value. Thus the first order condition for a risk neutral firm is

$$\text{(13)} \qquad \bar{p} - C'(y) - \sum_{u=2}^{n} \bar{u}_i\pi_i'(y) = 0.$$

Let $U(x)$ be a concave utility function of a risk averse firm. The optimal level of output y_a for such a firm must satisfy the equation

$$\frac{d}{dy} E(U(\pi)) = 0.$$

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3 The index in the sum starts at $i = 2$ since the input price is known with certainty at the beginning of production.

The above equation can be rewritten as follows

(14) $$E(U'(\pi_a)(p-C'(y_a) - \sum_{i=2}^{n} u_i\phi_i'(y_a))) = 0,$$

where $\pi_a = \pi(y_a)$. Equation (14) is equivalent to

$$E((p-\bar{p})U'(\pi_a)) - \sum_{i=2}^{n} \phi_i'(y_a)E((u_i-\bar{u}_i)U'(\pi_a))$$

$$+ [\bar{p}-C'(y_a) - \sum_{i=2}^{n} \phi_i'(y_a)\bar{u}_i]E(U'(\pi_a)) = 0.$$

Now the above equation for the expression in the brackets

$$\bar{p} - C'(y_a) - \sum_{i=2}^{n} \phi_i'(y_a)\bar{u}_i$$

$$= \frac{\sum_{i=2}^{n} \phi_i'(y_a)E((u_i-\bar{u}_i)U'(\pi_a))-E((p-\bar{p})U'(\pi_a))}{E(U'(\pi_a))} .$$

Since $\frac{\partial\pi}{\partial p} = y > 0$ and $\frac{\partial\pi}{\partial u_i} = -\phi_i(y) < 0$ then using the lemma contai in the appendix, the first term in the numerator on the left hand side of the above equation is positive while the second term is negative, and certainly $E(U'(\pi_a)) > 0$ because $U(x)$ is an increasing function. Thus we can assert that

(15) $$\bar{p} - C'(y_a) - \sum_{i=2}^{n} \phi_i'(y_a)\bar{u}_i > 0.$$

Let $K(y)$ be the function

$$K(y) = \bar{p} - C'(y) - \sum_{i=2}^{n} \bar{u}_i \phi_i'(y),$$

and let y_n be the optimal output of the risk neutral firm. Then from (13) we know that

(16) $$K(y_n) = 0,$$

while from (15),

(17) $$K(y_a) > 0.$$

Taking the derivative of $K(y)$, we have

$$K'(y) = - C''(y) - \sum_{i=2}^{n} \bar{u}_i \phi_i''(y).$$

It is obvious that $-K'(y)$ is the second derivative of the expected total variable cost function, which we know to be always positive. Then $-K'(y) > 0$ implying $K'(y) < 0$, and thus $K(y)$ is a decreasing function. From (16) and (17) we have

$$K(y_a) > K(y_n).$$

The above equation implies that $y_a < y_n$ because, as shown above, $K(y)$ is a decreasing function. Thus, the optimal level of production under risk aversion is less than that under a neutral attitude towards risk.

APPENDIX

The following Lemma is essential for the mathematical derivations.

<u>Lemma</u>. Let θ, $0 \leq \theta < \infty$ be a random variable with a density function $f(\theta)$, and let $U(x)$ be a twice continuously differentiable function of x such that $U'(x) > 0$ while $U''(x) < 0$. Assume that $\pi = \pi(\theta)$ is continuously differentiable function of θ, with a derivative $\pi'(\theta)$ which is positive (negative) for $0 \leq \theta < \infty$. Then the integral

$$I = \int_0^\infty (\theta - \bar{\theta})U'(\pi)f(\theta)d\theta$$

is negative (positive), where $\bar{\theta}$ denotes the mean of θ.

<u>Proof</u>. Let $G(\theta)$ be the function defined by

$$G(\theta) = \int_0^\theta (t-\bar{\theta})f(t)dt.$$

Clearly, we have $G(0) = 0$, $G(\infty) = 0$ and $G'(\theta) = (\theta-\bar{\theta})f(\theta)$. Thus $G(\theta)$ is decreasing for $\theta < \bar{\theta}$ and increasing for $\theta > \bar{\theta}$. Therefore $G(\theta) \leq 0$ for $0 < \theta < \infty$. The integral I can be rewritten in terms of $G(\theta)$ as follows

$$I = \int_0^\infty U'(\pi)G'(\theta)d\theta.$$

Integrating by parts, we get

$$I = -\int_0^\infty G(\theta)U''(\pi)\pi'(\theta)d\theta.$$

The boundary term in the integration by parts drops out since $G(0) = G(\infty) = 0$. Using the assumptions of the lemma and the above equation we deduce the conclusion of the lemma.

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