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Process Aggregation and Efficiency.

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Abstract

We analyze how the aggregation of production processes affects efficiency evaluations. A production unit can be inefficient even though all its production processes are efficient. We show how to test if observed "inefficiencies" may be caused by an aggregation and how to correct efficiency measures for such aggregation generated inefficiencies. The tests and the adjusted efficiency measurement programs involve hypothetical dis-aggregations designed to put the evaluated unit in its best possible light.

Keywords: Production Processes, Data Envelopment Analysis (DEA), Aggregation, Efficiency

1 Introduction

In efficiency and productivity analyses, it is necessary to work with aggregate production models. Non-homogenous inputs and outputs are aggregated into a few input and output categories, and spatially and temporally separated production processes are aggregated into one or at the most a few overall processes. In activity analysis or Data Envelopment Analysis (DEA) terminology, rows (variables) as well as columns (processes) are aggregated.

It is well-known that these aggregations affect the evaluations. We can usually identify more inefficiency the more we aggregate. Still, it is not an aspect of efficiency analyses that has received much attention in the literature. Little is known about the magnitude of the aggregation effect nor on how it can be evaluated.

The aim of this paper is to suggest *an approach that may help us evaluate the potential impact of process aggregations without actually having the dis-aggregated process data. Our approach involves hypothetical dis-aggregations to investigate if the conclusions derived from the aggregate information could be significantly altered by more dis-aggregated descriptions.*

There are at least three general reasons to be concerned with process aggregation. First, aggregations lead to *biased efficiency evaluations.* If a production unit adjusts production to variations in prices over time and space in a convex model, its profit numbers will improve (if they are available) but its efficiency scores based on physical production data will deteriorate since a convex technology favors producing the average output using the average input. We shall expand on this in great details below. Secondly, aggregations make it *difficult to connect with other evaluations* (and evaluators) of the same production system. Engineers using detailed process data and economists using aggregate data may get conflicting results. Similarly, marketing analysts can be impressed by a firms adaptability to changing market conditions and yet an aggregate evaluation may show deficient allocative efficiency because of the scale or scope properties of a convex model. Thirdly, aggregations may have *adverse incentive effects.* A contract with a production unit designed to improve technical and allocative efficiency, cf. for example Bogetoft(1994,95,97), may reduce responsiveness to market variations if it is based on aggregate data and straightforward production norms that do not take into account the aggregation effect. Hereby, profitability and more generally, goal fulfillment may suffer.

We note that biased results are most likely in organizations where there are *no price and profit data* to capture the gains from varying the production level and production mix over time and space. This includes most public service organizations where useful accounting systems are hard to design. Also, inadequate aggregations are most likely when *inputs and outputs cannot easily be transferred over time and space.* In such cases, firms cannot take advantage of the scale and scope opportunities by handling demand and supply variations via the inventory. This is a characteristic of part of the food industry - e.g. fresh vegetables and meat -, industries producing "peak capacity" - e.g. utility firms, ambulance services and the military - and most generally, service industries where the staff cannot accumulate its production in a low demand period or location and use it in high demand periods or regions. Since many of the cases analyzed by modern efficiency analysis approaches share these properties, we suggest that the aggregation problem deserves more than parsing interests.

A bit of *casual empiricism* can also provide some motivation. In a recent study, we used DEA and standard econometric approaches to evaluate the efficiency of European slaughter houses based on production data, turnovers etc. aggregated over plants and time to give yearly data at the company level. Somewhat surprisingly, we found that Danish slaughter houses - widely acclaimed to be among the most efficient in detailed process comparisons as well as being among the best world-wide to adjust production to changes in markets potentials - did not perform too well in the analyses, cf. Asmild and Bogetoft(1997). A possible explanation is that our use of data aggregated over time and space lead to seriously biased results by not accounting for the adaption to changes in the market conditions. Another study also indicates that the aggregation effect may be non-trivial. In Bogetoft and Wang(1999), we examined the productivity of 70 agricultural extension offices using traditional DEA models. We also examined the additional "inefficiencies" generated when the extension offices are aggregated into groups of two or three depending on geographic proximity. In total, 458 aggregations was considered. In 409 of these, the aggregation in itself generated "inefficiency", and in 100 of the aggregations, the aggregation generated "inefficiency" was as large as the average inefficiency of the individual offices.

We are not aware of papers that are directly related to the present. There are, however, at least three general approaches that are somewhat related to our approach.

One approach is to use *sensitivity analysis*. To determine a simple but sufficiently detailed model, one may examine how sensitive the outcome, efficiency scores etc., is to aggregations over time and space. (This is of course just one of several issue that can be addressed by sensitivity analysis. For other applications (in DEA models), see for example Charnes, Haag, Jaska and Semple(1992), Charnes and Neralic(1990), Charnes, Rousseau and Semple(1996) and Ray and Dahu(1992)) Our approach do not calculate the consequences of given aggregations. Rather, it seeks the dis-aggregation that has maximal affect on the outcome. It constructs a worst case scenario so as to put an evaluated unit in its best possible light.

Another approach is to investigate under which conditions it is possible to make "exact" aggregations. There is a large literature on the relationship between *aggregation of variables and separability of the production process.* (Although the process and variables aggregations are not the same, the former can be thought of as the latter if we interpret inputs and outputs from different processes as different inputs and outputs and ask when we can aggregate these over the time and space dimensions, i.e. when it is safe to aggregate the variables describing the same type of input or output in the different processes). Also, there are a few papers explicitly linking the aggregation and separability stories to the efficiency measurement problem, cf. Färe and Lovell(1988). We deviate here by examining specific production instances instead of general structures. Also, we are more concerned with the impact of in-exact aggregation than with the circumstances under which exact aggregations are possible. Still, we give a fundamental condition, coordination, that allows exact aggregations w.r.t. efficiency evaluations.

A third approach is to combine the methods from productivity analysis, including the estimation of production models using observed production plans, with the *production planning* literature, including the use of linear programming to model networks of production processes, cf. e.g. Färe and Grosskopf(1997). We deviate here by assuming that we do not have detailed observation of the process productions and by investigating how this could affect the outcome.

This paper is organized as follows. In the next Section, we illustrate the aggregation problems using simple examples. Then, in Section 3, we formalize the problem and provide a few analytical results. Some tests of the potential aggregation biases and some adjusted efficiency measurement programs are developed in Section 4 and 5. The modelling of the individual processes is discussed in Section 6. In Section 7, these results are combined with DEA and simple numerical illustrations are provided. Concluding remarks are given in Section 8.

2 Two Examples

To illustrate some of the misleading conclusions that may result from process aggregations, we first consider a few simple examples with convex production possibility sets and associated scale and scope properties favoring "average behavior". For simplicity, we will think of the different processes as representing different time periods of the same production process. Let there be two sub-periods and two Decision Making Units (DMUs). In both sub-periods, both DMUs have the same production possibility set.

Consider first the scope effect. Let inputs available be the same for each DMU in each period and let the output possibilities be as depicted in Figure 1a below. The realized productions and prices in the two sub-periods are

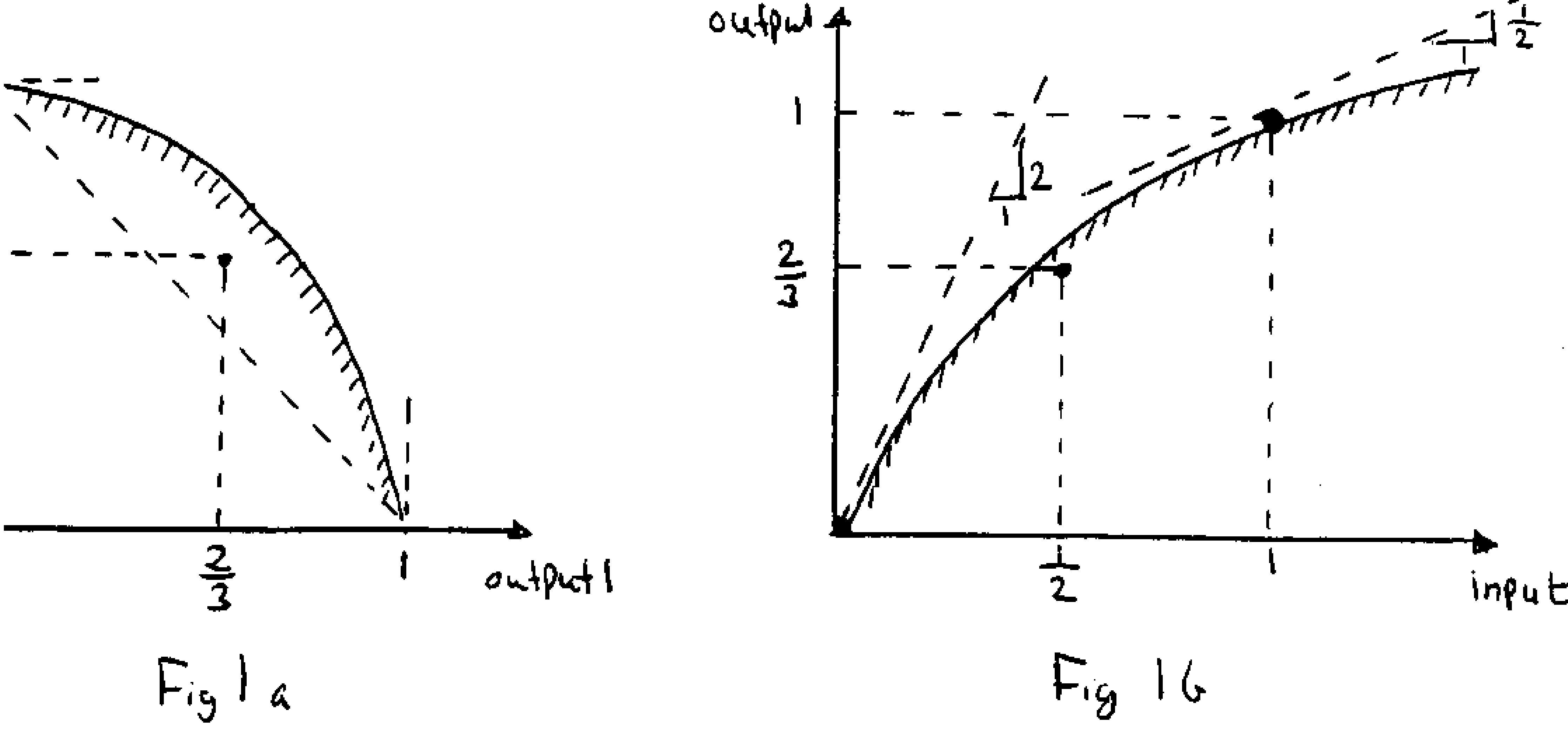
Subperiod:		1	2
Prices of	Product 1	1	0
	Product 2	0	1
DMU^1 production of	Product 1	1	0
	Product 2	0	1
DMU^2 production of	Product 1	$\frac{2}{3}$	$\frac{2}{3}$
	Product 2	$\frac{2}{3}$	$\frac{2}{3}$

We see that DMU^1 is technically and allocatively efficient in both sub-periods and that it makes a total profit of 2. DMU^2 on the other hand is neither technically nor allocatively efficient and it only makes a total profit of $\frac{4}{3}$. Yet, if we aggregate over time, DMU^1 has a total production of $(1,1)$ while DMU^2 has a total production of $(\frac{4}{3}, \frac{4}{3})$. Therefore, DMU^1 will mistakenly be classified as technically inefficient in an analysis that does not recognize the sub-period productions. DMU^2 would be classified as technically efficient (at least relative to DMU^1).

To illustrate the scale effects, assume now that one input is used to produce one output as illustrated in Figure 1b. The realized productions and prices in the two sub-periods are

Subperiod:		1	2
Prices of	Input	1	1
	Output	$\frac{1}{2}$	2
DMU^1	Input	0	1
	Output	0	1
DMU^2	Input	$\frac{1}{2}$	$\frac{1}{2}$
	Output	$\frac{2}{3}$	$\frac{2}{3}$

In this case, DMU^1 is technically and allocatively efficient while DMU^2 is neither. The profit numbers reflects this: DMU^1 earns 1 while DMU^2 only earns $\frac{2}{3}$. Yet, looking at the aggregate production, the decreasing return to scale properties make DMU^1 look technically (and allocatively) inefficient with a production plan $(1, 1)$ and DMU^2 (relatively) technically efficient as well as allocatively efficient with a production of $(1, \frac{4}{3})$ and time-averaged prices $(1, 1\frac{1}{4})$.



output
1
$\frac{2}{3}$
$\frac{1}{2}$
1
input
2
1
$\frac{1}{2}$
1
output1
Fig 1a
Fig 1b

Figure 1

As demonstrated by these examples, and formalized in the next Section, the ability to recognize production inputs and outputs sufficiently precise in time and space, may significantly affect the outcome of efficiency and productivity analyses. The difficulty in practice, however, is that the detailed and timely data are usually not available. Therefore, we cannot know with certainty whether an analysis based on aggregate information is biased. To compensate for this, we propose to test if there may exist fully efficient sub-period productions that would generate the observed aggregate outcome. If so, as it is the case in all the examples above, we will say that the aggregate outcome "is potentially process efficient". In turn, this suggests cautiousness in the conclusions.

3 Efficiency and Process Coordination

Let us assume that the production and price relations facing a DMU can be split into $|H|$ processes, $h \in H = \{1, ..., |H|\}$. For any such process h let $q^h \in \mathbb{R}^m$ be the production vector (with negative coordinates being inputs), let Q^h be the set of possible production plans, and when applicable, let $p^h \in \mathbb{R}^m_+ \backslash \{0\}$ be the non-negative ($\mathbb{R}_+$) prices of the products in the production plan. For simplicity, we assume throughout this paper that inputs and outputs are freely disposable, i.e. for any Q we assume that $q \in Q$, $q' \leq q \Rightarrow q' \in Q$.

For any set of production plans $Q \subset \mathbb{R}^m_+$, we let $WEff(Q)$ be the *weakly technically efficient* (or not strictly dominated) plans in Q

$$WEff(Q) = \{q \in Q \mid q + z \notin Q \text{ for all } z >> 0\}$$

where $z >> 0$ means that all coordinates of $z \in \mathbb{R}^m$ are strictly positive. By free disposability, this is equivalent to $WEff(Q) = \{q \in Q \mid q + \delta e \notin Q$ for all $\delta > 0\}$ where $e = (1, .., 1) \in \mathbb{R}^m$ and $\delta \in \mathbb{R}$. We shall say also that $q \in Q$ is *allocatively efficient* with respect to price vector $p \in \mathbb{R}^m_+ \backslash \{0\}$ if

$$q \in \arg\max_{q' \in Q} pq'$$

We emphasize that weak efficiency is weaker than the classical economic notion of efficiency. A production plan that can be improved in some but not all dimensions is weakly efficient. In the traditional notion of efficiency, the efficient productions $Eff(Q) = \{q \in Q \mid q + z \notin Q \text{ for all } z \geq 0, z \neq 0\}$ are all those for which we cannot increase any output (decrease any input) without decreasing (increasing) another output (input).

It is advantageous to use the weaker efficiency notion here. First of all it simplifies several of the results below where we would have to assume unique solutions or solutions with strictly positive prices if we worked with ordinary efficiency (to avoid picking up points on horizontal and vertical segments of the production possibility frontier), cf. for example Bogetoft and Pruzan(1997, Appendix A). Secondly, it is common to work with weaker notions of efficiency in the productivity analysis literature. This includes what we could call the Farrell(1957) efficient plans. Letting q_- and q_+ be the (negative) inputs coordinates and (positive) coordinate of q, respectively. The Farrell efficient plans based on contractions in the input space or expansions in the output space is then $FIEff(Q) = \{q \in Q \mid q - \delta(q_-, 0) \notin Q$ for all $\delta > 0\}$ and $FOEff(Q) = \{q \in Q \mid q + \delta(0, q_+) \notin Q \text{ for all } \delta > 0\}$. This is closely related to the weak efficiency concept - it only deviates when some of the inputs or outputs in q are zero. Thirdly, we believe that it makes conceptually sense in the present paper to work with weak efficiency. With convex sets, the weakly efficient plans are all those that result from optimal economic behavior (profit maximization) for some non-negative and non-zero price vector. We record this as a Lemma.

Lemma 1 *If Q is convex, a production plan $q \in Q$ is weakly technically efficient if and only if it is allocatively efficient with respect to some price vector $p \in \mathbb{R}^m_+ \backslash \{0\}$.*

Proof: To show the first part, assume that q is weakly efficient. It is therefore a boundary point, at by convexity it can be weakly separated from Q, i.e. there exist a non-zero price vector q such that

$$pq \geq pq' \quad \forall q' \in Q$$

We must just show that $p \geq 0$ Assume that this is not the case, i.e. there exist a coordinate j such that $p_j < 0$. Then, letting $q'_j \rightarrow -\infty$ the separation above cannot hold and we have a contradiction. (Note that we have defined free disposability as $Q = Q - \mathbb{R}^m_+$ here. Therefore, there exist feasible points with $q'_j \rightarrow -\infty$. If we had restricted inputs and outputs to be positive, we could have extended Q to $Q - \mathbb{R}^m_+$ and separated q from the latter set.) It follows that the price vector must be non-negative and since it is also non-zero, we have that q is allocatively efficient with respect to p.

Assume next that q is allocatively efficient with respect to $p \in \mathbb{R}^m_+ \backslash \{0\}$, i.e.

$$q \in \arg\max_{q' \in Q} pq'$$

If q is not weakly technically efficient we can find a q^* such that $q^* = q + \delta e$ is feasible with $\delta > 0$. However, this would mean that $pq^* > pq$ and we have a contradiction to q being allocatively efficient. □

Hence, the weakly technically efficient plans are allocatively efficient under some price vector and vice versa.

We can now formalize and generalize the results from Section 2. We do so by showing what is necessary and sufficient to ensure efficiency at the aggregate level, namely allocative efficiency at the process level with respect to the same set of prices.

Proposition 1 *If Q^h is convex for all $h \in H$, the aggregate plan $\sum_{h \in H} q^h$ is weakly technical efficient at the aggregate level:*

$$\textstyle\sum_{h \in H} q^h \in WEff(\sum_{h \in H} Q^h)$$

if and only if the process productions $q^h, h \in H$, are allocatively efficient at the process level with the same rate of substitutions in all processes:

$$\exists p \in \mathbb{R}^m_+ \backslash \{0\} : q^h \in \arg\max_{q \in Q^h} pq \quad \forall h \in H. \qquad \textit{(Coordination)}$$

Proof: When Q^h is convex for all $h \in H$, so is $\sum_{h\in H} Q^h$. Therefore, when the aggregate output is efficient, $\sum_{h\in H} q^h \in WEff(\sum_{h\in H} Q^h)$, we know from Lemma 1 that it is allocatively efficient with respect to some price vector

$$\exists p \in \mathbb{R}_+^m \backslash \{0\} : \textstyle\sum_{h\in H} q^h \in \arg \max_{q\in \sum_{h\in H} Q^h} pq \tag{1}$$

In turn, this implies

$$\exists p \in \mathbb{R}_+^m \backslash \{0\} : q^h \in \arg\max_{q\in Q^h} pq \quad \forall h \in H. \tag{2}$$

since if we use the same price vector as in (1) and q^h does not solve $\max_{q\in Q^h} pq$ for some h, we could have found a better solution in (1) by substituting the solution to $\max_{q\in Q^h} pq$ for the old q^h in (1). This shows that allocative efficiency at the process level with the same rate of substitution in all processes, is a necessary (only if) condition for aggregate efficiency.

The sufficiency also follows by a contradiction. Let $q^h \in \arg\max_{q\in Q^h} pq$, $h \in H$ be a solution to (2). Now if $\sum_{h\in H} q^h$ does not solve (1), there exists an alternative solution $\sum_{h\in H} \bar{q}^h$ such that $p\sum_{h\in H} \bar{q}^h > p\sum_{h\in H} q^h$. This means that for at least one h, we have $p\bar{q}^h > pq^h$ which contradicts that we had a solution to (2) to begin with. Hence, a solution to (2) gives us a solution to (1) and by Lemma 1, this give weak technical efficiency. This proves the sufficiency (if) part of the proposition. □

Proposition 1 shows that to get technical efficiency at the aggregate level, it is not enough to have allocative efficiency (or technical efficiency) at the process level with respect to process prices. That is, optimal (profit maximizing) behavior does not ensure aggregate technical efficiency. We need also that the processes are coordinated by a common price vector in the sense that the rate of substitution are the same in all processes. In more organizational terms, we can say that is not enough to have efficient processes, we need goal concordance among them as well. (Note that goal concordance here is with respect to a common weighting of the inputs and outputs and it is not necessary advantageous. If prices differs, concordance with respect to the same weights is sub-optimal).

In our first example above, productions were clearly not coordinated in DMU^1 and this was the source of the aggregation problem. It was also a lack of coordination that created the aggregation problem in our second example. Processes that are technically identically must operate at the same scale (or at least the same facet), and this was not the case in our second example. In summary, the coordination condition in Proposition 1 ensures that the scale and scope effects does not come into play and generate a discrepancy between process and aggregate performance. *In addition to optimal behavior in the different processes, we need price proportionality in the processes to aggregate information without obscuring the evaluations.*

One instance of Proposition 1 is particularly clear. If all processes are the same, $Q^1 = Q^2 = ... = Q^{|H|}$, a necessary condition for technical efficiency at the aggregate level is that they all operate if not at the same point, then at least at the same facet. This follows immediately from the coordination condition in Proposition 1. If the technologies are furthermore strictly convex (at least at the relevant (efficient) part), we have that the aggregate production plan is efficient if and only if all processes are using exactly the same (efficient) production plan.

This clearly illustrates how *desirable adjustments to temporal or spacial variations in prices at the process level will tend to show up as technical inefficiencies at the aggregate level.* Optimal economic behavior in the processes lead to aggregate inefficiency if prices over time and space are not - by some strike of luck - proportional. It is fair to say therefore that aggregate efficiency is quite unlikely and certainly not always desirable; it may come at the cost of inadequate adaptions to local variations in prices, or more generally, to variations in demand and supply.

We note that these claims regard aggregate efficiency in an absolute sense, i.e. when compared to a theoretical production model. In practice, efficiency is usually measured relative to other DMUs and since they may be "handicapped" by variations in the process prices also, relative aggregate efficiency may be more likely than absolute aggregate efficiency.

In this Section, we have identified the basic economic condition that allows us to get aggregate efficiency from process efficiency, namely coordination. With coordinated productions, the aggregation does not affect the efficiency evaluation. The aggregate production is efficient if and only if the process productions are efficient. Hence, under coordination, exact aggregation (w.r.t. efficiency) is possible.

4 Tests of Possible Process Efficiency

In practice, the detailed process information is often not available, and we cannot know with certainty if an inefficiency at the aggregate level is the result of sloppiness in the production processes or the result of an inappropriate aggregation, i.e. an aggregation of non-coordinated processes. We may, however, *ask if an observed inefficiency is sufficiently small to be explained by unobserved but potentially desirable adjustments of the processes to local variations in prices.* Taking this perspective a bit further, we may ask how much of aggregate inefficiency we can explain by the aggregation process alone.

In this and the next Section, we address these questions. We propose a few simple tests to determine if the observed aggregate production may be an aggregation of efficient process productions. If so, we say the outcome is "potentially process efficient". To test it, we let the processes be hypothetical and we construct process productions that will put a DMU in its best possible light.

In this Section, then, we assume that there is information about the aggregate production q as well as about the aggregate and disaggregate production possibility sets Q and $Q^h, h \in H$, where $Q = \sum_{h \in H} Q^h$. In Sections 6 and 7, we will discuss how to model the production possibility sets.

We say that $q \in Q$ is *potentially* $(Q^h, h \in H)$ *process efficient* if and only if

$$\exists\ q^h \in Q^h,\ \ h \in H : \textstyle\sum_{h \in H} q^h \leq q \ \ \textit{and}\ q^h \in WEff(Q^h)\ \forall h \in H$$

Hence, a potentially process efficient aggregate plan is one that results from aggregating efficient - but not necessarily coordinated - process productions.

To test if an aggregate production plan $q \in Q$ is potentially $(Q^h, h \in H)$ process efficient, we can solve the following program **(P1)**

$$\begin{array}{llll} \min & \Delta & & \\ \Delta, q^h, h \in H & & & \\ s.t. & q + \Delta e & \geq & \sum_{h \in H} q^h \\ & q^h & \in & WEff(Q^h) \quad h \in H \\ & \Delta & \in & \mathbb{R}^m_+ \end{array}$$

Note that the process productions $q^h, h \in H$ are choice variables here. We construct process productions that put q in its best possible light.

Let $V(\mathbf{P1})$ denote the value of this program. Clearly $V(\mathbf{P1}) \geq 0$. Furthermore

Proposition 2 *The aggregate production $q \in Q$ is {is not} potentially $(Q^h, h \in H)$ process efficient if and only if $V(\mathbf{P1}) = \{>\}\ 0$.*

Proof: In **(P1)**, we search the set of possible aggregate outcomes, $\sum_{h\in H} q^h$, resulting from efficient processes, $q^h \in Eff(Q^h)$, in an attempt to find one that q dominates, perhaps after an addition of slack Δe. If it is necessary to add slack Δe, q cannot be process efficient. However, if is not necessary to supplement q with slack Δe, q is process efficient. □

An advantage of **(P1)** is that it always has a feasible solution. On the other hand, since the purpose of **(P1)** is only to establish whether there exists efficient q^h's such that $q \geq \sum_{h\in H} q^h$, we may simplify the formulation by using any objective function, say 0, and eliminating Δ from the constraints. This leads to **(P1*)**

$$\begin{array}{lccl} \min & 0 & & \\ q^h, h \in H & & & \\ s.t. & q & \geq & \sum_{h\in H} q^h \\ & q^h & \in & WEff(Q^h) \quad h \in H \end{array}$$

If the program has a feasible solution (and the program value therefore is 0), the aggregate plan is potentially process efficient. If no feasible solution exist, the modified program gives us $+\infty$ (by traditional convention since $inf\ \emptyset = +\infty$) and q is not potentially process efficient.

Since we may not have handy expressions of $WEff(Q^h)$ available, it is useful to note than an alternative test program is **(P1')**

$$\begin{array}{lllcll} \min & \max & \sum_{h\in H} \Delta^h & & & \\ q^h, h \in H & \Delta^h, h \in H & & & & \\ s.t. & & q & \geq & \sum_{h\in H} q^h & \\ & & q^h & \in & Q^h & \forall h \in H \\ & & q^h + \Delta^h e & \in & Q^h & \forall h \in H \\ & & \Delta^h & \in & \mathbb{R}^m_+ & \forall h \in H \end{array}$$

The inner optimization program measures the distance (in direction e) between the contemplated process productions q^h and the efficient frontier of the process feasible regions. The outer optimization program seeks to minimize this distance. The test is unaltered, i.e. *q is {is not} potentially $(Q^h, h \in H)$ process efficient if and only if $V(P1') = \{>\}\ 0$.*

If the production possibilities are convex, we may reformulate this test program into a quadratic optimization program. Consider the set

$$C^h = \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^m \backslash \{0\} | \beta q \leq \alpha \ \forall q \in Q^h\}$$

Clearly C^h is a convex set - in fact a convex cone. Moreover, C^h gives a (dual) characterization of Q^h since

$$Q^h = \{q \in \mathbb{R}^m | \beta q \leq \alpha \ \ \forall (\alpha, \beta) \in C^h\}$$

(This can easily be verified by using the definition of C^h to get $\subseteq$ and by using that a point not in Q^h can be strictly separated from Q^h to prove $\supseteq$).

Using this characterization of Q^h we get a simple test for weak technical efficiency.

Lemma 2 *If Q^h is convex, we have*

$$q^h \in WEff(Q^h) \Leftrightarrow \min\{\alpha - \beta q^h | (\alpha, \beta) \in C^h, \beta \geq 0, \beta e = 1\} = 0$$

Proof: To prove this, note first that $\alpha - \beta q \geq 0 \ \ \forall (\alpha, \beta) \in C^h$. If $q^h \in WEff(Q^h)$ it can be weakly separated, i.e. by Lemma 1, $\exists\ (\alpha, \beta) \in C^h : \beta \geq 0,\ \beta \neq 0$ and $\alpha - \beta q^h = 0$. By rescaling and using that C^h is a cone, we can w.l.o.g. assume $\beta e = 1$. This shows ($\Rightarrow$). To prove ($\Leftarrow$) assume that q^h with $\min\{\alpha - \beta q^h | (\alpha, \beta) \in C^h, \beta \geq 0, \beta e = 1\} = 0$ is not weakly efficient. Let (α, β) be the solution to the minimization problem. Now, since q^h is not efficient, there exist $\hat{q}^h = q^h + \delta e \in Q^h$ for some $\delta > 0$, and we get $\alpha - \beta \hat{q}^h = \alpha - \beta(q^h + \delta e) = -\delta \beta e < 0$, i.e. we get a contradiction. □

Now, if we let

$$\begin{aligned} C^h_+ &= \{(\alpha, \beta) \in C^h | \beta \geq 0, \beta e = 1\} \\ &= \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^m | \beta q \leq \alpha \ \ \forall q \in Q^h, \beta \geq 0, \beta e = 1\} \end{aligned}$$

(which is still a convex set), we can reformulate the min max problem above as the following *quadratic optimization problem* **(P1")** with value V(**P1"**)

$$\begin{array}{llll} \min & \sum_{h \in H} (\alpha^h - \beta^h q^h) & & \\ q^h, (\alpha^h, \beta^h), h \in H & & & \\ s.t. & q & \geq \sum_{h \in H} q^h & \\ & q^h & \in Q^h & \forall h \in H \\ & (\alpha^h, \beta^h) & \in C^h_+ & \forall h \in H \end{array}$$

Again, the test principle is the same. *If Q^h is convex for all $h \in H$, q is {is not} potentially $(Q^h, h \in H)$ process efficient if and only if* $V(\mathbf{P1}^{''}) = \{>\}$ 0.

5 Process Corrected Efficiency Measures

Even if an aggregate plan may not be process efficient, the process perspective may still explain part of the aggregate inefficiency. To *measure the inefficiency that remains when variations among efficient processes have been accounted for*, we may measure the distance between the observed aggregate production plan and the set of minimal aggregations of efficient process productions. This is a relative easy task given the formulations above.

To illustrate this, we let $d \in \mathbb{R}^m_+ \backslash \{0\}$ be a direction in which we want to measure the inefficiency in the aggregate plan $q \in Q$. We can then use the *directional distance function* or excess function

$$\varepsilon(q,d) = \max\{\varepsilon | q + \varepsilon d \in Q\}$$

inspired by the Luenberger(1992)'s benefit function, cf. e.g. Bogetoft and Hougaard(1999) and Chambers, Chung & Färe (1995,98). The excess $\varepsilon(q,d)$ has a straightforward interpretation as the number of times the net-output bundle d can be produced in excess of what is actually produced. If $\varepsilon \geq 0$, we could have produced εd more. Hence, a large excess reflects a large slack and a considerable amount of inefficiency. Note also that this formulation is relatively general. If for example $d = (0, q_+)$, we get the output based Farrell measure as $(1+\varepsilon)$, and if $d = (-q_-, 0)$ we get the input based Farrell measure as $(1 - \varepsilon)$.

We can now measure the inefficiency in direction d that cannot be explained by a possible aggregation error as **(P2)**:

$$\begin{array}{lrcl} \min\limits_{\varepsilon, q^h, h \in H} & \varepsilon & & \\ s.t. & q + \varepsilon d & \geq & \sum_{h \in H} q^h \\ & q^h & \in & Eff(Q^h) \quad \forall h \in H \\ & \varepsilon & \in & \mathbb{R} \end{array}$$

In this program, we seek to minimize what we must add to the aggregate plan to make it potentially process efficient. Hence, ε measures the inefficiency that remains when the possible lack of process coordination has been accounted for. As a technical curiosity, we note that the traditional program considers expansions from the inside of the feasible region. Here, we use contractions, i.e. we minimize from the outside. The reason is that we are interested in the lower bound of the set of process efficient production plans. If $\epsilon < 0$ the aggregate plan is super efficient in the sense ϵd less could have been produced or ϵd more could have been consumed without affecting the potential process efficiency. If $\epsilon > 0$, the aggregate plan is not process efficient and we could have expanded the outputs and contracted the inputs with ϵd even if we only require process efficiency. Figure 2 illustrates these ideas.

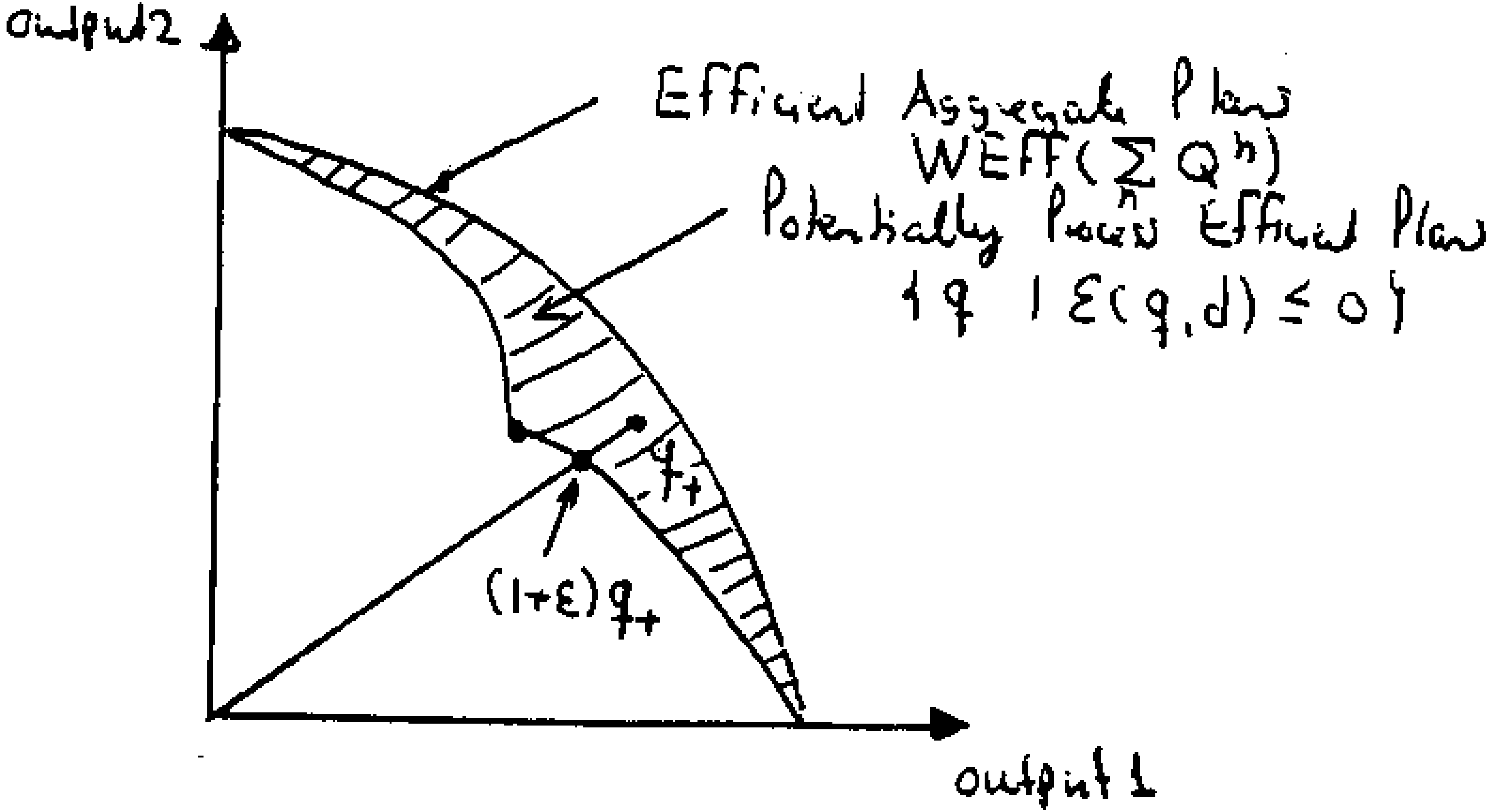

output2
Efficient Aggregate Plans
WEff($\sum_h Q^h$)
Potentially Process Efficient Plans
$\{q \mid \varepsilon(q,d) \leq 0\}$
q_t
$(1+\varepsilon)q_t$
output1

Figure 2

It is interesting to note that the set of aggregate production plans against which we compare here, i.e. the minimal process efficient plans plus those found by adding slack, may be a non-convex set even though all production possibility sets are convex. This is the feasible area except for the shaded one Figure 2. This non-convexity is a consequence of the processes being combined in 1:1 relations only and of the fact that we are looking at the minimal as opposed to the maximal frontier.

Again, it may be useful to reformulate the efficiency measurement program (**P2**) into a minmax program (**P2'**)

$$
\begin{array}{llllll}
\min & \max & \varepsilon + K \sum_{h \in H} \Delta^h & & & \\
\varepsilon, q^h, h \in H & \Delta^h, h \in H & & & & \\
st & & q + \varepsilon d & \geq & \sum_{h \in H} q^h & \\
 & & q^h & \in & Q^h & \\
 & & q^h + \Delta^h e & \in & Q^h & \forall h \in H \\
 & & \varepsilon & \in & \mathbb{R} & \\
 & & \Delta^h & \in & \mathbb{R}_+ & \forall h \in H
\end{array}
$$

where K is a sufficiently large number. By K being large, the program seeks first of all to make the processes efficient. Having ensured this, it seeks to reduce all aggregate outputs in the direction d. Lastly, we note that if Q^h

is convex for all $h \in H$, we can reformulate this as an quadratic program **(P2")**

$$\begin{array}{llll} \min\limits_{\varepsilon, q^h, (\alpha^h, \beta^h), h \in H} & \varepsilon + K \sum_{h \in H} (\alpha^h - \beta^h q^h) & & \\ s.t. & q + \varepsilon d & \geq & \sum_{h \in H} q^h \\ & q^h & \in & Q^h & \forall h \in H \\ & (\alpha^h, \beta^h) & \in & C_+^h & \forall h \in H \\ & \varepsilon & \in & \mathbb{R} \end{array}$$

where K is again sufficiently large to make the minimization of the last part of the objective the first priority.

Our focus in this paper is on evaluating an aggregate production plan q to see if a technical inefficiency can be explained by the aggregation effect, i.e. by a lack of process coordination. As a theoretical aside, however, it is worthwhile to note, that the same apparatus can model the *coordination value* in a multiple input-multiple output organization involving multiple production units. When the different production units have production possibility sets $Q^h, h \in H$ the total production possibility set is $\sum_{h \in H} Q^h$. A model of *the productive potential of a fully coordinated, centralized organization* is therefore

$$WEff(\sum_{h \in H} Q^h)$$

If on the other hand, the organization is decentralized and the production units are not coordinated, the output potential will be reduced. Let $\varepsilon(q, d)$ be the optimal value of ε in **(P2)** and let the improvement direction d be strictly positive $d_i > 0, i = 1, .., m$. The *productive potential with efficiency at the decentralized unit but no inter-unit coordination* is then

$$\{q \in \mathbb{R}^m | \varepsilon(q, d) \leq 0\}$$

This corresponds to the shaded area in Figure 2. If the input and output prices are the same in all production units, the difference between the (minimal) process efficient and the maximal aggregate outputs measures the value of process coordination and hereby of management at the firm level..

6 Process Models

The tests and the process adjusted efficiency measures developed above presume that information about 1) aggregate production q and 2) process production possibilities $Q^h, h \in H$ is available.

In a usual efficiency analysis, we would require information about a) process productions $q^h, h \in H$ and b) process production possibility sets Q^h or we would require information about c) aggregate information q and d) aggregate production possibilities Q. In terms of information need, the test and adjusted measures therefore require less information that a traditional detailed study and more information that a tradition entirely aggregated study.

We emphasize that the tests and the measures above do not impose any particular restrictions on how the process information $(Q^h, h \in H)$ is provided, except for the free disposability assumption and - in the dual formulations - an assumption of convexity. Hence, we can use purely theoretical models of the processes, we can rely on engineering information, detailed time and material requirement specifications, experimental results etc., and we can rely on a set of past observations of the processes, summarized into a description of the process production possibilities using a parametric or non-parametric productivity analysis technique, e.g. the estimation of cost functions using economic methods or the modelling of the processes as individual DEA models. Färe and Grosskopf(1997) uses the latter approach in combination with detailed information about any interaction among the processes, e.g. through intermediate products.

The process aggregation perspective may of course be relevant even in cases, where process information $Q^h, h \in H$ is not easily available. In some of these cases, we suggest that the process models can be modelled using aggregate information only. We now give a brief discussion of this.

Assume first that Q is given and $(0, ..., 0) \in Q$. If the process production possibility sets Q^h, $h \in H$, are not given as well, we suggest modeling Q^h by

$$Q^h := \sigma^h Q, \quad h \in H$$

where $\sigma^h \in [0, 1]$ is an indication of the relative size of the production facility using process $h \in H$ or the time spend on process $h \in H$, and $\sum_{h \in H} \sigma^h \geq 1$ (or just $= 1$,cf. below). The basic rationale for this class of models is that 1) if Q is convex and $(0, .., 0) \in Q$, we know that $\sigma^h Q \subseteq Q$, i.e. by claiming feasibility of $\sigma^h Q$, we do not extrapolate beyond the information in Q, and 2) by $\sum_{h \in H} \sigma^h \geq 1$, we ensure $\sum_{h \in H} Q^h \supseteq Q$.

If Q where given but $(0, ..., 0) \notin Q$, the best suggestion is probably to model Q^h in the same way. In this case, however, the motivation is less obvious. We would still get $\sum_{h\in H} Q^h \supseteq Q$, but we may end up extrapolating beyond Q since we may have $\sigma^h Q \not\subseteq Q$.

With no particular knowledge of the distribution of process sizes, we suggest using uniform sizes, $\sigma^h := |H|^{-1}$. Alternatively, one may endogenize the size distribution by picking it in the programs so as to put the DMU in its best possible light, e.g. by modifying **(P1)** to become **(P3)**

$$\begin{array}{llll}
\min\limits_{\Delta, \sigma^h, q^h, h\in H} & \Delta & & \\
s.t. & q + \Delta e & \geq & \sum_{h\in H} q^h \\
 & q^h & \in & WEff(\sigma^h Q^h) \quad h \in H \\
 & \sum_{h\in H} \sigma^h & \geq & 1 \\
 & \Delta & \in & \mathbb{R}_+ \\
 & \sigma^h & \in & \mathbb{R}_+ \qquad\qquad \forall h \in H
\end{array}$$

In **(P3)**, the size variables σ^h will automatically be chosen as small as possible. Therefore, we do not need the $\sigma^h \in [0,1]$ condition. We see also that when an optimal solution exists, one with $\sum_{h\in H} \sigma^h = 1$ exist as well. Therefore, we may without loss of generality assume $\sum_{h\in H} \sigma^h = 1$ throughout. This makes the interpretation of the σ^hs as relative sizes of facilities or as fractions of the time more natural. As previously, we may reformulate as minmax and quadratic programs.

Note that if we assume *constant return to scale* in Q, i.e. $q \in Q \Rightarrow kq \in Q$ $\forall k \geq 0$, it is inconsequential what we assume σ^h to be. We have $\sigma^h Q = Q$ for all $\sigma^h > 0$ and we get $\sum_{h\in H} Q^h = \sum_{h\in H} Q = Q$. In this case, we may just as well simplify and choose $\sigma^h = 1 \;\; \forall h \in H$.

An alternative and very attractive interpretation of these models is to think of Q as the set of feasible production plans if the same input-output combination is used throughout the period, and to think of σ^h as the time (fraction of the period) used to produce according to production plan $\vec{q}^h \in Q$, $h \in H$. Now, assuming *constant return to scale with respect to time*, we get $q^h \approx (\sigma^h, \vec{q}^h) \approx \sigma^h \, \vec{q}^h \in \sigma^h Q$ as above. Note that time or process intensity σ^h here is like a non-modelled input.

With no a priori knowledge of the processes, it is not only their feasible sets Q^h, $h \in H$ that we may not know. We may not even know *the number of processes* $|H|$.

Several factors affect the number of processes it is reasonable to contemplate. First, the number of distinct production locations or the number of times the production profile may be changed during the production observation period is relevant. Next the number of inputs and outputs in the production model, i.e. m, should be taken into account. Intuitively, the smallest productions that are consistent with the hypothesis of process efficiency will be a convex combination of the frontier productions in the most dispersed directions, i.e. in the directions of the coordinate axes. Thirdly, and must generally, we suggest to try to rationalize aggregate inefficiencies with the least possible number of processes. Restricting the number of processes (or the variation between them, cf. below) makes it less likely to accept the hypothesis of process efficiencies, i.e. it makes it harder to reconcile aggregate inefficiencies with the idea of process efficiencies. If we choose a small number of processes, we reduce the risk of favoring the disaggregate perspective.

One way to proceed is to endogenize the choice of $|H|$ just as one may endogenize the choice of sizes $\sigma^h, h \in H$. A related approach is to allow a large number of processes but to restrict how much they can reasonable deviate from each other because of switching costs, market stability etc. Restrictions on process variations could for example be modelled as constraints on $(q^h - q^k)$, $h, k \in H$ or in dual form, constraints on $(\beta^h - \beta^k)$, $h, k \in H$. This can be done as discussed in the DEA literature by introducing information about the ranking of prices, e.g. $\beta_j^h \geq \beta_k^h \; \forall h \in H$, the magnitude of price ratios, e.g. $\beta^h \in P \; \forall h \in H$ where $P \subseteq \mathbb{R}_+^m$ is as price cone, etc., cf. e.g. Ali, Cook and Seiford(1991), Charnes, Cooper, Wei and Huang(1989) and Golany(1988). Such additional information is particularly important here since it can restrain the obvious curse of the process perspective when we do not know the Q^h, $h \in H$ set, namely an exploding doubt and ability to see any plan as the result of optimizing behavior.

To sum up, we believe that the approach of this paper is relevant even in cases, where there is little a priori information about the production possibility sets of the processes.

7 Test Programs in DEA Framework

Up until now, we have assumed that the aggregate production possibility set $Q^h, h \in H$ or at least Q were given. We will now illustrate how to combine our tests with an estimation of Q. To do so, we will use DEA to estimate Q.

To emphasize the connection with DEA, we will distinguish between inputs and outputs in this Section. Inputs are denoted by $\tilde{x}$ when they are observed and x when they are constructed. Similarly, $\tilde{y}$ and y denote observed and constructed output plans. We assume that data is available on the r aggregate inputs $\tilde{x}^i \in \mathbb{R}^r_+$ used and s aggregate outputs $\tilde{y}^i \in \mathbb{R}^s_+$ produced in each of n DMUs, DMU^i , $i \in I = \{1,..,n\}$. In the terminology above, $r + s = m$, and we have for each $i \in I = \{1,..,n\}$ observed the aggregate production plan $q = (-\tilde{x}^i, \tilde{y}^i) \in \mathbb{R}^r_- \times \mathbb{R}^s_+$.

Following the original DEA model by Charnes, Cooper and Rhodes(1978,79), we assume that the underlying but unknown aggregate production possibility T has the following properties:

1. Free disposability:

   $\forall x, y, x', y' : (x, y) \in T,\ x' \geq x,\ y' \leq y \Rightarrow (x', y') \in T.$

2. Convexity:

   T is convex

3. Constant return to scale:

   $\forall (x, y) \in T,\ k \geq 0 : k(x, y) \in T.$

We assume that all observed production plans $(\tilde{x}^i, \tilde{y}^i)$ $i \in I$ belongs to T. To estimate T we use the minimal extrapolation principle, i.e. we construct our estimate $\widehat{T}$ of T as the smallest set containing data and satisfying conditions 1-3 above. It is easy to see that this leads to

$$\widehat{T} = \{(x, y) \in \mathbb{R}^r_+ \times \mathbb{R}^s_+ | \exists \lambda \in \mathbb{R}^n_+ : x \geq \textstyle\sum_{i \in I} \lambda^i \tilde{x}^i \text{ and } y \leq \sum_{i \in I} \lambda^i \tilde{y}^i \}$$

Now to examine if the production plan of DMU^j, $(\tilde{x}^j, \tilde{y}^j)$, can be process efficient, we insert the estimated $\widehat{T}$ instead of Q in **(P1')**. Also, because of the constant return to scale property, we let $Q^h = Q$. This gives us ($\mathbf{P_{DEA}}$)

$$
\begin{array}{llllll}
\min\limits_{x^h, y^h, \gamma^h} & \max\limits_{\lambda^h, \Delta^h} & \sum_{h\in H} \Delta^h & & & \\
s.t. & & \widetilde{x}^j & \leq & \sum_{h\in H} x^h & \\
& & \widetilde{y}^j & \geq & \sum_{h\in H} y^h & \\
& & x^h & \geq & \sum_{i\in I} \gamma^{hi}\widetilde{x}^i & \forall h \in H \\
& & y^h & \leq & \sum_{i\in I} \gamma^{hi}\widetilde{y}^i & \forall h \in H \\
& & x^h - \Delta^h e_x & \geq & \sum_{i\in I} \lambda^{hi}\widetilde{x}^i & \forall h \in H \\
& & y^h + \Delta^h e_y & \leq & \sum_{i\in I} \lambda^{hi}\widetilde{y}^i & \forall h \in H \\
& & (x^h, y^h) & \in & \mathbb{R}^r_+ \times \mathbb{R}^s_+ & \forall h \in H \\
& & \gamma^h, \lambda^h & \in & \mathbb{R}^n_+ & \forall h \in H \\
& & \Delta^h & \in & \mathbb{R}_+ & \forall h \in H
\end{array}
$$

where $e_x = (1, .., 1) \in \mathbb{R}^r_+$ and $e_y = (1, .., 1) \in \mathbb{R}^s_+$. The equivalence is obvious once it is observed that the old hypothetical production plan q^h corresponds to $(-x^h, y^h)$ and that the old condition $q^h + \Delta^h \in Q$ corresponds to the condition $x^h - \Delta^h e_x \geq \sum_{i\in I} \lambda^{hi}\widetilde{x}^i$ on the input side and $y^h + \Delta^h e_y \leq \sum_{i\in I} \lambda^{hi}\widetilde{y}^i$ on the output side. As above, $(\widetilde{x}^j, \widetilde{y}^j)$ is potentially process efficient if and only if the optimal value of this program is 0.

For given values of γ^h and $(x^h, y^h), h \in H$, the inner maximization problem is just a standard linear programming problem:

$$
\begin{array}{lllll}
\max\limits_{\lambda^h, \Delta^h} & \sum_{h\in H} \Delta^h & & & \\
s.t. & \Delta^h e_x + \sum_{i\in I} \lambda^{hi}\widetilde{x}^i & \leq & x^h & \forall h \in H \\
& \Delta^h e_y - \sum_{i\in I} \lambda^{hi}\widetilde{y}^i & \leq & -y^h & \forall h \in H \\
& \lambda^h & \in & \mathbb{R}^n_+ & \forall h \in H \\
& \Delta^h & \in & \mathbb{R}_+ & \forall h \in H
\end{array}
$$

Introducing dual variables $\mu^h \in \mathbb{R}^r_+$ for the first set of constraints and $\nu^h \in \mathbb{R}^s_+$ for the second set of constraints, we get

$$
\begin{array}{llllll}
\min\limits_{\mu^h, \nu^h} & \sum_{h\in H} \mu^h x^h & - & \sum_{h\in H} \nu^h y^h & & \\
s.t. & \mu^h\widetilde{x}^i - \nu^h\widetilde{y}^i & \geq & 0 & & \forall i \in I, h \in H \\
& \mu^h e_x + \nu^h e_y & \geq & 1 & & \forall h \in H \\
& (\mu^h, \nu^h) & \in & \mathbb{R}^r_+ \times \mathbb{R}^s_+ & & \forall h \in H
\end{array}
$$

This program picks process input and output prices so as to minimizes the sum of the deficit created by these subject to the conditions that none of the realized productions plans can be profitable. Now, combining with the outer minimization problem, we get the following *quadratic* formulation ($\mathbf{P_{DEA}}$")

$$
\begin{array}{llcll}
\min\limits_{x^h, y^h, \mu^h, \nu^h, \gamma^h} & \sum_{h\in H}\mu^h x^h & - & \sum_{h\in H}\nu^h y^h & \\
s.t. & \sum_{h\in H} x^h & \geq & \widetilde{x}^j & \\
 & \sum_{h\in H} y^h & \leq & \widetilde{y}^j & \\
 & x^h & \geq & \sum_{i\in I}\gamma^{hi}\widetilde{x}^i & \forall h\in H \\
 & y^h & \leq & \sum_{i\in I}\gamma^{hi}\widetilde{y}^i & \forall h\in H \\
 & \mu^h\widetilde{x}^i - \nu^h\widetilde{y}^i & \geq & 0 & \forall i\in I, h\in H \\
 & \mu^h e_x + \nu^h e_y & \geq & 1 & \forall h\in H \\
 & (x^h, y^h) & \in & \mathbb{R}^r_+ \times \mathbb{R}^s_+ & \forall h\in H \\
 & (\mu^h, \nu^h) & \in & \mathbb{R}^r_+ \times \mathbb{R}^s_+ & \forall h\in H \\
 & \gamma^h & \in & \mathbb{R}^n_+ & \forall h\in H
\end{array}
$$

Hence, to test if DMU^j may be process efficient using a DEA model, we must minimize a quadratic function over a polyhedral convex set. Positive values correspond to a rejection of the hypothesis that DMU^j is potentially process efficient. We see that ($\mathbf{P_{DEA}}$") corresponds to (**P1"**) with the exception that we have $\alpha^h = 0$ here since the process production possibility sets Q^h are constant return to scale sets such that all hyperplanes supporting efficient productions will contain $0 \in \mathbb{R}^m$ also. (If we changed condition 3 to hold for all $k \in [0,1]$ or simply for $k = 1$ we would get the so-called decreasing return to scale and varying return to scale models introduced by Banker, Charnes and Cooper(1984), and we would again need the α^h constants).

The other programs could be adapted to the DEA setup in a similar fashion. For now, we turn instead to a simple numerical illustration.

Consider the following single input two outputs aggregate productions plans from five DMUs:

DMU^i	1	2	3	4	5
Inputs $\widetilde{x}^i$	1	1	1	1	1
Output 1 $\widetilde{y}^i_1$	2	0	1	3/2	4/3
Output 2 $\widetilde{y}^i_2$	0	2	1	1/4	4/3
Aggrate Eff. ?	Yes	Yes	No	No	Yes

DMU^3 and DMU^4 are clearly inefficient in the aggregate model, cf. also

Figure 3a below. We shall now investigate if this may be the result of process aggregations.

Assume that we estimate the aggregate production possibility set using the DEA model above. Also, as above, we let the processes have the same production possibility set. Finally, we assume that it is reasonable to allow two processes. The DEA-based potential process efficiency test program ($\mathbf{P_{DEA}}$") for DMU^3 now reads:

$$
\begin{array}{ll}
\min & z = \mu^1 x^1 + \mu^2 x^2 - \nu_1^1 y_1^1 - \nu_2^1 y_2^1 - \nu_1^2 y_1^2 - \nu_2^2 y_2^2 \\
x^h, y^h, \mu^h, \nu^h, \gamma^h, h = 1,2 & \\
s.t. & x^1 + x^2 \geq 1 \\
& y_1^1 + y_1^2 \leq 1 \\
& y_2^1 + y_2^2 \leq 1 \\
& x^1 \geq \gamma^{11}1 + \gamma^{12}1 + \gamma^{13}1 + \gamma^{14}1 \\
& y_1^1 \geq \gamma^{11}2 + \gamma^{12}0 + \gamma^{13}1 + \gamma^{14}\frac{3}{2} \\
& y_2^1 \geq \gamma^{11}0 + \gamma^{12}2 + \gamma^{13}1 + \gamma^{14}\frac{1}{4} \\
& x^2 \geq \gamma^{21}1 + \gamma^{22}1 + \gamma^{23}1 + \gamma^{24}1 \\
& y_1^2 \geq \gamma^{21}2 + \gamma^{22}0 + \gamma^{23}1 + \gamma^{24}\frac{3}{2} \\
& y_2^2 \geq \gamma^{21}0 + \gamma^{22}2 + \gamma^{23}1 + \gamma^{24}\frac{1}{4} \\
& \mu^1 1 - \nu_1^1 2 - \nu_2^1 0 \geq 0 \\
& \mu^1 1 - \nu_1^1 0 - \nu_2^1 2 \geq 0 \\
& \mu^1 1 - \nu_1^1 1 - \nu_2^1 1 \geq 0 \\
& \mu^1 1 - \nu_1^1 \frac{3}{2} - \nu_2^1 \frac{1}{4} \geq 0 \\
& \mu^1 1 - \nu_1^1 \frac{4}{3} - \nu_2^1 \frac{4}{3} \geq 0 \\
& \mu^2 1 - \nu_1^2 2 - \nu_2^2 0 \geq 0 \\
& \mu^2 1 - \nu_1^2 0 - \nu_2^2 2 \geq 0 \\
& \mu^2 1 - \nu_1^2 1 - \nu_2^2 1 \geq 0 \\
& \mu^2 1 - \nu_1^2 \frac{3}{2} - \nu_2^2 \frac{1}{4} \geq 0 \\
& \mu^2 1 - \nu_1^2 \frac{4}{3} - \nu_2^2 \frac{4}{3} \geq 0 \\
& \mu^1 + \nu_1^1 + \nu_2^1 \geq 1 \\
& \mu^2 + \nu_1^2 + \nu_2^2 \geq 1 \\
& x^1, y_1^1, y_2^1, x^2, y_1^2, y_2^2 \geq 0
\end{array}
$$

Solving this quadratic optimization problem, we find that the optimal solutions are given by

$$
\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix},
$$

$$\begin{pmatrix}\mu^1\\ \nu^1\end{pmatrix} = \begin{pmatrix}\tau\\ \frac{\tau}{2}\\ \frac{\tau}{4}\end{pmatrix}, \quad \begin{pmatrix}\mu^2\\ \nu^2\end{pmatrix} = \begin{pmatrix}\tau\\ \frac{\tau}{4}\\ \frac{\tau}{2}\end{pmatrix}, \quad z = 0$$

for any $\tau \geq 4/7$. This shows that DMU^3 is in fact process efficient. Of course, this is not surprising in this simple case where we could draw the test-problem; the production plan of DMU^3 is equal to the sum of the two efficient plans $(\frac{1}{2}\tilde{x}^1, \frac{1}{2}\tilde{y}^1)$ and $(\frac{1}{2}\tilde{x}^2, \frac{1}{2}\tilde{y}^2)$, and therefore DMU^3 must be process efficient as well. The set of process efficient plans are depicted in Figure 3a below. We see that DMU^4 is not process efficient.

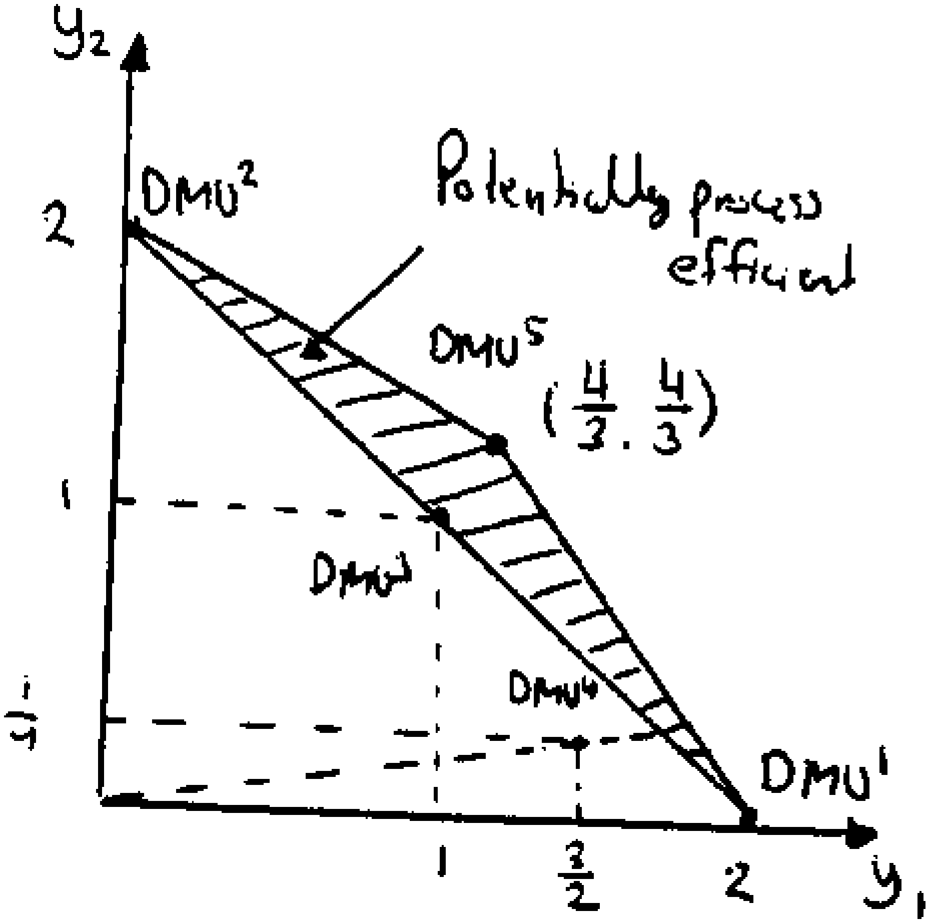


Fig 3a

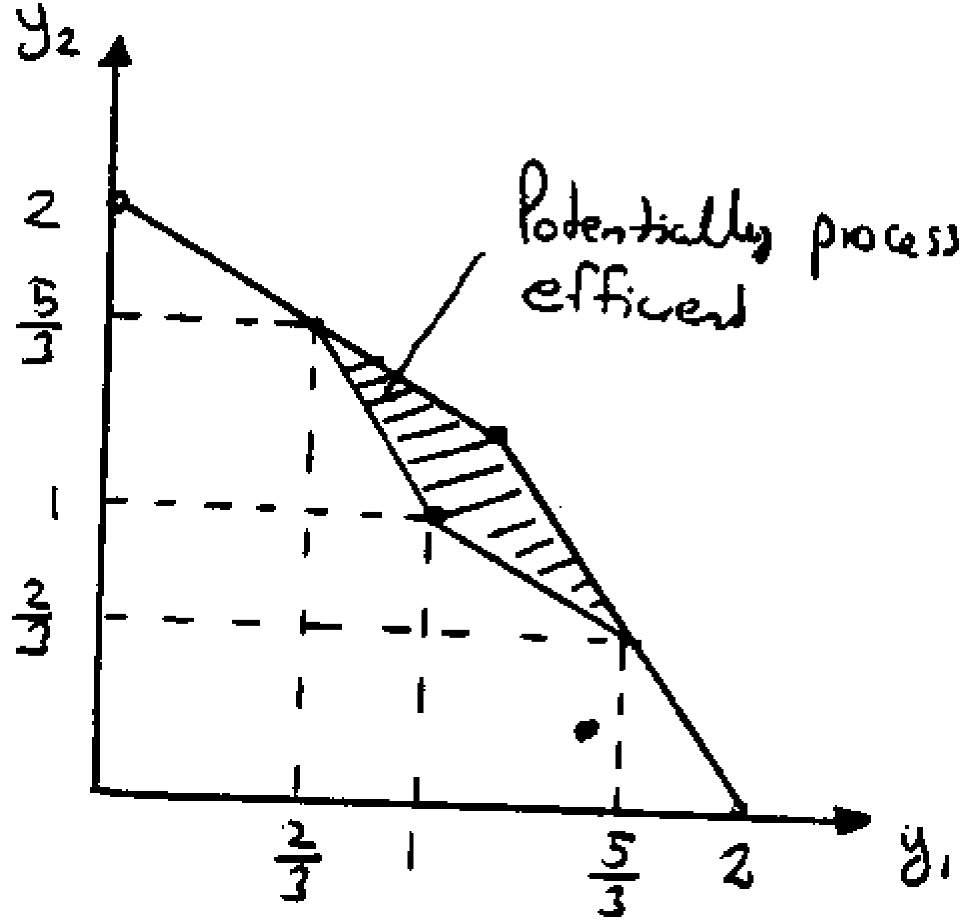


Fig 3b

Figure 3

Using the traditional Farrell output based efficiency measure corresponding to the direction $d = (0, q_+)$, we get $\varepsilon = 0$ for DMU^3 in the process corrected model **(P2)** as opposed to $\varepsilon = 1/3$ in the un-corrected model. The corrected ε score for DMU^4 is $1/7 = 0.14$, while the un-corrected score is $3/13 = 0.23$. This is obvious from the geometry of the problem, and it can be seen by solving the **(P2")** model. For DMU^4 and using the parametrization above, we get:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ \frac{12}{7} \\ 0 \end{pmatrix}, \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} \\ 0 \\ \frac{2}{7} \end{pmatrix}, \gamma^1 = \begin{pmatrix} \frac{6}{7} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 \\ \frac{1}{7} \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \mu^1 \\ \nu^1 \end{pmatrix} = \begin{pmatrix} \tau \\ \frac{\tau}{2} \\ \frac{\tau}{4} \end{pmatrix}, \begin{pmatrix} \mu^2 \\ \nu^2 \end{pmatrix} = \begin{pmatrix} \tau \\ \frac{\tau}{4} \\ \frac{\tau}{2} \end{pmatrix}, \varepsilon = \frac{1}{7}$$

for any $\tau \geq 4/7$.

Of course the convexity here of the set of production plans that are not process efficient - or just so - (i.e. the production possibility set minus the shaded area) is a result of our ability to scale up and down. This means that any convex combination $\alpha q^1 + (1-\alpha)q^2$ can be thought of as a sum: $\alpha q^1+(1-\alpha)q^2 = q^{*1}+q^{*2}$, where $q^{*1} = \alpha q^1 \in \alpha Q = Q$ and $q^{*2} = (1-\alpha)q^2 \in (1-\alpha)Q = Q$. If we assume instead that the processes can only work within the restriction of the DEA model in which the input level is $\frac{1}{2}$, then we get a much reduced set of process efficient plan, and a non-convex set of "not or just" process efficient plans, cf. Figure 3b above.

8 Conclusions

Most plants, firms or organizations include several processes operated in parallel as well as over time. When these processes are described in aggregate terms, the adaption to variations in supply and demand over space and time will be suppressed. In fact, swift and attractive process variations may lead to non-favorable aggregate evaluations. We showed that *to be efficient at the aggregate level, all processes should be coordinated*, i.e. be allocatively efficient with respect to the same price vector.

To cope with this aggregation problem, we suggested making imaginary disaggregations. We *find the hypothetical process productions that make a DMU look as favorable as possible.* If it is possible to find process production plans that are all efficient and that aggregate into the inefficient outcome actually observed, we say that the aggregate outcome "is potentially process efficient". In such cases, we suggest being cautious when making conclusions based solely on aggregate data.

There are many relevant *extensions* of the research initiated here. One is to use the hypothetical disaggregation procedure to the problem of variable aggregation, e.g. the aggregation of different types of capital. If we can find hypothetical disaggregations that are consistent with the aggregate data but which put a DMU in a significantly different light, caution is called for. Another extension is to introduce alternative efficiency concepts, production models and dualizations. A final and from an applied point of view probably the most significant extension would be to investigate more thoroughly how to handle a lack of information about the production possibilities in the processes. Endogenous choice of the processes and in particular their numbers is attractive, but it makes it necessary also to use additional information, say partial price or preference information, to restrain the flexibility of having a large number of processes.

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