(*This document has been specially formatted for double sided printing*)

Faculty Paper Series

Faculty Paper 00-4 November 1999

Sustainability as Intergenerational Fairness

By

Richard T. Woodward

Assistant Professor

Department of Agricultural Economics

Texas A&M University

College Station, TX 77843-2124

r-woodward@tamu.edu

Sustainability and Intergenerational Fairness*

By

Richard T. Woodward
Assistant Professor, Department of Agricultural Economics
Texas A&M University

College Station, TX 77843-2124
r-woodward@tamu.edu

Abstract: This paper presents an economic model of *sustainability* defined as intergenerational fairness. Assuming that intergenerational fairness is an obligation of each generation, a recursive optimization problem is obtained. The problem has the advantage that uncertainty can readily be incorporated in the model and it can be solved numerically for a wide range of specifications. The possibility of tradeoffs between efficiency and sustainability are discussed. Under plausible conditions, it is shown that a sustainability obligation is met only if there is the expectation of economic growth.

November 1999

---

* The author thanks Richard Bishop, Jean-Paul Chavas, Bill Provencher, Mario Miranda, Antony Scott and anonymous reviewers for useful comments and/or conversations. Partial support for this research was provided by the Texas Agricultural Experiment Station and the University of Wisconsin Sea Grant Institute, Federal Grant number NA46RG0481, project number R/PS-46.

Sustainability as Intergenerational Fairness:
Efficiency, Uncertainty and Numerical Methods
By Richard T. Woodward

1. Introduction

Throughout the history of economic thought there has been enormous interest in the issue of intergenerational equity (Ramsey 1928, Koopmans 1960). In recent years, most of this discussion has taken place using the terms *sustainable development* or *sustainability* and has paid particular attention to the role of natural resources and the environment in sustaining economic wellbeing. Both optimal control (e.g., Solow 1974) and overlapping generations (e.g., Howarth 1991) models have been used to address whether a sustainable economy is feasible and efficient. An excellent review of this literature is provided by Toman, Pezzey and Krautkraemer (1995).

The present paper contributes to this literature in three ways. First, we incorporate into our model an interpretation of sustainability based on the Foley's (1967) principle of fairness -- a generation is defined as behaving sustainably if it does not expect to be envied by future generations. Given Rawls' (1971) characterization of justice as fairness, our approach is conceptually quite similar to a Rawlsian maximin objective that has predominated in the economics literature.[1] Following Riley (1980) and similar to writings by political philosophers (Laslett and Fishkin, 1992), sustainability is treated as an obligation of the current generation to future generations. Since the welfare of each generation is assumed to be altruistic, we obtain results similar to those of Calvo (1978).

Second, we explicitly incorporate risk into the analysis. Ironically, although long-term uncertainty is one of the central issues associated with sustainability, it has received scant treatment in the economics literature. Notable exceptions are Howarth (1995, 1997), Toman (1994) and Asheim and Brekke (1993). While Howarth (1995) suggests the use a sustainability criterion equivalent to the one we propose below, both he and Toman end up appealing to rules of thumb for policy formation to achieve sustainability. Although such rules may be useful in practice, in this paper we seek to understand their theoretical underpinnings so as to better evaluate their ability to achieve the goals of efficiency and sustainability. Our framework has many similarities to Asheim and Brekke (1993), though they use a non-altruistic framework.

Finally, we show that numerical methods can be used to find optimal-sustainable policies. This represents an important contribution in that it substantially broadens the scope of the problems that economists can consider. Analytical methods are limited because closed-form solutions can be obtained only for relatively simple problems. The introduction of numerical methods, therefore, can broaden the spectrum of problems that

---

[1] The problem of intergenerational equity has been studied using a maximin objective at least since Solow (1974) and continues to this day (Lauwers 1997 and Pezzey and Withagen 1998).

might be considered. These methods are then used to evaluate sustainability in simple one- and two-dimensional economies.

2. The Model

We assume that the welfare of generation t can be written as an additive function of the infinite stream of utility of all future generations, i.e.,

$$U\left({}_t z, {}_t x, {}_t e\right) = E_t \sum_{s=t}^{\infty} b^{s-t} u\left(z_s, x_s, e_s\right) \tag{1}$$

where $x_t \in X$ is a vector describing generation t's endowment, z_t is the vector of choices made by the t^{th} generation and $e_t \in \Lambda$ is the vector of i.i.d. stochastic shocks that occur after the choices, z_t, have been made. The notation ${}_t z$ indicates the infinite series of choices, z_t, z_{t+1}, z_{t+2}, …, and likewise for ${}_t x$ and ${}_t e$. The parameter $b<1$ is the societal discount factor[2] and E_t is the expectation operator contingent on the information available to generation t as captured in the endowment vector x_t. The functions $u(\cdot)$ and $U(\cdot)$ will be called the generational utility and welfare functions respectively. We assume that there is a set of m stationary transition functions, $x_{t+1}^i = g^i\left(z_t, x_t, e_t\right)$, i=1,…, m, which we will frequently write in vector notation

$$x_{t+1} = g\left(z_t, x_t, e_t\right). \tag{2}$$

The set of feasible choices depends on the state of the economy, $\Gamma(x_t) \subset \mathbb{R}^n$.

Assuming each generation chooses to maximize its welfare, (1) can be rewritten recursively,

$$V\left(x_t\right) = \max_{z_t \in \Gamma\left(x_t\right)} E_t u\left(z_t, x_t, e_t\right) + bE_t V\left(x_{t+1}\right) \text{ s.t. (2).} \tag{3}$$

Sustainability as intergenerational fairness

Most of the economic analysis concerned with *sustainability* has taken its motivation from Rawls' principle of justice.[3] Accordingly, economists have evaluated numerous problems in which the objective is to maximize the minimum utility across all future generations (e.g., Solow 1974, Hartwick 1977). A significant contribution to this literature was made by Calvo (1978) who modified the model by assuming that welfare is equal to the discounted sum of future utility (see also Rodriguez 1981 and Asheim 1988).

---

[2] Pezzey (1997) explores conditions where the discount factor may vary over time. While this innovative approach merits further investigation, we follow the standard approach of using a constant-discount factor.

[3] Chichilnisky (1996, 1997) provides an important alternative approach to the issue of sustainability. Using two axioms that rule out dictatorship of either the present or the indefinite future, she derives an objective function made up of the weighted sum of a present-value function and the least utility of the distant future. Neither a Rawlsian objective function nor the sustainability-constrained objective put forth here satisfy these axioms. On the other hand, Chichilnisky's objective does not necessarily lead to intergenerationally fair outcomes as defined here.

Calvo's extension allows optimal maximin plans to be dynamically consistent and can lead to growth over time.

In the present paper we use a slightly different approach, basing our analysis on the principle of intergenerational fairness. Foley (1967) defines an allocation as fair, "if and only if each person in the society prefers his [or her] consumption bundle to the consumption bundle of every other person in the society" (p. 74). In a static economy, an allocation of a multidimensional endowment *x* between *n* individuals is fair if each individual, *i*, *j*, prefers his or her bundle to that of every other individual, i.e., $u_i(x_i) \geq u_i(x_j)$ for all *i*, *j*.[4]

Following Foley, we define choices by the current generation as *intergenerationally unfair* if, given z_t, either the current generation envies future generations or future generations envy the present, and there exists an alternative feasible choice such that there is no envy. We define a set of choices, z_t, as *sustainable* or *consistent with sustainability* if they are intergenerationally fair.

A set of choices, z_t, is fair to future generations if

$$\text{FF}_t\text{:} \quad E_t u(z_t, x_t, e_t) + b\, E_t U({}_{t+1}z, {}_{t+1}x, {}_{t+1}e)\Big|_{z_t} \leq E_t U({}_{t+j}z, {}_{t+j}x, {}_{t+j}e)\Big|_{z_t} \text{ for all } j=1, 2, \ldots .^5 \qquad (4)$$

To take the expectation in (4), generation *t* must make some assumptions about how future generations will make choices. For example, the endowment of generation *t*+2 depends not only on the choices and endowment of generation *t*, but also on the choices of generation *t*+1. The assumption regarding future choices would typically take the form of a *policy rule*, a mapping from the endowment vector, *x*, to a choice vector, *z*. In this paper we will assume that future generations follow a rule that treats their descendants fairly.

Assumption a. FF_s is satisfied for all future generations, *s*=*t*+1,*t*+2, … .

If assumption a does not hold, then virtually any effort to treat distant generations fairly might be undone by the unfair choices of intervening generations and choices made in *t* would be dynamically inconsistent. The following proposition shows that if assumption a holds then sustainability is achieved if each generation is incrementally fair to the following generation.

Proposition 1: FF_t is satisfied for all *t* if and only if

$$\text{IFF}_t\text{:} \quad E_t u(z_t, x_t, e_t) + b\, E_t U({}_{t+1}z, {}_{t+1}x, {}_{t+1}e)\Big|_{z_t} \leq E_t U({}_{t+1}z, {}_{t+1}x, {}_{t+1}e)\Big|_{z_t} \qquad (5)$$

is satisfied for all *t*.

---

[4] Following Foley's original definition, the issue of fairness received substantial attention from economists during the 1970s, leading to a number of alternative definitions (Varian 1974, Pazner 1977, Pazner and Schmeidler 1974, 1978). See Thomson and Varian (1985) for a summary. Chavas (1994) represents a recent application of the framework of fairness to policy analysis in agriculture.

[5] This definition was also proposed by Howarth (1995) and is the discrete-time and stochastic analogue of Riley's (1980) definition.

Proof: $FF_t \Rightarrow IFF_t$: FF_t requires that the fairness inequality hold for all $j=t+1, t+2,\ldots$, while IFF_t requires only that the fairness inequality is satisfied for $j=t+1$.

$IFF_t \Rightarrow FF_t$: IFF_{t+1} implies

$$E_{t+1}u(z_{t+1}, x_{t+1}, e_{t+1}) + b\,E_{t+1}U(z_{t+2}, x_{t+2}, e_{t+2})\Big|_{z_{t+1}} \le E_{t+1}U(z_{t+2}, x_{t+2}, e_{t+2})\Big|_{z_{t+1}}.$$

Taking expectations based on the information available to generation t and contingent on the decision z_t, we have,

$$E_t\left[E_{t+1}u(z_{t+1}, x_{t+1}, e_{t+1}) + b\,E_{t+1}U(z_{t+2}, x_{t+2}, e_{t+2})\Big|_{z_{t+1}}\right]\Bigg|_{z_t} \le E_t\left[E_{t+1}U(z_{t+2}, x_{t+2}, e_{t+2})\Big|_{z_{t+1}}\right]\Bigg|_{z_t} \quad (6)$$

Using the law of iterated expectations, this inequality can be rewritten

$$E_t\,U(z_{t+1}, x_{t+1}, e_{t+1})\Big|_{z_t} \le E_t\,U(z_{t+2}, x_{t+2}, e_{t+2})\Big|_{z_t}$$

Hence, IFF_t and IFF_{t+1} together imply

$$E_t u(z_t, x_t, e_t) + bE_t\,U(z_{t+1}, x_{t+1}, e_{t+1})\Big|_{z_t} \le E_t\,U(z_{t+2}, x_{t+2}, e_{t+2})\Big|_{z_t}.$$

By induction, the same relationship holds replacing $t+2$ with $t+3$ and so on. ||

For non-altruistic preferences, $b=0$, Proposition 1 is equivalent to Asheim and Brekke's (1993) Lemma 3. This proposition leads to what Howarth (1992) refers to as a "chain of obligation" in which an obligation to treat our immediate descendants fairly implies an obligation to all future generations. But Proposition 1 is more general -- not only does an obligation to the next generation imply that all future generations should be treated fairly, but an obligation of fairness to all future generations is satisfied if each generation is suitably fair to the following generation.

As we have defined it, sustainability is a symmetric criterion, implying a lack of envy by future generations of the present and vice versa. The problem of intergenerational choice, however, is fundamentally asymmetric. While the present is able to act without regard to the interests of the future, the future has no choice but to accept the actions of the present (Bromley 1989). If the present generation is acting in its best interest it will maximize its welfare, minimizing the extent to which it envies future generations. This temporal advantage ensures that, to the extent possible, the present should not envy the future.

The temporal disadvantage of the future, on the other hand, means that there is no guarantee that optimal behavior will lead to choices that are fair to future generations. Laslett (1992) maintains that the rights of earlier generations must be matched by duties to generations yet to come. However, there is nothing in the standard present-value optimization criterion that would require that future generations be treated fairly. To ensure that sustainability is achieved, therefore, the optimization problem that the current generation solves must be altered.

3. Sustainability-constrained optimization

Intergenerational fairness requires that, to the extent possible, the current generation is both fair to the future and to fair to itself. This is achieved if all generations solve the following sustainability-constrained optimization problem,

$$\begin{aligned} &V^S(x_t) = \max_{z_t} E_t u(z_t, x_t, e_t) + bE_t V^S(x_{t+1}) \; s.t. \\ &\text{i)}\; z_t \in \Gamma(x_t) \\ &\text{ii)}\; x_{t+1} = g(z_t, x_t, e_t) \\ &\text{iii)}\; V^S(x_t) \le E_t V^S(x_{t+1}). \end{aligned} \tag{7}$$

The first and second constraints identify intra- and intertemporal feasibility respectively. The third constraint is the sustainability constraint. We denote the value function of this problem $V^S(x)$ to differentiate it from $V(x)$ defined in (3).

The existence of a solution to (7) is ensured by an assumption of free disposal. Let $\underline{u}$ be the lowest possible level of instantaneous utility across the entire domain, i.e., $\underline{u} = \inf_{z,x,e} u(z,x,e)$.

> Assumption b. For all $x \in X$ there exists some $\underline{z} \in \Gamma(x)$ such that $u(\underline{z},u,e)=\underline{u}$ for all $e \in \Lambda$.

Assumption b simply means that any generation can always reach the lowest possible level of utility by acting in a suitably wasteful manner. It can easily be shown (see Lemma 3 in Appendix 1) that choosing $\underline{z}$ forever satisfies the three constraints of (7).

Incorporating sustainability as a constraint will be disconcerting to some readers. It might be argued that if society *chooses* to constrain itself then there must be some other, more fundamental, objective function that it is truly maximizing.[6] Dasgupta and Mäler (1995) argue that this underlying function should be modeled to allow the consideration of tradeoffs between sustainability and other goals. Sen (1997) provides two reasons why explicit analysis of the constraint might be useful. First, he shows that modeling self-imposed constraints directly can be functionally important. Secondly,

> Even if it were the case that -- 'ultimately' -- everything were determined by 'basic' preferences exclusively over *culmination outcomes*, it would still be interesting and important to see how the derived preferences ("nonbasic" but functionally important) actually work in relation to the choice act (p. 749, emphasis in original).

---

[6] While many economists view a constraint-based approach with suspicion, one referee has pointed out that using a social welfare maximization approach to address principles of fairness is "out of fashion in moral philosophy and political theory." In that literature the constraint-based approach is quite well developed.

Hence, while we do not discard the objective function approach as irrelevant, we argue that modeling sustainability as a constraint, can yield important insights into how policy might evolve *if* intergenerational fairness is held as a generational obligation.[7]

4. The solution of sustainability-constrained optimization problems

Having proposed (7) as an interesting social optimization problem, we now need to consider whether such a problem can actually be solved. This is not a trivial question. The sustainability-constrained value function, $V^S(x)$, appears four times in the statement of the problem, on both the right- and left-hand sides of the objective function and twice again in the sustainability constraint. Yet this function, like the unconstrained value function in standard dynamic programming problems, is unknown *a priori*. Except in very restrictive cases, it is impossible to solve (7) analytically and typically there will be no closed form for $V^S(\cdot)$.[8] Fortunately, as we discuss here and prove in Appendix 1, a unique sustainability-constrained value function exists for a wide variety of economies and can be found using numerical methods.

The algorithm that can be used to solve sustainability-constrained optimization problems proceeds in two basic steps. The first step is to find the solution to the unconstrained optimization problem, i.e., (7) without the sustainability constraint, (iii).[9] We will call the value function associated with the unconstrained optimization problem $V^*(x)$.

Then, using the unconstrained value function as our initial guess, $V^0=V^*$, the sustainability-constrained value function, $V^S(x)$, is found by recursively solving the problem

$$\begin{aligned} &V^k(x_t) = \max_{z_t} Eu(z_t, x_t, e_t) + bEV^{k-1}(x_{t+1}) \ s.t. \\ &\text{i) } z_t \in \Gamma(x_t) \\ &\text{ii) } x_{t+1} = g(z_t, x_t, e_t) \\ &\text{iii) } Eu(z_t, x_t, e_t) + bEV^{k-1}(x_{t+1}) \le EV^{k-1}(x_{t+1}). \end{aligned} \tag{8}$$

at each point in the state space. Proposition 4 in Appendix 1 proves that this algorithm leads to the unique sustainability-constrained value function, V^S, as $k\rightarrow\infty$.

---

[7] Since the constraint is self-imposed it is possible that a generation may choose to violate it if sustainability were in conflict with some other objective such as Pareto efficiency. We discuss the possibility of such a conflict below.

[8] Two examples of simple stochastic models for which an analytical solution is possible are provided by Asheim and Brekke (1993) for the case with b=0. Under certainty, the Dasgupta and Heal (1974, 1979) capital-resource economy that has been studied by many authors can also be solved analytically (e.g., Stiglitz 1974, Pezzey and Withagen 1998).

[9] This problem could be solved by a number of methods (see Judd 1998) including successive approximations of the value function (Bertsekas 1976).

While numerical methods can be used to solve many problems, they do face some notable limitations. Most importantly, since computers must operate in finite space, only problems that are bounded can be solved using these methods.

> Assumption c. The sets X, Λ, and $\Gamma(x)$ are all closed and bounded and the instantaneous utility function $u(x,z,e)$ is defined over the entire domain.

This assumption is not met in all economies of interest. For example, it rules out the use of a logarithmic utility function defined over the non-negative real numbers since $\lim_{x\to 0}\ln(x)$ is undefined. Furthermore, it rules out problems in which the solution requires unbounded accumulation of one or more assets as in Dasgupta and Heal's (1974, 1979) capital-resource economy. Nonetheless, some problems that are naturally unbounded can be solved approximately by innocuously modifying state space and/or utility function.

5. Sustainability under uncertainty

It is worth pausing momentarily to consider the role of uncertainty in our sustainability-constrained model. While the use of the expectation operator seems a natural extension of the subjective expected utility framework to the problem of intergenerational fairness, two conceptual complications arise. First, z_t and $\boldsymbol{e}_t$ determine not only the endowment in t+1 but, in effect, the very generation that arrives in t+1. Howarth (1995, 422) argues, therefore, that the expectation operator is adding across "logically distinct state-contingent generations." If this is true and sustainability requires the current generation to be fair to any possible future generation, then a sustainability criterion must hold not on average, but for all possible values of $\boldsymbol{e}$. A disadvantage of such a criterion is that it could be impossible to achieve or could require choices that would in most states leave both current and future generations worse off than without a sustainability obligation. Moreover, a requirement that $V(x_t)\leq V(g(z_t,x_t,\boldsymbol{e}_t))$ for all possible $\boldsymbol{e}_t$ would likely be unfair to the current generation. Such a constraint would almost certainly lead to a situation in which the current generation envies the future.

The second complication that arises in the treatment of uncertainty is the applicability of the subjective expected utility (SEU) framework to represent the preferences of each generation. There is an enormous body of evidence suggesting that individuals frequently violate the axioms of the SEU hypothesis (see Machina 1987). This is particularly true when faced with highly uncertain problems and the normative foundation for policy choice in such problems has been questioned (Manski 1996, Woodward and Bishop 1997). Asheim and Brekke (1993) point out that the tendency to place extra weight on highly negative consequences with small probabilities is also relevant when considering the issue of sustainability. While we believe that these problems with the SEU structure are important and merit further discussion, we retain the SEU framework here on the belief that improved understanding of sustainability will be achieved incrementally.

Our definition of sustainability has direct implications for the time-path of welfare under risk. Consider the case where x is one-dimensional. The sustainability constraint requires that $E_t V^S(x_{t+1}) \geq V^S(x_t)$. If $V^S(\cdot)$ is monotonically increasing and strictly concave, then, by Jensen's inequality, sustainability is satisfied only if the current generation expects the endowment to grow.

The extension of this result to the case of a multiple dimensional state space requires more careful specification of what is meant by growth. Let the endowment vector x_t be composed of m elements, $x_t^1,\ldots x_t^m$, each of which is defined so that in the neighborhood around the actual current endowment, x_t, sustainable welfare is increasing in x_t^i for all i. Taking a first order approximation of the change in $V^S(x)$ with respect to a change from x_t to x_{t+1} yields

$$\Delta V^S \approx \sum_i \frac{\partial V^S(x_t)}{\partial x^i} \cdot \left(x_{t+1}^i - x_t^i\right).$$

Dividing through by the $\partial V^S/\partial x^1$ yields an approximate change in the value of the endowment measured in terms of the marginal value of x^1,

$$\Delta V^S \Big/ \frac{\partial V^S(x_t)}{\partial x^1} \approx \left[\sum_i \tilde{p}_t^i \cdot \left(x_t^i - x_{t+1}^i\right)\right] \tag{9}$$

where $\tilde{p}_t^i = \dfrac{\partial V^S/\partial x^i}{\partial V^S/\partial x^1}$ and $\tilde{p}_t^1 = 1$. Generation t's expectation of the change in the value of the endowment can, therefore, be approximated,

$$\frac{E_t \Delta V^S}{\partial V^S(x_t)/\partial x^1} \approx \int_e \left[\sum_i \tilde{p}_t^i \cdot \left[x_t^i - g^i(x_t, z_t, e_t)\right]\right] h(e_t; x_t)\, de_t\,, \tag{10}$$

where $h(\cdot)$ is the joint probability density function of e_t conditional on x_t. If the right-hand side of (10) is equal to zero then, measured in period t prices, the expected value of generation t+1's endowment is approximately equal to the value of generation t's endowment. However, if $V^S(\cdot)$ is strictly concave the sign of the error due to the linear approximation (10) is negative. Hence, if concavity holds, then sustainability is *not* satisfied if generation t expects that, on average, the value of generation t+1's endowment will be just equal to that of its own endowment.

There has been substantial attention of the use of the framework of national accounting to evaluate the sustainability of economies (e.g., Hartwick 1977, 1990; Solow 1993). The basic lesson of this literature (under very restrictive conditions regarding population, technology and preferences and with an important caveat due to Asheim (1994) and Pezzey (1994)[10]) has been that the maintenance of the value of an economy's endowment is an indicator of sustainability. We find here a new implication for planning: under risk, sustainability can typically be achieved only if planners aim not for the maintenance of the economy's value, but for its growth.

---

[10] Asheim (1994) and Pezzey (1994) show that if the economy's value is calculated using market prices, then the value of the economy's assets can be growing over a period during which the consumption is at a level that cannot be sustained indefinitely.

6. Sustainability-constrained optimization and Pareto efficiency

Having proposed an alternative to the standard present-value optimization problem, we now consider the compatibility of this norm with Pareto efficiency. Let z_t^S be the optimal-sustainable policy for generation t. One way this policy might be inefficient is if there is an alternative policy, z_t that would either increase the welfare of generation t, or that of generation t+1, without diminishing the welfare of the other. That is, z_t is *stepwise Pareto superior* to z_t^S if

$$E_t u(z_t, x_t, e_t) + bE_t V^S\left(g(z_t, x_t, e_t)\right) \geq E_t u\left(z_t^S, x_t, e_t\right) + bE_t V^S\left(g\left(z_t^S, x_t, e_t\right)\right) \tag{11a}$$

and

$$E_t V^S\left(g(z_t, x_t, e_t)\right) \geq E_t V^S\left(g\left(z_t^S, x_t, e_t\right)\right) \tag{11b}$$

with a strict inequality holding in at least one case. If such an inefficiency were identified, then it seems likely that policy makers would want to reconsider their commitment to sustainability. As Proposition 2 shows, such inefficiencies are possible and are revealed in the shadow price on the sustainability constraint, l.

> Proposition 2: Assuming that the solution to the sustainability-constrained optimization problem (7), $\left(z_t^S, l\right)$, is a saddle point,[11] then, if l>1, this solution is not efficient and, if l<1, the solution is stepwise efficient.

The proof is provided in Appendix 1.

Stepwise inefficiencies arise if the sustainability constraint rules out Pareto superior choices. This conflict between efficiency and sustainability might occur for a variety of reasons. First, it can occur if the $V^S(x)$ is bounded. For example, consider an infinite-horizon cake-eating economy in which z represents consumption of a single nonrenewable stock, x, so that $x_{t+1}-x_t = z$, and with u(z=0)=0 and u'>0. Since no finite level of consumption can be sustained indefinitely, the sustainability-constrained optimal level of consumption is zero for all generations so that $V^S(x)=0$ for all $x\geq 0$. A Pareto improvement over the sustainability-constrained optimum, therefore, would be for any generation to consume a portion of the stock. But if z_t>0 then, generation t's welfare will exceed the sustainable welfare of all future generation's, violating the sustainability constraint. As we will see when we consider a renewable resource economy below, when the sustainability-constrained value function is bounded from above, the potential for inefficiencies is great.

In a diverse economy, however, it is quite unlikely that stepwise inefficiencies will arise. We will show that if there exists in an economy a means by which the welfare of all future generations can be increased, then the sustainability-constrained optimum cannot be stepwise inefficient. It is common for such opportunities to exist. For example, consider an economy in which utility is solely a function of the amount of corn consumed, c_t. Next period's seed stock, R_{t+1}, is inversely related to the amount of corn consumed today,

---

[11] This restriction is satisfied in many, but by no means all problems of interest. See Takayama (1985) for a discussion of conditions when a saddle point is not guaranteed.

$R_{t+1}=rR_t-c_t$, where $r>1$ is the crop's rate of growth. In such an economy, it is possible to increase consumption in all future periods by $(r-1)\Delta c_t$ simply by reducing consumption today by $\Delta c_t \le c_t$. Proposition 3 shows that when such an asset exists, the sustainability-constrained optimum will be efficient.

A *sustainably-productive alternative* to a choice z_t, given the endowment x_t, is an alternative $z_t' \in \Gamma(x_t)$, such that for some $d>0$

$$E_t U\left({}_{t+j}z, {}_{t+j}x, {}_{t+j}e\right)\Big|_{z_t'} \ge E_t U\left({}_{t+j}z, {}_{t+j}x, {}_{t+j}e\right)\Big|_{z_t} + d \quad \text{for all } j=1,2.,\ldots.$$

where the expectation operator on both sides is based on the same policy rule.

To ensure continuity, we make the following assumptions.

Assumption d: Utility can be discarded so that for any combination of z_t, x_t and e_t, and for any $a \in [\underline{u}, u(z_t, x_t, e_t)]$, there exists an alternative choice z_t' such that $u(z_t', x_t, e_t)=a$ and $g^i(z_t', x_t, e_t)=g^i(z_t, x_t, e_t)$ for all i.

Proposition 3: Suppose assumption d holds and there exist sustainably-productive alternatives for all $z_t \in \Gamma(x_t)$. If z_t^S is the sustainability-constrained optimal choice at x_t, then it is stepwise Pareto efficient.

The proof is provided in Appendix 1.

Proposition 3 shows that the potential for stepwise inefficiencies is quite limited. If there exists any way to increase the welfare of all future generations, regardless of the relative cost to the current generation, then this type of inefficiency is avoided. Hence, if there is an asset in the economy like corn in the example above that allows future production possibilities to be increased at the cost of current utility, then this conflict between efficiency and sustainability cannot arise.

7. Optimal-sustainable management in a one-dimensional resource economy

We now demonstrate the properties of sustainability-constrained economies in one- and two-dimensional economies. We begin with an economy dependent upon a single renewable resource, x. In each period the decision maker chooses z_t, the portion of the available stock to consume immediately. The remaining stock, $(1-z_t)(x_t+e_t)=\tilde{x}_t$, grows according to a logistic growth function,

$$x_{t+1} = \tilde{x}_t + r\,\tilde{x}_t\left(1-\frac{\tilde{x}_t}{\bar{x}}\right) \tag{12}$$

where r is the growth rate, $\bar{x}$ is the unexploited steady state and e_t is a normally distributed i.i.d. shock with mean zero and standard deviation s.[12] The portion of the stock that is consumed in t yields utility, $u(z_t,x_t,e_t)=(z_t(x_t+e_t))^{1-\gamma}$ with $\gamma<1$.

---

[12] For this model, the normal distribution is approximated using a third order Gaussian quadrature approximation using values from Miranda (1994).

We consider first the case of a policy maker in a deterministic economy (s=0). Figure 1 presents the time path of the resource that would follow from the maximization of each generation's welfare without imposing a sustainability constraint (which we call the PV-optimal policies). As expected (Clark 1976), the PV-optimal trajectory converges to a steady-state level, in this case x_m=1.253, less than the level at which the maximum sustainable yield is achieved, x_{msy}=1.40.

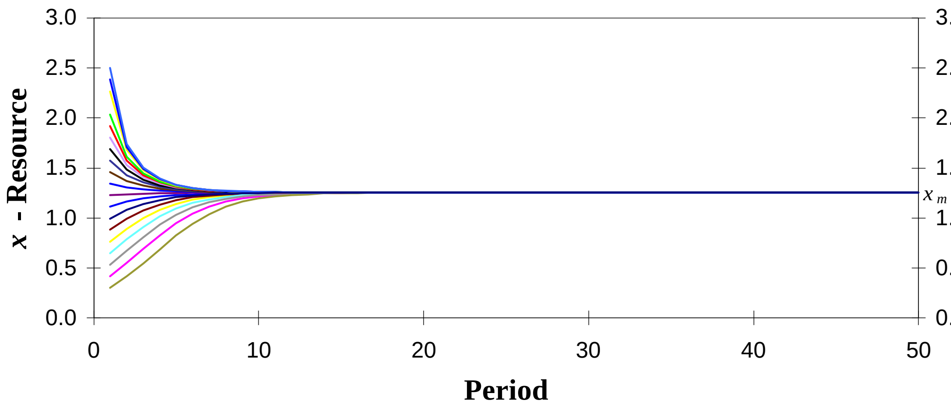

Figure 1: Evolution of the resource endowment in the unconstrained one-dimensional resource economy. Parameters of the model: b=0.9, r=0.8, $\bar{x}$=2, γ=0.9. Parameters of the numerical program (see Appendix 2): order of the Chebyshev polynomial, 30; bounds on state space, 0.3 and 2.5; convergence criterion 10^{-8}.

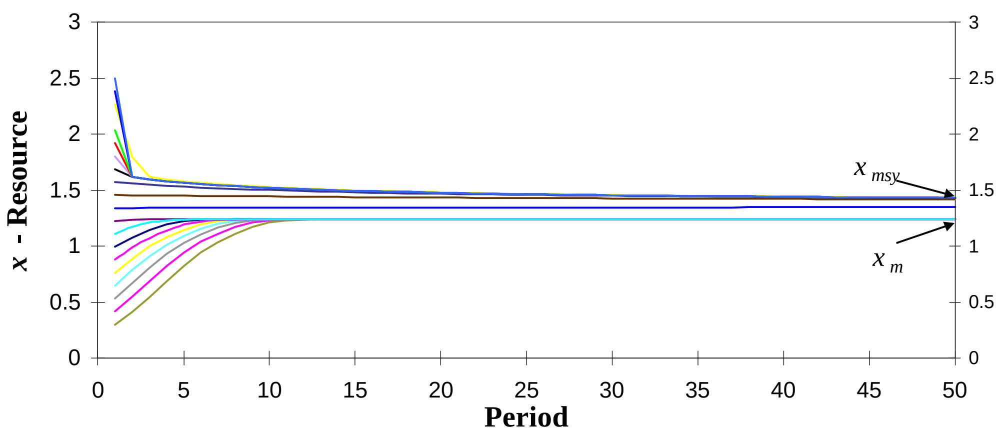

Figure 2: Resource endowment over time in the sustainability-constrained one-dimensional resource economy. For parameter values see Figure 1.

Some of the implications of imposing a sustainability constraint can be anticipated immediately and are equivalent to Calvo's (1978) results. Because the unconstrained value function is monotonically increasing in x, at any point from which the PV-optimal policy leads to growth in the resource stock over time, the sustainability constraint would not bind. Over this range, therefore, the PV-optimal and sustainability-constrained optimal (S-optimal) policies would coincide. Sustainability and PV-optimality only diverge, therefore, for initial stocks above x_m.

Figure 2 presents the paths for this economy that follow from applying the S-optimal policy rule. As anticipated, paths that begin below x_m are identical to those in Figure 1. For initial stocks between x_m and x_{msy}, the S-optimal policy leads to the exact maintenance of that stock level, coinciding with Calvo's (1978) maximin criterion.

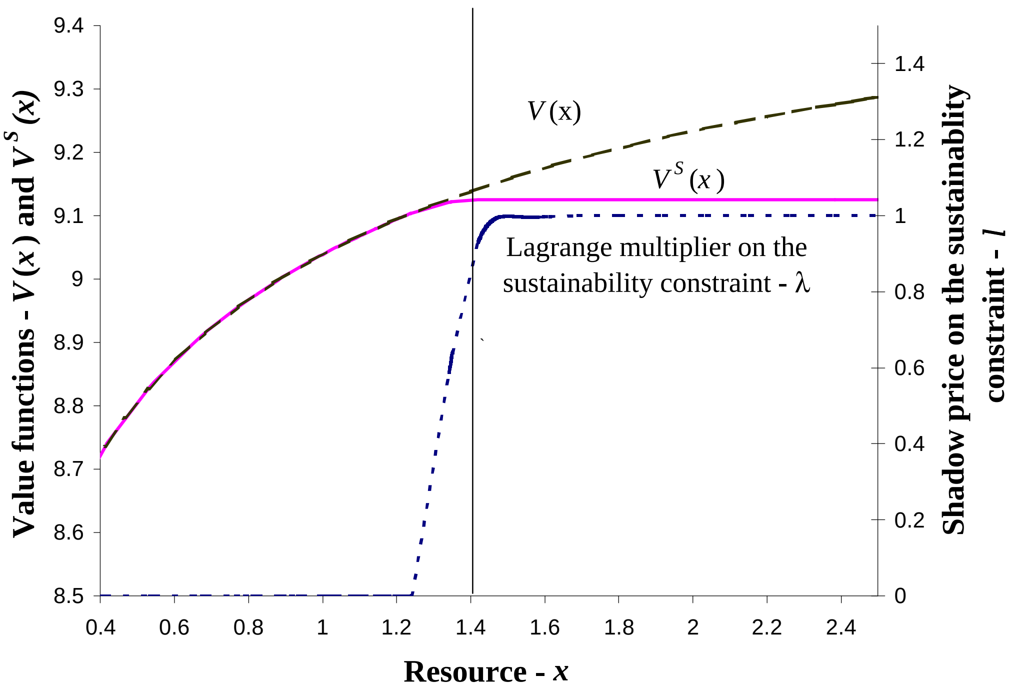

Figure 3: Value functions as a function of the resource stock in unconstrained and constrained optimation problems. For parameter values see Figure 1.

If the initial stock is above x_{msy}, then the S-optimal policy rule leads to a gradual decline in the resource stock toward x_{msy} -- these paths merit special consideration. By definition, no path can lead to a constant stream of harvests that is higher than the maximum sustainable yield. Hence, $V^S(x)$ must reach a maximum at x_{msy} (Figure 3). If the initial resource stock is above x_{msy}, therefore, the S-optimal policy cannot lead to an initial level of utility levels greater than the maximum sustainable level. Hence, if $x_0>x_{msy}$, then the only policy consistent with the sustainability constraint is to waste the excess stock. This is obviously inefficient since a Pareto superior path would involve immediately consuming the stock in excess of x_{msy}, followed by the maximum sustainable level of consumption from the second period onward. This inefficiency is confirmed in Figure 3 where the values of the l are shown to be equal to 1.0 when $x>x_{msy}$.[13]

[13] The small range of values greater than x_{msy} for which l is slightly less than 1.0 is due to approximation errors in the numerical algorithm.

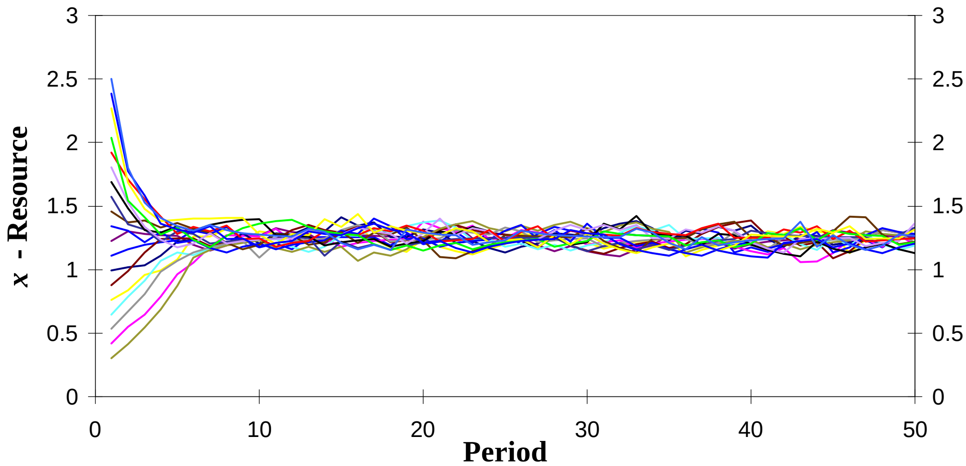

Figure 4: Selected paths of the resource stock value in the unconstrained one-dimensional resource economy under risk (s =0.05). For other parameter values see Figure 1.

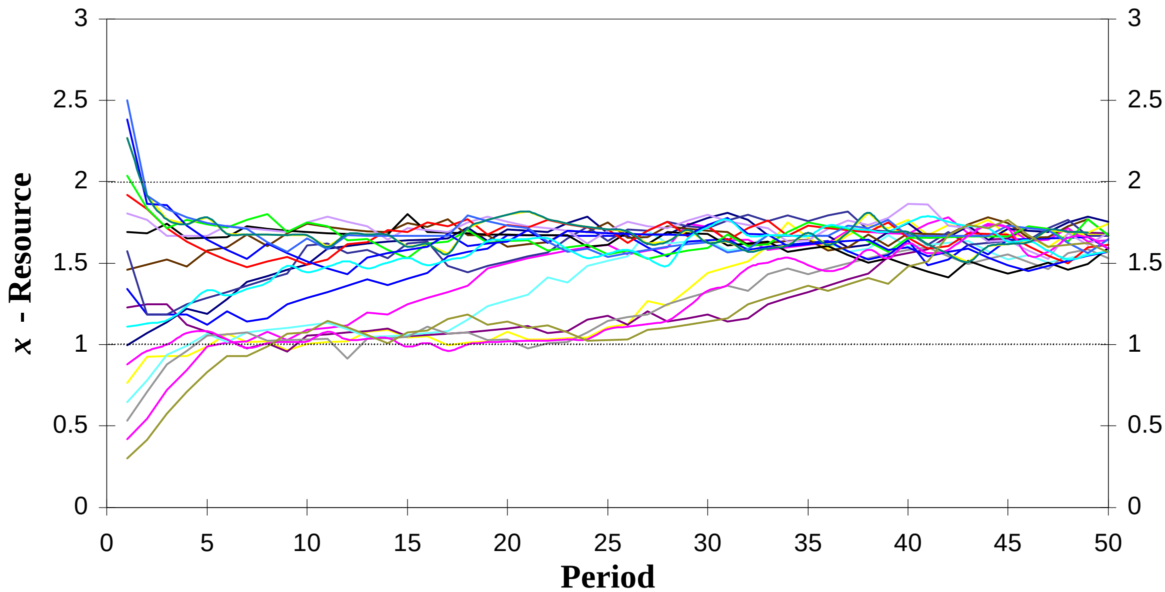

Figure 5: Selected paths of the resource stock value in the sustainability-constrained resource economy under risk (s = 0.05). For other parameter values see Figure 1.

Figures 4 and 5 present the PV- and S-optimal paths for the resource-based economy under risk (s =0.05). The PV-optimal paths follow a pattern similar to that observed in Figure 1 but with noise due to the shocks. The addition of risk to the constrained model, however, leads to an important difference in the S-optimal path. Under certainty, S-optimal management of the resource led to a range of steady-states from x_m to x_{msy}. However, when risk is introduced in Figure 5, there is a gradual upward trend in the resource stock over time. In period 10, for example, the average resource stock across the twenty paths presented is 1.41 and by the end of the fifty-period simulation this average has increased to 1.64.[14]

The gradual increase in the stock under risk confirms the theoretical expectation of growth in sustainability-constrained problems. It also suggests, however, that the presence of risk increases the likelihood that the sustainability constraint can lead to inefficiencies. We have seen that if the stock is greater than x_{msy} there is a conflict between efficiency and sustainability. The S-optimal policy under risk leads the economy toward this point.

8. Optimal-sustainable management of a two-dimensional resource economy

We now consider a two-dimensional capital-resource economy in which, in addition to the renewable resource now identified as x_1, the economy is also dependent upon a reproducible capital stock, x_2. Using a Cobb-Douglas production function, the existing capital stock and withdrawals from the resource stock, $z_1(x_1+e)$, together produce a fungible output. This output can then either be consumed, say c, or invested in the capital stock for the next period, z_2, so that $c+z_2=\left(z_1\left(x_1+e\right)\right)^a\left(x_2\right)^{1-a}$ and $x_{2t+1}=x_{2t}+z_{2t}$. Again utility is a concave function of consumption, $u(c) = c^{1-\gamma}$.

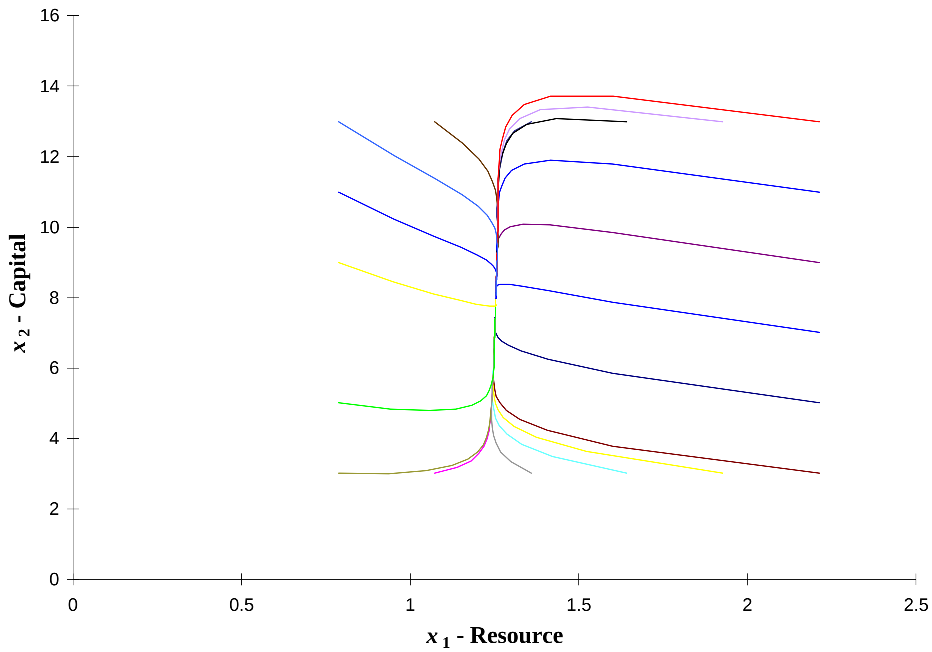

Figure 6: Selected paths of the capital-resource economy in the unconstrained economy. Parameters used: the order of the Chebyshev polynomial is 20 in each dimension; the bounds are 0.5 and 2.5 and on x_1 and 1.0 and 15.0 on x_2; and a=0.5. For all other parameters see Figure 1.

---

[14] The shapes of neither the constrained nor the unconstrained value functions change significantly by adding the uncertainty. Since for $x>x_{msy}$ $V^S(x)$ is horizontal, growth beyond x_{msy} would not be expected.

We consider first the deterministic case. In the absence of the sustainability constraint, the optimal trajectories lead to a single steady-state equilibrium as seen in Figure 6. For the parameter values used, the steady state occurs at about $x_1=x_m=1.25$ and $x_2=7.90$.[15] The value function for this model is monotonically increasing in both x_1 and x_2. Hence, paths that increase both assets are consistent with sustainability while paths that reduce both assets violate sustainability. Paths beginning in the northwest or southeast corners lead to increments in one asset and reductions in the other. It is along these paths that the question of sustainability is most interesting.

The S-optimal trajectories in the two-dimensional economy are displayed in Figure 7. For any path beginning in the state space below the dotted line, the sustainability constraint does not bind and the PV-optimal and S-optimal paths coincide. When starting above the dotted line, any reduction in one asset must be compensated by increments to the other. Because of the opportunities for substitution, the effect of the constraint on the management is slight when compared with the one-dimensional case. For all initial points considered, the S-optimal paths lead to equilibrium stocks of x_1 near x_m, increasing only slightly for higher initial capital stocks. Because of the ability to substitute capital for the natural resource, the S-optimal use of the natural resource is much more efficient in this model. The constraint does, however, substantially alter the management of x_2. The consumption of capital that is seen in Figure 6 is disallowed in the sustainability-constrained model. In effect, therefore, the sustainability constraint becomes a constraint on capital stock, rather than a constraint on the resource.

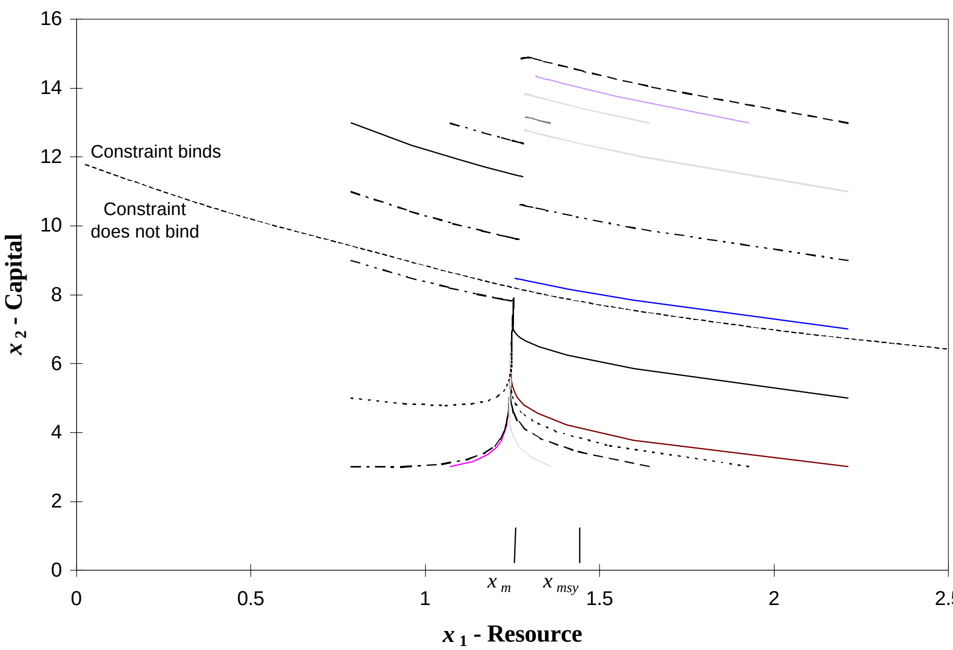

Figure 7: Selected paths of the capital-resource economy in the sustainability-constrained economy. For parameter values see Figure 6.

---

[15] While the specific results are sensitive to parameter values chosen, the trends exhibited in the figures are insensitive to parameter choice over a wide range.

Because current consumption can always be converted into a greater capital stock, there are always sustainably-productive alternatives in this economy. Hence, we know from Proposition 3 that the solution to the problem will be stepwise Pareto efficient. This is reflected in the Lagrange multiplier on the sustainability constraint, which reaches a maximum of only 0.25 in the state-space considered.

The introduction of risk in this model is illustrated in Figure 8.[16] Averaging across twenty simulated paths, we find that, as anticipated, risk again leads to growth in the value of the resource endowment over time. In this case, the growth in value is due entirely to increases in the capital stock as the path of the resource stock is nearly identical in the two models.

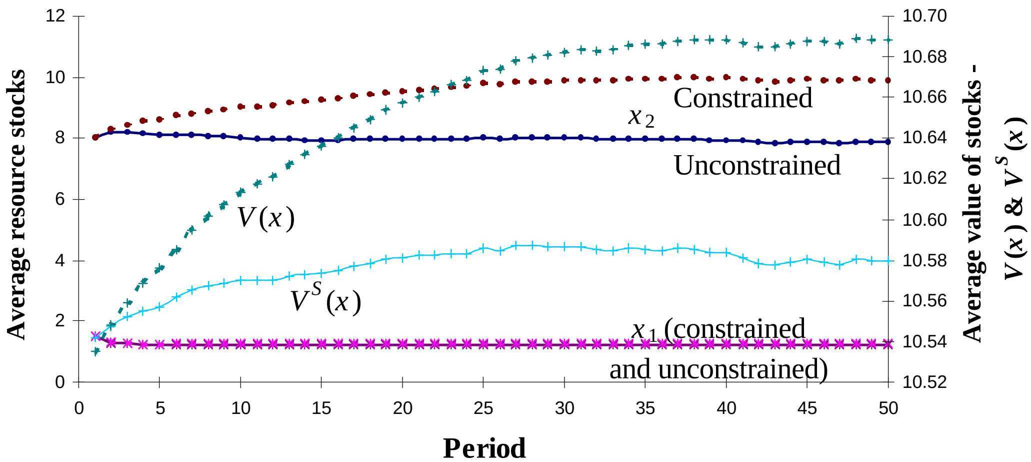

Figure 8: Average of the state variables and values of the endowment, $V(x_t)$ and $V^S(x_t)$, across 20 policy paths following the PV- and S-optimal policies in a stochastic two-dimensional economy. For parameter values see Figure 6 and footnote 16.

9. Conclusions

Sustainability continues to be prominent in public debates about economic development and natural-resource management. Economics has much to offer to these debates. The discipline can offer clarity as to its meaning, explore the implications of commitments to sustainability, and provide guidance as to how sustainability can be achieved. This paper contributes in each of these areas.

---

[16] In the numerical solution e_{it} is approximated by a third order Gaussian quadrature approximation in each dimension using values from Miranda (1994). In both dimensions the e_{it} is assumed to have a mean of zero and a standard error of 0.01. Greater variability in e did not change the qualitative nature of the results but introduced significant "noise", making it more difficult to see the relevant trends.

First, we argue that the principal motivation behind concerns about sustainability is a concern that current policies are intergenerationally unfair. We posit, therefore, that the meaning of sustainability can be clarified using Foley's (1967) economic definition of fairness. Assuming policy makers hold intergenerational fairness to be a generational obligation, we arrive at our framework of sustainability-constrained optimization. We find two advantages to this model. First, we believe that our recursive framework is a intuitively attractive mathematical formalization of arguments made by others that there is an obligation to treat future generations fairly (e.g., Weiss 1989). Secondly, the model has a practical advantage in that it can be solved numerically by adapting common tools of dynamic programming.

Two theoretical implications of sustainability are discussed. First, we find that under risk, planning for sustainability requires the expectation of growth. This means that policies in pursuit of sustainability must actually seek sustainable growth. Secondly, we consider the possibility that the norm of sustainability might conflict with the pursuit of economic efficiency. While we find that the sustainability constraint can lead to conflicts with the norm of Pareto efficiency, we also show that such conflicts are avoided in economies with productive opportunities. Hence, we are confident that in most situations policy makers can pursue the goal of sustainability without conflicting with the goal of efficiency.

Finally, we provide the groundwork for models that might advance the search for sustainable policies in more complex economies. The numerical methods introduced here vastly expand the types of problems that can be considered in analysis of sustainability. Clearly, the range of problems that might be considered using these methods is still limited, but it is enormous compared to those for which analytical solutions can be obtained.

Appendix 1

Propositions and Proofs

10. Proof that the successive approximation algorithm will solve (7).

Proposition 4: Under assumptions b and c, the iterative solution of (8) with $V^0(x)=V^*(x)$, converges to $V^S(x)$ as $k\to\infty$.

The final proof is provided below. We develop the proof following the basic structure put forth by Bertsekas (1976). If V^k is the solution to (8), we can write

$$V^k = T(V^{k-1}), \tag{13}$$

where T is defined by (8) and maps from the function V^{k-1} onto the function V^k. T is referred to as the sustainability-constrained Bellman's operator. The operator T has an important monotonicity property.

Lemma 1**.** For any bounded functions $V^A:X\to\mathbb{R}^1$, and $V^B:X\to\mathbb{R}^1$, such that $V^B(x)\geq V^A(x)$ for all $x\in X$

$$T(V^B(x))\geq T(V^A(x)) \text{ for all } x\in X.$$

Proof: For a particular point, x_t, let Problem A be the maximization of $Eu(z,x_t,e)+bEV^A(g(z,x_t,e))$ subject to the associated feasibility and sustainability constraints, and let Problem B the similar problem with V^B substituted for V^A. Suppose z^A solves Problem A and z^B solves Problem B. Since $V^B(x)\geq V^A(x)$, for all x and z^A satisfies the sustainability constraint, it follows that

$$Eu(z^A,x_t,e)+bEV^A(g(z^A,x_t,e))\leq Eu(z^A,x_t,e)+bEV^B(g(z^A,x_t,e)).$$

and, for $b<1$, that

$$Eu(z^A,x_t,e)\leq(1-b)EV^A(g(z^A,x_t,e))\leq(1-b)EV^B(g(z^A,x_t,e)).$$

Hence, z^A is also a feasible solution to Problem B. By definition of the maximization, therefore, it follows that for all points in $x_t\in X$

$$Eu(z^B,x_t,e)+bEV^B(g(z^B,x_t,e))\geq Eu(z^A,x_t,e)+bEV^A(g(z^A,x_t,e)) \;\|$$

Lemma 2: If $V^0(\cdot)=V^*(\cdot)$, the solution to the unconstrained optimization problem, then $V^{k+1}(x)\leq V^k(x)$ for all $x\in X$, $k=0,1,\ldots$.

Proof: Let $T^*(V)$ be the standard Bellman's operator, equivalent to the sustainability-constrained Bellman's operator in (13) without the sustainability constraint. Because of the additional constraint in T, for any function $V:X\to\mathbb{R}^1$, $T^*(V)\geq T^1(V)$. V^* is defined by the functional fixed point where $T^*(V^*)=V^*$ (Bertsekas, 1976). Hence, it follows that $T^1(V^*)\leq V^*$. By Lemma 1 it follows that $T^2(V^*)\leq T(V^*)$ and, by induction, $T^{k+1}(V)\leq T^k(V)$ for all k. ||

For purposes of discussion, it is now helpful to define

$$V^{\infty} \equiv \lim_{k\to\infty} T^{k}\left(V^{*}\right). \tag{14}$$

We now prove the existence and uniqueness of V^{∞} and then show that this limit is equal to V^{S}, the solution to (7).

Lemma 3**.** If assumptions b and c are held, then V^{∞} is bounded from above by V^{*} and from below by $\underline{u}/(1-b)$.

Proof: The upper bound on V^{∞} follows immediately from Lemma 2.

The lower bound on V^{∞} is guaranteed by assumption b. If $u(\underline{z},x,e)=\underline{u}$ for all x and e, then the worst possible stream of utility has a present value of stream of

$$\sum_{t=0}^{\infty} b^{t}\underline{u} = \underline{u}/(1-b) \equiv \underline{V}. \tag{15}$$

We need only show that $\underline{z}$ satisfies the sustainability constraint. Using (15), the sustainability constraint for a choice of $\underline{z}$ can be written

$$\underline{u}=(1-b)\underline{V}\leq(1-b)EV^{k-1}(g(x_t,\underline{z},e_t)).$$

This holds since $V^{k-1}(x)\geq\underline{V}$ for all x. ||

Proposition 4a: If assumptions b and c hold, then $V^{\infty}=\lim_{k\to\infty} T^{k}\left(V^{*}\right)$ exists.

Proof: By Lemma 3, for all k, $T^{k}(V^{*}) \in [\underline{V},T^{k-1}(V^{*})]$. By Lemma 2, at the k^{th} stage of the algorithm, for all $x\in X$ there are two possible outcomes, either $T^{k+1}(V^{*}(x))<T^{k}(V^{*}(x))$ or $T^{k+1}(V^{*}(x))=T^{k}(V^{*}(x))$. If the former holds for all k, then the algorithm will converge to the lower bound $\underline{V}$. If the later case holds for some k, then $T^{k+j}(V^{*})=T^{k}(V^{*})$, for all $j=1,2,\ldots$.||

Proposition 4b: If assumptions b and c hold, then V^{∞} is equal to V^{S}, the solution to (7).

Proof: If $V^{S}(x)<V^{\infty}(x)$, for a given $x\in X$, then it is possible that there exists a policy that satisfies the constraints of (7) and yields a higher value than $V^{S}(x)$. But this contradicts the fact that V^{S} is the maximum value. Hence, $V^{S}\geq V^{¥}$.

By definition $T^{k}(V^{S})=V^{S}$ and by Lemma 1, $T^{k}(V^{*})\geq T^{k}(V^{S})$. Hence, for all k, $V^{k}\geq V^{S}$ and, accordingly, $V^{\infty}\geq V^{S}$. ||

Proof of Proposition 4: Propositions 4a and 4b imply that Proposition 4 holds. ||

11. Proof of Proposition 2, the relationship between efficiency and λ.

In the proof of Proposition 2 we use the following lemma.

Lemma 4: If the sustainability constraint is binding at the optimum of (7), then an alternative policy, $z_t\in \Gamma(x_t)$, can be stepwise Pareto superior to the optimal policy z_t^{S} only if $E_t u(z_t, x_t, e_t) > E_t u(z_t^{S}, x_t, e_t)$.

Proof: For generation t, the policy z_t is preferred to z_t^{S} if (11a) and (11b) are satisfied, one with a strict inequality. There are three ways that such an inefficiency might occur:

i. (11a) holds with a strict inequality and (11b) with an equality;

ii. (11a) holds with an equality and (11b) with a strict inequality;

iii. (11a) holds with a strict inequality and (11b) with a strict inequality.

If $E_t u(z_t, x_t, e_t) \leq E_t u(z_t^S, x_t, e_t)$, then (11a) can only hold with a strict inequality if $E_t V^S(g(z_t, x_t, e_t)) > E_t V^S(g(z_t^S, x_t, e_t))$, ruling out i.

Since z_t^S is the solution to (7), $E_t u(z_t^S, x_t, e_t) \leq (1-b) E_t V(g(z_t^S, x_t, e_t))$. Suppose that $E_t u(z_t, x_t, e_t) \leq E_t u(z_t^S, x_t, e_t)$, and that (11b) holds with a strict inequality. Then

$$E_t u_t(z_t, x_t, e_t) < (1-b) E_t V^S(g(z_t, x_t, e_t)).$$

Hence, z_t does not violate the sustainability constraint, contradicting the assumption that z_t^S is optimal. ||

Proof to Proposition 2: Let $\mathcal{L}(z_t, l)$ be the Lagrangian of (7), i.e.,

$$\mathcal{L}(z_t, l_t) = E_t u(z_t, x_t, e_t) + b E_t V^S(x_{t+1}) - l\left[E_t u(z_t, x_t, e_t) + (b-1) E_t V^S(x_{t+1})\right].$$

For notational convenience, let $u_t^S = E_t u(z_t^S, x_t, e_t)$, $V_{t+1}^S = E_t V^S(g(z_t^S, x_t, e_t))$, $u_t = E_t u(z_t, x_t, e_t)$ and $V_{t+1} = E_t V^S(g(z_t, x_t, e_t))$, where $z_t \in \Gamma(x_t)$. The saddle-point condition implies that

$$u_t(1-l) + [b + l(1-b)]V_{t+1} \leq u_t^S(1-l) + [b + l(1-b)]V_{t+1}^S. \tag{16}$$

By Lemma 4, z_t we need only consider z_t such that $u_t > u_t^S$. Introducing such a case, (16) can be simplified to

$$\frac{(V_{t+1} - V_{t+1}^S)}{(u_t - u_t^S)} \leq \frac{(l-1)}{[b + l(1-b)]}.^{17} \tag{17}$$

Call the left- and right-hand sides of (17) LHS and RHS respectively. If for some $z_t \in \Gamma(x_t)$, LHS≥0 then z_t^S is stepwise Pareto inefficient. LHS≥0 ⇒RHS≥0, which for $0<b<1$, holds only if $l \geq 1$.

Similarly, $l < 1$ ⇒ RHS<0 ⇒ LHS<0 for all $z_t \in \Gamma(x_t)$ such that $u_t > u_t^S$. Hence, z_t^S is stepwise Pareto efficient. ||

12. Proof of Proposition 3, that productive economies are stepwise Pareto efficient

Before proving Proposition 3, we need to show that under the stated conditions the sustainability-constrained value function will not be bounded at the current endowment.

---

[17] Note that the denominator of the RHS of (17) is strictly positive for all values of $0<b\leq 1$ and $l \geq 0$ and the denominator of the LHS is strictly positive by the assumption that $u_t > u_t^S$.

Lemma 5: If assumption d holds and there exists a sustainably-productive alternative to the optimal choice z_t^S at x_t then $V^S(x)$ is not bounded at $V^S(x_t)$.

Proof: Because of the sustainability constraint, if z_t^S is the solution to (7), then $E_tV^S(x_{t+1})\geq V^S(x_t)$. We need only consider the possibility that $E_tV^S(x_{t+1})=V^S(x_t)$ since only then is it possible that $V^S(x_t)$ is the upper bound on $V^S(x)$. However, if a sustainably-productive alternative exists, then there is some choice $z_t'\in\Gamma(x_t)$ such that $E_t U\left({}_{t+j}z, {}_{t+j}x, {}_{t+j}e\right)\Big|_{z_t'} \geq V^S\left(x_t\right)+d$ for all j. Since utility is disposable, all future generations can achieve welfare equal to $V^S(x_t)+d$, implying that $E_tV^S(g(z_t',x_t,e_t))>V^S(x_t)$ and, therefore, there are some e_t such that $V^S(g(z_t',x_t,e_t))>V^S(x_t)$. ||

We can now prove the proposition, which we restate here for completeness.

Proposition 3:

Proof: Suppose not. By Lemma 4, if z_t^S is stepwise inefficient, then there must exist some alternative, z_t, such that $E_tu(z_t,x_t,e_t)>E_tu(z_t^S,x_t,e_t)$ and $E_tV^S(g(z_t,x_t,e_t))\geq E_tV^S(g(z_t^S,x_t,e_t))$. However, using assumption d and Lemma 5, if the economy has sustainably-productive potential then there exists a choice, z_t', alternative to z_t, such that $E_tu(z_t',x_t,e_t)=E_tu(z_t^S,x_t,e_t)$ and $E_tV^S\left(g\left(z_t',x_t,e_t\right)\right)>E_tV^S\left(g\left(z_t,x_t,e_t\right)\right)$. This implies that $E_tu(z_t',x_t,e_t)+bE_tV^S(g(z_t',x_t,e_t))>E_tu(z_t^S,x_t,e_t)+bE_tV^S(g(z_t^S,x_t,e_t))$ and that the sustainability constraint is not violated, contradicting the assertion that z_t^S is optimal.||

Appendix 2:
Technical details of the numerical algorithms for the solution of sustainability-constrained optimization problems

The algorithm used to solve sustainability-constrained optimization problems requires the repeated solution of (8). Discrete dynamic optimization problems, where the state variables take on a finite number of values, can be solved exactly. However, if the state space is continuous, as in the problems considered here, the solution is only an approximation of the true solution. The reason for this is that T must be applied point by point and, therefore, can only be evaluated at a finite number of points. Because there is no closed-form specification of the value function, V^k, interpolation or rounding is required.

In the models solved here, the value function is approximated over the entire domain using a Chebyshev polynomial of the state variables. This approximation method is preferable in many circumstances because a higher degree of precision can be obtained at less computational expense (Judd 1998). The use and implementation of Chebyshev polynomials are discussed in Press et al. (1989). In the k^{th} iteration of the algorithm, (8) is solved at a finite set of points in the state space, $\hat{X}$, and then, based on these points, coefficients, c_k, are calculated that can be used to estimate the value function across the entire range of X. The approximate Bellman's operator, therefore, is essentially a mapping from c_k to c_{k+1}, so that

$$\hat{V}(x;c_{k+1}) = \max_z EU(z,x) + d\hat{V}(x;c_k) \text{ s.t. constraints, for all } x \in \hat{X} . \tag{18}$$

This procedure is repeated until a convergence criterion, $\|\hat{V}(x;c_{k+1}) - \hat{V}(x;c_k)\| < \tau$, is met. The optimization problems at each point in the state space are solved using NPSOL (Gill et al. 1986), a sequential quadratic programming algorithm. The unconstrained problem was solved using the successive approximation method (Bertsekas 1976).

Many of the results presented are of simulated time-paths from a variety of starting points in the state space. NPSOL is used again here to determine the policies at each point using the final estimate of the value function. The optimal policy is then fed into the state equations to determine the state in the next period.

In the two-dimensional model the reproducible capital stock, x_2, has no natural upper bound and optimal strategies sometimes lead to accumulation of that asset beyond any arbitrarily established bound. This is problematic since it is impossible to guarantee that the resulting value functions and policies are not functions of the numerical bound that must be chosen. Sensitivity analysis was used to verify that the results were not sensitive to the bounds imposed. In the simulated paths, paths are frozen if they reach the boundary of the defined state space.

In the solution of the models an additional choice variable is used. This variable simply scales utility between zero and one, thereby satisfying assumption b. Except where the sustainability constraint was inconsistent with stepwise Pareto efficiency, the optimal value of this variable is always 1.0. Where values less than one are chosen, the shadow price on the sustainability constraint is equal to 1.0.

13. References

Asheim, Geir B. "Rawlsian Intergenerational Justice as a Markov Perfect Equilibrium in a Resource Technology." *Review of Economic Studies*. 55(July 1988):469-83.

Asheim, Geir B. "Net National Product as an Indicator of Sustainability." *Scandinavian Journal of Economics* 96(1994):257-65.

Asheim, Geir B. and Kjell Arne Brekke. "Sustainability when Resource Management has Stochastic Consequences." Norwegian Central Bureau of Statistics, Research Department Discussion Paper no. 86, (1993).

Bromley, Daniel W. "Entitlements, Missing Markets, and Environmental Uncertainty." *Journal of Environmental Economics and Management*. 17(September 1989):181-94.

Bertsekas, Dimitri P. *Dynamic Programming and Stochastic Control*. New York: Academic Press, 1976.

Calvo, Guillermo A. "Some Notes on Time Inconsistency and Rawls' Maximin Criterion." *Review of Economic Studies*. 45(February 1978):97-102.

Chavas, Jean Paul. "Equity Considerations in Economic and Policy Analysis." *American Journal of Agricultural Economics*. 76(December 1994):1022-33.

Chichilnisky, Graciela. "An Axiomatic Approach to Sustainable Development." *Social Choice and Welfare*. 13(April 1996):231-57.

Chichilnisky, Graciela. "What Is Sustainable Development?" *Land Economics*. 73(November 1997):467-91.

Clark, Colin W. *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*. New York: Wiley, 1976.

Dasgupta, Partha S. and Geoffrey M. Heal. "The Optimal Depletion of Exhaustible Resources." *Review of Economic Studies* (Symposium 1974):3-28.

Dasgupta, Partha S. and Geoffrey M. Heal. 1979. *Economic Theory and Exhaustible Resource.* Cambridge, England: Cambridge University Press.

Dasgupta, Partha and Karl-Göran Mäler. Poverty, Institutions, and the Environmental Resource Base *Handbooks in Economics, volume IIIA*. Kenneth J. Arrow and Michael D. Intriligator, eds. New York: Elsevier, 1995.

Foley, Duncan K. "Resource Allocation and the Public Sector." *Yale Economic Essays* 7(1967):45-98.

Gill, Philip E., Walter Murray, Michael A. Saunders and Margaret H. Wright. *User's Guide for NPSOL* (*Version 4.0*): *A Fortran Package for Nonlinear Programming*. Technical Report SOL 86-2, Systems Optimization Laboratory, Stanford University, 1986.

Hartwick, John M. "Intergenerational Equity and the Investing of Rents from Exhaustible Resources." *American Economic Review*. 67(December 1977):972-74.

Hartwick, John M. "Natural Resources, National Accounting and Economic Depreciation." *Journal of Public Economics*. 43(December 1990):291-304.

Howarth, Richard B. "Intergenerational Competitive Equilibria under Technological Uncertainty and an Exhaustible Resource Constraint." *Journal of Environmental Economics and Management*. 21(November 1991):225-43.

Howarth, Richard B. "Intergenerational Justice and the Chain of Obligation." *Environmental Values*. 1(Summer 1992):133-40.

Howarth, Richard B. "Sustainability under Uncertainty: A Deontological Approach." *Land Economics*. 71(November 1995):417-27.

Howarth, Richard B. "Sustainability as Opportunity." *Land Economics*. 73(November 1997):569-79.

Judd, Kenneth L. *Numerical Methods in Economics*. Cambridge, Mass.: MIT Press, 1998.

Koopmans, Tjalling C. "Stationary Ordinal Utility and Impatience." *Econometrica* 28(April 1960):287-309.

Laslett, Peter. "Is there a Generational Contract?" *Justice Between Age Groups and Generations*. Laslett, Peter and James S. Fishkin, eds., pp. 24-47. New Haven, Connecticut: Yale University Press, 1992.

Laslett, Peter and James S. Fishkin, eds. *Justice Between Age Groups & Generations*. New Haven, Connecticut: Yale University Press, 1992.

Lauwers, Luc. "Rawlsian Equity and Generalised Utilitarianism with an Infinite Population." *Economic Theory*. 9(January 1997):143-50.

Machina, Mark J. "Choice Under Uncertainty: Problems Solved and Unsolved." *Journal of Economic Perspectives* 1(Summer 1987):121-154.

Manski, Charles F. "Treatment Under Ambiguity." Social Systems Research Institute, Working paper 9619R, University of Wisconsin-Madison, 1996.

Miranda, Mario J. "Lecture Notes in Dynamic Computational Economics." Unpublished manuscript, The Ohio State University, 1994.

Pazner, Elisha A. "Pitfalls in the Theory of Fairness." *Journal of Economic Theory*. 14(April 1977):458-66.

Panzer, Elisha A. and David Schmeidler. "A Difficulty in the Concept of Fairness." *Review of Economic Studies*. 41(July 1974):441-43.

Pazner, Elisha A. and David Schmeidler. "Egalitarian Equivalent Allocations: A New Concept of Economic Equity." *Quarterly Journal of Economics*. 92(November 1978):671-87.

Pezzey, John C. V. "Non-Declining Aggregate Wealth is not Equivalent to Sustainability." Paper presented at the AERE session on the Economics of Sustainable Development, A.S.S.A. Annual Meeting, Washington D.C., 6-8 January, 1994.

Pezzey, John C. V. "Sustainability Constraints versus "Optimality" versus Intertemporal Concern, and Axioms versus Data." *Land Economics*. 73(November 1997):448-66.

Pezzey, John C. V. and Cees A. Withagen. "The Rise, Fall and Sustainability of Capital Resource Economies." *Scandinavian Journal of Economics*. 100(June 1998):513-27.

Press, William H., Brian P. Flannery, Saul A. Teukolsky and William T. Vetterling. *Numerical Recipes: The Art of Scientific Computing (FORTRAN Version)*, Cambridge University Press, Cambridge, 1989.

Ramsey, Frank P. "A Mathematical Theory of Saving." *Economic Journal* 38(1928):543-559.

Rawls, John. *A Theory of Justice*. Cambridge, Mass.: Harvard University Press, 1971.

Riley, John G. "The Just Rate of Depletion of a Natural Resource." *Journal of Environmental Economics and Management*. 7(December 1980):291-307.

Rodriguez, Alvaro. "Rawls' Maximin Criterion and Time Consistency: A Generalization." *Review of Economic Studies*. 48(October 1981):599-605.

Sen, Amartya. "Maximization and the Act of Choice." *Econometrica*. 65(July 1997):745-79.

Solow, Robert M. "Intergenerational Equity and Exhaustible Resources." *Review of Economic Studies* (Symposium 1974):29-45.

Solow, Robert. "An Almost Practical Step toward Sustainability." *Resources Policy*. 19(September 1993):162-72.

Stiglitz, Joseph E. "Growth with Exhaustible Natural Resources: Efficient and Optimal Growth Paths." *Review of Economic Studies* (Symposium 1974):123-37.

Takayama, Akira. *Mathematical Economics* (2nd edition). New York: Cambridge University Press, 1985.

Thomson, William and Hal R. Varian. "Theories of Justice Based on Symmetry." *Social Goals and Social Organizations: Essays in Memory of Elisha Pazner*. Leonid Hurwicz, David Schmeidler and Hugo Sonnenschein, eds. Cambridge: Cambridge U. Press, 1985.

Toman, Michael A. "Economics and "Sustainability": Balancing Trade Offs and Imperatives." *Land Economics*. 70(November 1994):399-413.

Toman, Michael A., John Pezzey and Jeffrey Krautkraemer. "Economic Theory and 'Sustainability'" *Handbook of Environmental Economics*. Daniel Bromley, ed., pp. 139-165. Cambridge: Blackwell, 1995.

Varian, Hal R. "Equity, Envy, and Efficiency." *Journal of Economic Theory*. 9(September 1974):63-91.

Weiss, Edith Brown. *In Fairness to Future Generations; International Law, Common Patrimony, and Intergenerational Equity*. Tokyo: The United Nations University, 1989.

Woodward, Richard T. and Richard C. Bishop. "How To Decide if Experts Disagree: Uncertainty-Based Choice Rules in Environmental Policy." *Land Economics* 73(November 1997):492-507.