Estimating Elasticities of Substitution Using Data Envelopment Analysis

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Selected Poster prepared for presentation at the 2016 Agricultural & Applied Economics Association Annual Meeting, Boston, MA, July 31 – Aug. 2.

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BACKGROUND

- Knowledge of the level of substitutability of inputs is crucial for farm management decision-making, but this knowledge is often limited and/or incomplete.
- The elasticity of substitution measures the substitutability of inputs - it shows the effect on inputs from a change in the marginal rate of technical substitution (production perspective) or a change in input prices (cost perspective), with output held constant.
- Hicks developed the elasticity of substitution to describe the substitutability of inputs for a two-input production technology. Morishima extended the Hicksian elasticity to describe the substitutability of inputs for an n-input production technology (Blackorby and Russell 1989).
- Modeling parametric production and cost functions is the traditional approach to estimating these elasticities, but this approach is problematic in the absence of time-series data.
- An alternative strategy is to estimate elasticities using a non-parametric approach, Data Envelopment Analysis (DEA). With a limited number of observations, DEA can provide information regarding optimal production behavior.
- The input-oriented, Banker-Charnes-Cooper (BCC) technical efficiency model and the cost efficiency model are two types of DEA models analogous to the economic production maximization and cost minimization models, respectively.

OBJECTIVES

- Derive (Hicksian) elasticities of substitution for inefficient firms from the technical efficiency and cost efficiency DEA models assuming variable returns to scale
- Illustrate the use of elasticities of substitution with an empirical example using enterprise data from the Kansas Farm Management Association (KFMA)

TECHNICAL EFFICIENCY MODEL

BCC (min): $\min_{\theta_o,\lambda} \theta_o$

Subject to:

$$\boldsymbol{\lambda}'\boldsymbol{x_k} \le \theta_o x_{k,o} \qquad \forall \text{ k inputs} \qquad \rightarrow \boldsymbol{v_k}$$

$$\boldsymbol{\lambda}'\boldsymbol{y_m} \ge y_{m,o} \qquad \forall \text{ m outputs} \qquad \rightarrow \boldsymbol{u_m}$$

$$\boldsymbol{e}'\boldsymbol{\lambda} = 1 \qquad \rightarrow u_0$$

$$\boldsymbol{\lambda} \ge 0$$

- The objective of the BCC model is to estimate the technical efficiency of a firm or decision-making unit (DMU), θ_o, relative to all other DMUs in the sample, by choosing weights, $\boldsymbol{\lambda}$, that places it on the technological or production frontier (Cooper et al. 2007).
- Technical efficiency is defined as virtual output divided by virtual input $\left(\theta_o = \frac{\sum_{m=1}^{n} u_m y_{m,o}}{\sum_{k=1}^{n} v_k x_{k,o}}\right)$, where y_m is the DMU's m^{th} output, x_k is the DMU's k^{th} input, u_m is the DMU's m^{th} output weight corresponding to y_m, and v_k is the firm's k^{th} input weight corresponding to x_k.
- The constraints force composite inputs to be less than or equal to the technologically efficient input level, force composite outputs to be greater to or equal to the technologically efficient output level, allow for variable returns to scale, and ensure non-negativity of weights.
- The input and output vectors of weights, $\boldsymbol{u_m}$ and $\boldsymbol{v_k}$, represent shadow prices (dual variables) to the first and second constraints, with u_0, a free variable, the shadow price to the third constraint.

COST EFFICIENCY MODEL

Cost (min): $\min_{z,\lambda} \boldsymbol{w}'\boldsymbol{z}$

Subject to:

$$z_k - \boldsymbol{\lambda}'\boldsymbol{x_k} \ge 0 \qquad \forall \text{ k inputs} \qquad \rightarrow \boldsymbol{v_k}$$

$$\boldsymbol{\lambda}'\boldsymbol{y_m} - y_{m,o} \ge 0 \qquad \forall \text{ m outputs} \qquad \rightarrow \boldsymbol{u_m}$$

$$\boldsymbol{e}'\boldsymbol{\lambda} = 1 \qquad \rightarrow u_0$$

$$\boldsymbol{\lambda}, \boldsymbol{z} \ge 0$$

- The objective of the cost efficiency model is to choose **z** and $\boldsymbol{\lambda}$ that minimizes the DMU's cost, relative to all other DMUs in the sample, where **w** is the DMU's input costs and **z** is the cost-minimizing level of inputs.
- Cost efficient DMUs are those that are technically efficient, and also exhibit allocative efficiency. Cost efficiency is defined as the ratio of the cost-minimizing level of input use divided by the DMU's observed input costs $\left(\frac{\boldsymbol{w}'\boldsymbol{z}}{\boldsymbol{w}'\boldsymbol{x_o}}\right)$.
- The constraints force the cost minimizing inputs to be greater than or equal to the composite inputs, and the composite outputs to be greater than or equal to the DMU's observed output. The convexity constraint allow for variable returns to scale.
- As in the technical efficiency model, the input and output vectors of weights, $\boldsymbol{u_m}$ and $\boldsymbol{v_k}$, represent shadow prices (dual variables) to the first and second constraints, with u_0, a free variable, the shadow price on the third constraint.

HICKSIAN TECH. EFFICIENCY ELASTICITIES

- The Hicksian elasticity of substitution for the production problem shows the rate of change of the ratio of inputs divided by the rate of change of the marginal rate of technical substitution $\left(\sigma_{i,j}^{H} = \frac{dln\left[{}^{x_j}/_{x_i}\right]}{dln\left[{}^{f_i}/_{f_j}\right]}\right)$ (Chambers 2007).
- Based on this measurement, the Hicksian elasticity for technical efficiency, for inefficient DMUs, can be derived directly from the Lagrangian of the BCC minimization problem:

$$\sigma_{i,j}^{H} = \left[\frac{\left(v_i(\boldsymbol{\lambda}'\boldsymbol{x_i}) - v_j(\boldsymbol{\lambda}'\boldsymbol{x_j})\right)(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o})}{(\boldsymbol{\lambda}'\boldsymbol{x_i}\boldsymbol{\lambda}'\boldsymbol{x_j})\left(v_j(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{j,o}) - v_i(\boldsymbol{\lambda}'\boldsymbol{x_k} - \theta_o x_{i,o})\right)}\right] \text{(Miller 2016).}$$

- This elasticity shows the degree of input substitutability that an inefficient DMU can make (at optimality) and remain on the technically efficient frontier.
- This measurement is only valid for inefficient DMUs. For efficient firms, that exist on the technically efficient frontier, continuous derivatives cannot be derived. Thus, different methods must be employed, but that was beyond the scope of the present research.

HICKSIAN COST EFFICIENCY ELASTICITIES

- The Hicksian elasticity of substitution for the cost minimization problem is analogous to the technical efficiency problem – it shows the rate of change of the ratio of input prices divided by the rate of change of the ratio of marginal costs with respect to input quantities $\left(\sigma_{i,j}^{HC} = \frac{dln\left[{}^{w_j}/_{w_i}\right]}{dln\left[{}^{c_i}/_{c_j}\right]}\right)$ (Chambers 2007).

HICKSIAN COST EFFICIENCY ELASTICITIES

- The Hicksian elasticity for cost efficiency, for inefficient DMUs, can be derived directly from the Lagrangian of the cost efficiency problem:

$$\sigma_{i,j}^{HC} = \left(\frac{z_j(w_j - v_j)}{w_i z_i} - \frac{z_j(w_j - v_j)}{w_j z_j}\right)^{-1} - \left(\frac{z_i(w_i - v_i)}{w_i z_i} - \frac{z_i(w_i - v_i)}{w_j z_j}\right)^{-1} \text{(Miller 2016).}$$

- This elasticity shows the degree of input substitutability that an inefficient DMU can make (at optimality) and remain on the cost efficient frontier.
- This measurement is only valid for inefficient DMUs. For efficient firms, that exist on the cost efficient frontier, continuous derivatives cannot be derived. Thus, different methods must be employed, but that was beyond the scope of the present research.

MORISHIMA COST EFFICIENCY ELASTICITIES

- The Morishima elasticity of substitution for the cost minimization problem shows the rate of change of the ratio of the i^{th} and j^{th} inputs divided by the rate of change of the j^{th} input price $\left(\sigma_{i,j}^{MC} = \frac{dln\left[{}^{x_i}/_{x_j}\right]}{dln[w_j]}\right)$ (Chambers 1988). Unlike the Hicksian elasticity, the Morishima is a non symmetric (i.e. $\sigma_{i,j}^{MC} \ne \sigma_{j,i}^{MC}$) measurement.
- The Morishima elasticity for the cost efficiency, for inefficient DMUs, can be derived directly from the Langrangian of the cost efficiency problem:

$$\sigma_{i,j}^{MC} = \frac{w_j z_j}{(w_i - v_i)z_i} - \frac{w_j z_j}{(w_j - v_j)z_j} \text{ (Miller 2016).}$$

- This elasticity shows the degree of input substitutability that an inefficient DMU can make to its i^{th} and j^{th} inputs, holding output constant, with all other inputs free to adjust.
- This measurement is only valid for inefficient DMUs. For efficient firms, that exist on the cost efficient frontier, continuous derivatives cannot be derived. Thus, different methods must be employed, but that was beyond the scope of the present research.

EMPIRICAL APPLICATION: DATA

- For an empirical illustration of the elasticities derived, the efficiency of dryland corn production under reduced tillage was examined for 119 corn-planting farms in Kansas in 2014.
- Enterprise-level input and output data was collected from the KFMA, an organization affiliated with Kansas State University that provides financial data and planning for farmers (KFMA 2015).
- Input data included total expenses for fuel, fertilizer, herbicide, seed, labor (both hired and unpaid labor, machinery (including machinery rental and repair) and land (in total acres). Output was measured as total value of dryland corn production.
- Input variables were measured with a quantity index, with total input expenses divided by per-acre input costs (collected from the KFMA). The output variable was not transformed, since corn price was assumed to be the same for all farms in the study.

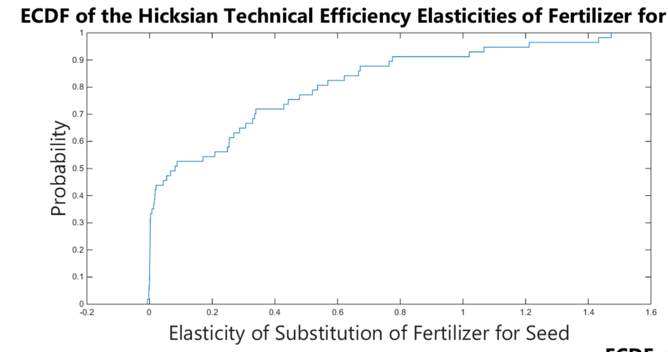
EMPIRICAL APPLICATION: RESULTS

- The technical and cost efficiency DEA models were estimated for each farm using the General Algebraic Modeling System (GAMS). The results of this estimation were used to compute Hicksian production and cost elasticities as well as Morishima cost elasticities in MATLAB for each farm following the derivations shown earlier.
- The results of the estimation of Hicksian and Morishima elasticities show that, on average and for this set of farms, only a slight degree of substitutability (or complementarity) exists between inputs. This indicates that changes in an input's relative marginal productivity or price does not substantially alter the proportion of inputs applied.
- Though the sample means of the elasticities indicate limited substitutability of inputs, individual farms exhibit a diversity of responses. Similarly, elasticity results differ across the different estimation models.

EMPIRICAL APPLICATION: RESULTS

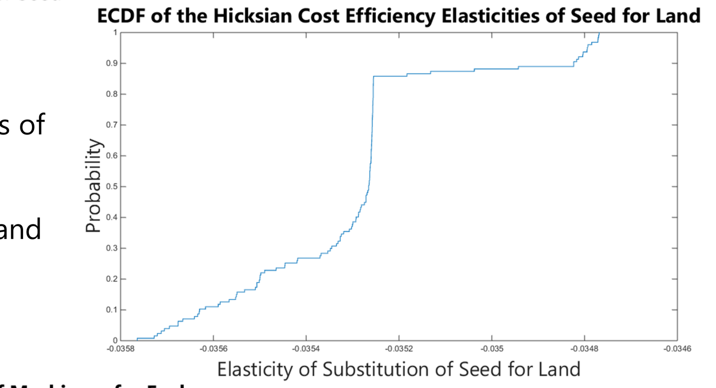
- The means of the technical efficiency elasticities indicate that most inputs tend to act as complements, the means of the Hicksian cost efficiency elasticities indicate that most inputs tend to act as substitutes, and the means of the Morishima cost efficiency elasticities indicate that some inputs act as substitutes and others act as complements.
- Presented below are some of the more typical results from the estimation procedure (shown as empirical cumulative distribution (ECDF) functions):

ECDF of the Hicksian Technical Efficiency Elasticities of Fertilizer for Seed



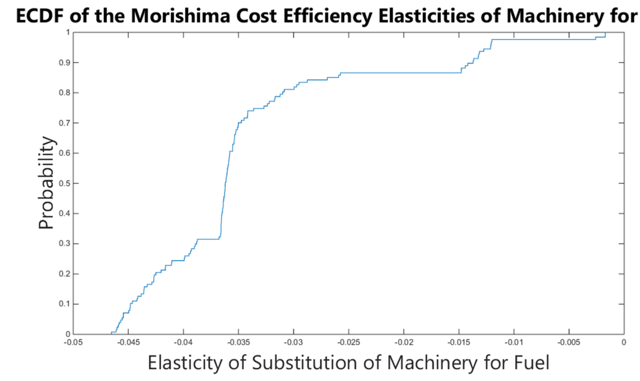
The Hicksian technical efficiency elasticities of fertilizer for seed have a mean value of 0.28, with most farms presenting positive values, indicating that fertilizer and seed are complements.

ECDF of the Hicksian Cost Efficiency Elasticities of Seed for Land



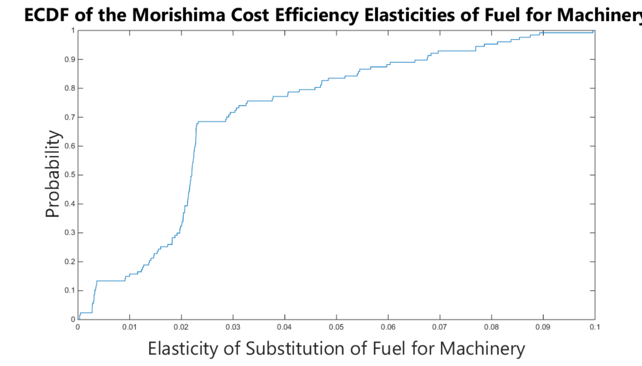
The Hicksian cost efficiency elasticities of seed for land have a mean value of -0.035, with most farms presenting negative values, indicating that seed and land are substitutes.

ECDF of the Morishima Cost Efficiency Elasticities of Machinery for Fuel



The Morishima cost efficiency elasticities of machinery for fuel have a mean value of -0.034, with most farms presenting negative values, indicating that machinery and fuel are substitutes.

ECDF of the Morishima Cost Efficiency Elasticities of Fuel for Machinery



The Morishima cost efficiency elasticities of fuel for machinery have a mean value of 0.028, with most farms presenting positive values, indicating that fuel and machinery are complements. This contradicts the Morishima cost efficiency elasticities of machinery for fuel, highlighting the non-symmetric nature of the Morishima elasticities (the effect from a change in the price of machinery is different from the effect from a change in the price of fuel).

CONCLUSION

- This study developed a procedure by which Hicksian production and cost elasticities and Morishima cost elasticities for inefficient DMUs can be derived using traditional technical and cost efficiency DEA frameworks.
- An empirical example using corn enterprise data for Kansas farms under reduced tillage served as an illustration of these elasticities. The results of the empirical example indicate limited substitutability across inputs.
- Future research should focus on estimation of Hicksian and Morishima elasticities for efficient DMUs (i.e. DMUs that reside on the technical or cost frontiers).

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