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On the Measurement of Poverty and Deprivation

by

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ON THE MEASUREMENT OF POVERTY AND DEPRIVATION

The literature on poverty measures that followed the seminal work of Amartya Sen (1976) has generally taken (with only few exceptions) the basic approach put forward by Sen: The measure of aggregate poverty was defined as a weighted sum of the individual poverty; the individual poverty itself was measured by the income gap up to the (predetermined) poverty line; the weights of the aggregate measure were determined so that the measure will satisfy a set of axioms.

Three basic axioms or desired properties have been proposed by Sen to determine the weights and thus also the functional form of the aggregate poverty measure. These are:

(F) The Focus axiom: The poverty measure must be determined by the incomes of the poor only.

(M) Monotonicity Axiom: Given other things, a reduction in the income of any poor individual must (strictly) raise the measure of aggregate poverty.

(T) Transfer Axiom: Given other things, a transfer of income from a poor individual to any one who is richer must strictly raise the measure of aggregate poverty.

These axioms induce a rather general structure on the functional form of poverty measures (see Bigman, 1985), and considerable number of alternative poverty measures have consequently been proposed that satisfy the three axioms but differ, sometimes quite markedly, in their functional form from the index proposed by Sen[1] (see Kakwani, 1980; Thon, 1979; Foster, Greer and Thorbecke, 1984).

This approach can be criticized for a number of reasons:

(i) The general specification of the aggregate poverty measure is assumed to be a weighted sum of the individual poverty, without being derived from established paradigms of welfare theory.

(ii) The measure of individual poverty is assumed to be the poverty gap, This has not been (and cannot be - as we shall see later on) justified however by the established and widely accepted notions on individual preferences;

(iii) The additive form of the index that has been assumed may impose restrictive assumptions on the form of the welfare function which is implicit in the poverty measure.

(iv) The three axioms determine a wide group of functions from which aggregate poverty measusres can be selected. This has led Sen (1979) to conclude that pluralism is inherent in the excercise of poverty index construction. This pluralism may prevent, however, the analyst and the policy maker from drawing clear conclusions in regard to the desirability of alternative income profiles and hence of different government policies.

The objectives of this paper are: (i) to specify a general set of axioms or desired properties which are standard in the analysis of income inequality and poverty measures and to determine the conditions under which these axioms would lead to the specific axioms (F), (M) and (T), and to an additive structure of the aggregate poverty measure - as has been proposed by Sen; (ii) To propose a general, Dalton (1920) - type definition of poverty as the welfare losses resulting from the income gaps, and to determine the general form of poverty indices which corresponds to this definition. In so doing, I will display dual approaches to the derivation of poverty measures both drawing on the concept of welfare losses, and illustrate these measures on a diagram; (iii) To prove that the welfare approach to the specification of poverty measures leads to a corresponding single family of poverty measures and thus to a single measure of income inequality. Furthermore, I will also show that of the various poverty measures that are used in applied work or has been proposed in the literature, only one is a member of this family, while all other indices violate one or more of the axioms.

I. Preliminary: The Basic Properties and the General Structure of Poverty Measures

The following notations apply: S_n denotes the set of n individuals and $\underline{Y}^n = (y_1,\ldots,y_n)\varepsilon\Omega_+^n$, an income profile of these individuals, Ω_+^n being the non-negative n-dimensional Euclidean sub-space. z denotes the "poverty line", i.e., the minimum income below which an individual is considered poor. I assume z to be well defined and known. z partitions the set S_n into two mutually exclusive and exhaustive subsets of "poor" and "rich" individuals. Let S_p denote the set of p poor individuals and $\underline{Y}^p = (y_1,\ldots,y_p)\varepsilon\Omega_+^p$ their income profile; and let S_r denotes the set of $r = n-p$ rich individuals and $\underline{Y}^r = (y_{p+1},\ldots,y_n)\varepsilon\Omega_+^r$ their income profile. By definition $\underline{Y}^r \geq z.\underline{1}^r$ where $\underline{1}^r$ is the appropriate vector of ones.

"Poverty-measure" is a real valued function $P(z,\underline{Y}):\Omega_+^n \times \Omega_+^1 \Longrightarrow R$ having the following properties:

Axiom 1: Non-negativity - $P(z,\underline{Y})$ is non-negative for all addmissible (i.e., non-negative) z and $\underline{Y}$, with equality holding if and only if $y_i \geq z$ for all $i \varepsilon S_n$.

Axiom 2: Monotonicity - $P(z,\underline{Y})$ is non-increasing in $\underline{Y}$ and strictly decreasing in $\underline{Y}^p$.

Axiom 3: Scale Independence - $P(z,\underline{Y})$ is homogeneous of degree zero in $(z,\underline{Y})$

Axiom 4: Anonymity - $P(z,\underline{Y}) = P(z,\pi\underline{Y})$, where π is an arbitrary permutation matrix of size nxn

Axiom 5: Separability - The poverty line z partitions S_n into two strictly separable subsets of poor and rich individuals.

Axiom 6: Heredity - if $P(z,\underline{Y})$ has certain properties over S_n then it has the same properties over all subsets of S_n.

Axiom 1 is simply the requirement that the poverty index attains its minimum value of zero when all individuals have incomes above the poverty line. Axiom 2 states that any increase in the income of a poor individual must strictly reduce the measure of poverty, whereas an increase in the income of a rich individual may either reduce this measure or leave it unchanged. Axiom 3 states that individuals have no money illusion, and $P(z,\underline{Y})$ is consequently independent of the units of measurement.[2] Axiom 4 requires poverty to be a function of the size of the incomes only, independent of the personal lables assigned to incomes.

The separability property in Axiom 5 is the key to the derivation of aggregate poverty measures. It states that the conditional ordering defined by the poverty measure over the income profiles of the poor is independent of the components of $\underline{Y}^r$. Similarly, the conditional ordering over the income profiles of the rich is independent of the components of $\underline{Y}^p$. It should be noted, though, that strict monotonicity of $P(z,\underline{Y})$ in $\underline{Y}^p$ (Axiom 2) suffice, in itself, to ensure that the set S_p is strictly separable in $P(z,\underline{Y})$ from its complement in S_n (see Bigman, 1985).

An important corollary of the separability axiom is that poverty measures having this property can be represented as

$$P(z,\underline{Y}) = \phi(z,P^p(z,\underline{Y}^p),P^r(z,\underline{Y}^r)) \quad (1)$$

where ϕ is increasing in both P^p and P^r (see Gorman, 1968). Thus $\underline{Y}^p$ and $\underline{Y}^r$ can be aggregated into and represented by two independent composite variables P^p and P^r. Axiom 6 is stated as a separate requirement although strict separability together with certain "regularity" conditions can secure that property of poverty measures (see Blackorby, Primont and Russell, 1978, Ch. 3).

The axiom stated thus far are standard. They induce, however, a rather specific structure on the functional form of poverty measures, as we see in the following corollaries.

COROLLARY 1: $P(z,\underline{Y})$ is strictly increasing in z, provided that not all individuals are rich.

PROOF: This is a direct corollary of axioms 2 and 3 which imply that a fall by α percent in z is equivalent to a rise by $\alpha/(1-\alpha)$ percent in all incomes ||.

Although this requirement may appear obvious, not all poverty measures have this property. Elsewhere (Bigman, 1985) I have shown that the commonly used "poverty gap" measure may <u>increase</u> as z declines.

COROLLARY 2: The Focus Axiom - $P(z,\underline{Y})$ is a function of the incomes of the poor individuals only.

PROOF: Axioms 1 and 6 imply that in the representation of the poverty measure in Eq. (1), the measure $P^r(z,\underline{Y}^r)$ is, by definition, zero. $P(z,\underline{Y})$ is thus a monotonic increasing function of $P^p(z,\underline{Y}^p)$ and hence a function of $\underline{Y}^p$ only. ||

The measure P^p, which is the aggregation function of $\underline{Y}^p$ in P, can therefore itself serve as a poverty measure. It can be constructed as follows: (see Blackorby, Primont and Russell, Ch. 3): For a given structure of $P(z,\underline{Y})$, let $P^p(z,\underline{Y}^p)$ be defined as:

$$P^p(z,\underline{Y}^p) = P(z,(Y^p,z.\underline{1}^r)), \qquad (2)$$

the vector of incomes $(\underline{Y}^p,z.\underline{1}^r)$ being the vector $(y_1,\ldots,y_p,z,\ldots,z)$. A consequence of this corollary is that aside from the poverty line and poverty incomes, the only other properties of income profiles $\underline{Y}\varepsilon\Omega^n_+$ relevant for poverty indices are the parameters $\underline{p}$ and $\underline{n}$.

COROLLARY 3: Aggregate poverty measures having the properties specified in the above axioms, can be written as

$$P(z,\underline{Y}) = \Psi(z,P_1(z,y_1),\ldots,P_p(z,y_p)) \qquad (3)$$

where each of the $P_i(z,y_i)$ is independent of the components of $\underline{Y}$ other than y_i itself.

PROOF: Strict monotonicity of $P(z,\underline{Y})$ in $\underline{Y}^p$ implies that both P and P^p are strictly decreasing with any increase in the income of a poor individual. Each singleton in S_p is thus strictly separable in P and P^p. P^p is therefore <u>completely strictly separable</u> in S_p (see Blackorby Primont and Russell Ch. 4) and can thus be written in the form of Eq. (3) ||.

$P_i(z,y_i)$ is the social evaluation of the individual measure of poverty. Similarly to the above construction of P^p in Eq.(2), this measure can be constructed as:

$$P_i(z,y_i) = P(z,\ldots,z,y_i,z,\ldots,z)) \qquad (4)$$

$$1,\ldots,i,i+1,\ldots,n$$

Hence, the only properties of $\underline{Y}\epsilon\Omega_+^n$ which are relevant for the measures P_i are y_i itself and the dimension parameters p and n, i.e.,

$$P_i(z,y_i) = P_i(z,y_i,p,n) \tag{5}$$

COROLLARY 4: The poverty measure can be specified as a function of the relative income gaps.

PROOF: The scale independence axiom implies that the poverty measusre can be written as

$$P(z,\underline{Y}) = P(1,(\tfrac{1}{z}.\underline{Y}))$$

The individual poverty measure can thus be written as

$$P_i = P[1,(1,\ldots,1,\frac{y_i}{z},1,\ldots,1)]$$

$$= P(1,(\underline{1}^n - e_i^n.(\frac{g_i}{z}))] \tag{6}$$

where $\underline{1}^n$ is an n-dimensional vector of ones and e_i^n is the n-dimensional unit vector with 1 in the $i^{\underline{th}}$ place and zero elsewhere. By a simple transformaiton we can define the latter expression to be a function of $(\frac{g_i}{z})$. Hence,

$$P_i^t(\frac{g_i}{z},p,n) \overset{def}{=} P_i[1,(\underline{1}^n-\theta_i^n(\frac{g_i}{z}))] \tag{7}$$

Since this is true for every element P_i of the aggregate poverty element, it is true also for $P(z,\underline{Y})$ itself.||

The functional form of the poverty index suggested by these corollaries is still too general, however. Sen assumed an additive structure and most later writers have adopted this assumption. In the general structure of the poverty measure that we have obtained thus far, additivity can be the result of an additional axiom of linear homogeniety, viz.

Axiom 7: $P(z,\underline{Y})$ is homogeneous of degree one in $(P_1,\ldots,P_p)$ (doubling all the individual poverty ceteris paribus doubles the aggregate index).

Poverty measures having this property can be written, via Eqs.(3) and (7), as

$$P(z,\underline{Y}) = \sum_{i=1}^{p} \Psi_i P_i(\frac{g_i}{z},p,n) \tag{8}$$

where $\Psi_i = \partial\Psi/\partial P_i$: $i=1,\ldots,p$. This structure need not be additive <u>separable</u>, however, because the weight Ψ_i need not be independent of P_j for any $i\neq j$.

An additive structure also characterizes indices that are <u>additive decomposable</u>, i.e., indices having the following property:

Axiom 8: Additive Decomposability - Given a partition of $\underline{Y}$ into F non-empty groups $\underline{Y} = (\underline{Y}_1,\ldots\underline{Y}_F)$, there are weights $w_1,\ldots,w_F$ such that:

$$P(z,\underline{Y}) = \sum_{f=1}^{F} w_f . P^f(z,\underline{Y}_f) \qquad (9)$$

If, in particular, $\underline{Y}$ is partitioned into its individual elements then there are weights $w_1,\ldots,w_n$ such that

$$P(z,\underline{Y}) = \sum_{i=1}^{p} w_i . P_i(\frac{g_i}{z},p,n). \qquad (10)$$

Another property usually required of inequality measures but has relevance also for poverty measures is the following

Axiom 9: Population Replication Principle -

For indices of <u>relative</u> poverty: $P(z,(\underbrace{\underline{Y},\ldots,\underline{Y}}_{q})) = P(z,\underline{Y})$

For indices of <u>absolute</u> poverty: $P(z,(\underbrace{\underline{Y},\ldots,\underline{Y}}_{q})) = q.P(z,\underline{Y})$

An obvious corollary of the last two axioms is that the weights must sum up to unity. Althouhg additive decomposability appears to be a desirable property for geographical or demographical analyses of poverty, all but one of the additive indices that have been proposed in the literature - including that of Sen, are <u>not</u> decomposable. The only exception is the index proposed by Foster, Greer and Thorbecke. All others are, however, homogeneous linear in the individual poverty gaps, and thus the representation in Eq. 8 is suitable for them.

To complete the specification of the aggregate measure of poverty one has to define a measure of individual poverty and to determine the weights.[3] Sen defined the measure of individual poverty to be the individual income gap up to the poverty line, and determined the weights to be the rank order of the interpersonal ordering of the poor by income (his axiom R). The resulting aggregate poverty measure,which summs up the weighted individual measures of poverty, is shown to have the desired properties specified in Axioms (F), (M), and (T). The other poverty measures adopted Sen's definition of the individual poverty gap but differed in their specification of the weights. The next section takes a different approach. It derives a general form of a poverty index on the basis of well defined properties of the social welfare and the individual utility functions. This will allow us to examine the assumptions implicit in each and every index and offer principles for deriving a poverty index that will accord with the established paradigms of welfare theory.

II. Dual Measures of Poverty and Deprivation

Poverty measures can be linked to the community's welfare function via a Dalton-(1920)-type definition of poverty as the welfare losses resulting from the income gaps of the poor, i.e.,

$$P(z,\underline{Y})=1-\frac{W(U_1(y_1),\ldots,U_p(y_p),U_{p+1}(y_{p+1}),\ldots,U_n(y_n))}{W(U_1(z),\ldots,U_p(z),U_{p+1}(y_{p+1}),\ldots,U_n(y_n))} \tag{11}$$

W is a general (ordinal) social evaluation function and the U_i's are the individual utility functions. W and all the U_i's are assumed to be twice continuously differentiable, monotonic strictly increasing and strictly concave. These properties of the welfare and utility functions imply that poverty measures having the general form of Eq. (11) will satisfy the non-negativity (Axiom 1) and monotonicity (Axiom 2) axioms.[4] Strict separability of P in S_p and S_r (Axiom 5) and separable (i.e. not interdependent) individual utility functions is both necessary and sufficient, under the definition in Eq. (11), for W itself to be strictly separable in these two subsets. Hence, W can be represented as

$$W = \rho\{W^p(u_1,(y_1),\ldots,U_p(y_p));W^r(U_{p+1}(y_{p+1}),\ldots,U_n(y_n))\} \qquad (12)$$

where ρ is increasing. Moreover, the Focus Axiom (corollary 2) indicates that the poverty measure is not affected by the components of $\underline{Y}^r$ in that $P(z,(\underline{Y}^p,\underline{Y}^r) = P(z,(\underline{Y}^p,z.\underline{1}^r))$ for all $\underline{Y}^r \geq z.\underline{1}^r$. We can therefore focus our attention on the aggregator function W^p of $[U_1(y_1),\ldots,U_p(y_p)]$ in W, and on the corresponding aggregator function P^p of $(y_1,\ldots,y_p)$ in P, and define poverty as the welfare losses of the poor individuals only, i.e.,

$$P^p(z,\underline{Y}^p) = 1-\frac{W^p(U_1(y_1),\ldots,U_p(y_p))}{W^p(U_1(z),\ldots,U_p(z))} \qquad (13)$$

The aggregator function W^p can be constructed as follows:

$$W^p(U_1(y_1),\ldots,U_p(y_p)) = W(U_1(y_1),\ldots,U_p(y_p),U_{p+1}(z),\ldots,U_n(z))$$

The individual utility function which corresponds to this specification is:

$$U_i = U(\mathrm{Min}\{y_i, z\}) \quad : \quad i=1,\ldots,n$$

This representation of the individual utility function is similar to that proposed by Hagenaars (1984). Hence, the only parameters of the vector $\underline{Y}^r$ relevant for W^p and thus also for P^p are the dimension parameters $\underline{p}$ and $\underline{n}$. On these grounds, let us therefore <u>define</u> the overall poverty measure as

$$P(z,\underline{Y}) = H.P^p(z,\underline{Y}^p) \tag{14}$$

where H = p/n is the Head-Count ratio. This definition is essentially a generalization of Sen's Normalization Axiom which requires, in the special case that all the poor have precisely the same income[5], that the poverty measure will have the form.

$$P(z,\underline{Y}) = H.G, \tag{15}$$

where G is the income gap ratio: $[(z-\bar{y}_p)/z]$, $\bar{y}_p$ being the average income of the poor. Sen thus <u>assumes</u> that in that special case the average income gap ratio G represents the poverty measure P^p, and that the individual gap ratio $(z-y_i)/z$ represents the individual measure of poverty - $P_i(z,y_i)$. The definition in Eq. (14) is more general in that it allows other representaitons of the poverty measures. Later on in the paper I will explicitly examine the assumptions on the individual utility function and on the social welfare function that are implicit in

Sen's representation of the individual and the aggregate measures of poverty.

To determine the functional form and the mathematical properties of P^p on the basis of its definition in Eq.(13) and the standard properties of "well-behaved" social and individual welfare functions, let us assume that individuals are identical in all respects except perhaps for their income, in the sense that their utility functions $U(y)$ are identical. (This "symmetry" axiom of the welfare function corresponds to the Anonymity axiom of the poverty measure). The <u>representative income of the poor</u> is defined as that income y_p^* which, if received by all the poor, would be ranked as socially equally desirable as the current distribution[6], i.e.,

$$W^p(U_1(y_1),\ldots,U_p(y_p)) = W^p(U_1(y_p^*),\ldots,U_p(y_p^*)) \qquad (16)$$

Let us further assume that W^p is positively linearly homogeneous (PLH) in $(U_1,\ldots,U_p)$[7]. In this case $W^p(U_1(y_1),\ldots,U_p(y_p) = U(y_p^*).W^p(\underline{1}^p)$, and the poverty measure is thus given by

$$P^p(z,\underline{Y}^p) = 1 - \frac{U(y_p^*)}{U(z)}. \qquad (17)$$

The defficiency of this definition (as well as of the original definition in Eqs.(11) or (13)) is that this normalization is not invariant to linear transformation of the functions U or W (see Atkinson, 1970)[8]. It highlights, however, an important property of poverty measures: If the poverty measure is defined as in Eq.(17) and if the individual utility functions are strictly concave, then the aggregate measure must be strictly convex. As a consequence, if we raise the equally distributed equivalent income or, more generally, the income of any poor individual, the aggregate poverty measure would decline but, ~~with~~ additional increases of equal amounts in ~~of~~ that person's income, would reduce the aggregate poverty measure at decreasing rates. Strict convexity of the aggregate poverty measure is thus a manifestation of the decreasing marginal utility of income in strictly concave utility functions and, conversely, strictly concave utility functions determine an aggregate poverty measure which is strictly convex. Given the definition of poverty measures in Eq.(17) [or in Eq.(13)], let us now examine some of the specific measures of poverty that has been proposed.

Blackorby and Donaldson (1980) explicitly assumed the poverty gap to be homogeneous linear in the poverty gap $(z-y_p^*)/z$, so that "doubling the percentage shortfall (in income) ceteris paribus doubles the index" (1980, p. 1055). Their index is thus given by

$$P_{BD} = H.[\frac{z-y_p^*}{z}] \tag{18}$$

The corresponding poverty measure of the poor P^p is thus merely the income gap ratio $(z-y_p^*)/z$. Let us re-write their index as:

$$P_{BD} = = H.[1 - \frac{y_p^*}{\bar{y}_p} . \frac{\bar{y}_p}{z}], \tag{19}$$

Let

$$I_p = 1 - \frac{y_p^*}{\bar{y}_p} \tag{20}$$

be an inequality measure of the poor's incomes having the form of the inequality index proposed by Atkinson (1970), and let

$$G = \frac{z-\bar{y}_p}{z} \tag{21}$$

be the average income gap ratio of the poor. The poverty measure P_{BD} can thus be written as

$$P_{BD} = H[1-(1-I_p)(1-G)]$$

$$= H[G+(1-G)I_p] = H.G\ [1+\frac{(1-G)}{G}\ .\ I_p] \tag{22}$$

If I_p is the Gini measure of inequality taken over the vector of poor incomes, P_{BD} would then be (a close approximation of) Sen's measures.[9] Blackorby and Donaldson thus seem to propose a general class of poverty measures, that may differ one from the other only in their measure the income of inequality among the poor.

Clark, Hemming and Ulph (1981) proposed a family of poverty measures which is also homogeneous linear in the poverty gap. This measure has the form

$$P_{CHU} = H.G.(\frac{g_p^*}{\bar{g}_p}) = H.G.(1+I_g) \tag{23}$$

where $\bar{g}_p$ is the mean poverty gap, g_p^* the representative income gap of the poor, and $I_g = [\frac{g^*_p}{\bar{g}_p} - 1]$ is an index of inequality of the income gaps. They also show that their measure contains as, a special case, Sen's poverty measure when I_g is the Gini coefficient as applied to the income gaps. It can be easily shown that the measure in eq. (23) contains not only Sen's measure but also the measures of Kakwani, Thon, Anand, and several others.(Interestingly, it does not, however contain the specific measure proposed by Clark et al. themselves, as we shall see later on). This is so because the definition of the poverty measure in Eq.(23) is in fact the dual or the mirror image of the definition of poverty measures in Eq.(18). This can be seen by writing the poverty measure in either one of the following two ways: One, as in Eq.(18), i.e.,

$$P = H.G.\{1+[(1-G)/G].I_p\} \tag{24}$$

and the other as in eq.(23), i.e.,

$$P' = = H.G.(1+I_g) \tag{25}$$

If y^*_p is a weighted average of the individual incomes the two definitions are identical because in this case $g^*_p = z-y^*_p$ and hence,

$$\frac{g^*_p}{\bar{g}_p} = \frac{z-y^*_p}{z-\bar{y}_p} = \frac{z}{z-\bar{y}_p} - \frac{y^*_p}{\bar{y}_p} \cdot \frac{\bar{y}_p}{z-\bar{y}_p} = = [1 + \frac{(1-G)}{G} \cdot I_p] \tag{26}$$

In other words to any measure of income inequality included in the definition of the poverty measure in Eq.(18), corresponds a measure of inequality of the poverty <u>gaps</u> included in the definition in Eq.(25). If, in addition $g_p^* = z - y_p^*$ then the two measures are identical and

$$I_g = \frac{1-G}{G} \cdot I_p.$$

An important corollary of these dual specifications of the pertinent poverty indices is the following:

COROLLARY 5: If the poverty measure can be expressed as either

$$P(z,\underline{Y}) = H.G.[1+\frac{1-G}{G} \cdot I_p]$$

or

$$P(z,\underline{Y}) = H.G.[1+I_g]$$

or both

then necessary and sufficient condition for this index to satisfy axioms (F), (M) and (WT) (weak transfer) is that the inequality index (either I_g or I_p) is S-convex (i.e., agree with the Lorenz quasi ordering)

COMMENT: The corollary is limited to the <u>weak transfer</u> axiom (WT) which constrains the transfers to poor individuals only and requires the recepient to remain poor even after the transfer. The reason is that if, as a result of the transfer, the recepient crosses the poverty line, the weights of all the remaining poor (which may depend on the number of poor individuals) can consequently decrease and the poverty measure can therefore fall, in contradiction to Axiom (T). This may be the case of the Head Count index, of Sen's index, of the index proposed by Clark et al. (see Thon, 1982) etc.

PROOF: To prove the corollary, notice that the Focus Axiom follows directly from the definitions of H, G, and the inequality measures I_g or I_p. Regressive transfers among the poor after which the recepient still remained poor, will change neither H nor G. The inequality measure however will rise if and only if I_g (and I_p) is S-convex. To verify monotonicity, write the index as:

$$P(z,\underline{Y}) = H.(\frac{g_p^*}{z}),$$

and notice that g_p^* must rise monotonically with any fall in the income of a poor individual. If, however, the income of a rich individual falls, neither g_p^* nor G would rise - once z is fixed ||.

The welfare analysis of poverty indicates that the measure of poverty should express the total welfare losses which are due in part to the poverty gap and in part to the income inequality among the poor. The specific form of the measure must be determined so as to represent well behaved individual utility and social welfare functions and, at the same time, be independent of linear transformations of these functions. A suggestive candidate for a poverty measure is the one defined in Eq.

(18) which has a great deal of similarity to the inequality measure suggested by Atkinson (1970). One example, is therefore the family of indices proposed by Blackorby and Donaldson. It is easy to verify that the indices of Sen, Thon, Kakwani, Anand as well as the Head-Count and the Poverty-Gap indices all belong to this family. Consequently, these indices too can be written, interchangeably, either in the form of Eq.(22) or in the form of Eq.(23), each index having, however, a different measures of inequality thereby representing a different degree of 'inequality aversion'.

Clark <u>et al.</u> assume an individual deprivation function of the form

$$d(g_i) = (1/\alpha)g_i^{\alpha} \quad : i=1,\ldots,p.$$

where the inequality aversion should be $\alpha > 1$ for <u>strict</u> cancavity. They further assumed a symmetric and additive social welfare function of the form

$$-W(\underset{\sim}{g},\alpha) = \sum_{i=1}^{p} d(g_i)$$

where $\underset{\sim}{g} = (g_1,\ldots,g_p)$ is the vector of the income gaps. Without making, however, any further use of that social welfare function, they define the poverty index as:

$$P(z,\underset{\sim}{g}) = H.G.\left(\frac{g_p^*}{\bar{g}_p}\right). \tag{27}$$

This definition is, however, not consistent with their own earlier definition of the individual deprivation and social welfare functions. The reason: If poverty represents the loss of welfare due to the income gaps, so that $P(z,g) = - W(g,\alpha)$, then this measure must be strictly <u>convex</u> when the deprivation functions are strictly concave in income. But the definition of the index in Eq.(27) and the definition of the 'equally distributed equivalent poverty gap' as:

$$g_p^* = [(1/p) \sum_{i=1}^{p} g_i^{\alpha}]^{1/\alpha},$$

that has been proposed by Clark <u>et. al</u>., imply a poverty measure which is homogeneous <u>linear</u> in the poverty gaps for all $\alpha \geq 1$ and thus does not exhibit the decreasing marginal utility of income. I will return to this issue in section IV. This comment highlights, however, an important defficiency of the poverty measures defined in either Eq.(18) or Eq.(23): Indices which are PLH in the poverty gap, fail to reflect well behaved individual utility and social welfare functions because they do not exhibit the decreasing marginal utility of income, and the corresponding "increasing poverty aversion"[12].

III. Illustrations and Extensions

To illustrate the two components of poverty measures, namely, the one expressing the poverty-gap and the other expressing the inequality aversion, write Eq.(22) as

$$P(z,\underline{Y}) = H.G + H.(1-G).I_p \qquad (29)$$

The element H.G expresses the welfare losses due to the poverty-gap. As noted earlier, Sen's Normalization Axiom, requires the poverty measures to be equal to H.G if all the poor have the same income. The second element $H(1-G).I_p$ expresses the welfare losses due to the inequality in the distribution of incomes among the poor. These two components are illustrated in Figure 1: The line $\overline{OF}$ represents an equal distribution of the poverty line income z. $\overline{BF}$ thus measures the poverty-line income z relative to the mean income. The set of poor individuals is determined at that point on the Lorenz curve in which the tangent is parallel to $\overline{OF}$. This would be point M in the Figure. The Head-Count measure is thus given by the distance $\overline{OA} = F(z)$, F being the cumulative distribution of individuals. The poverty-gap ratio $(z-\bar{y}_p)/z$. is given by the ratio $\overline{EF}/\overline{BF}$ which is also equal to $\overline{MN}/\overline{AN}$, since

$$G = \frac{z-\bar{y}_p}{z} = \frac{z-\phi(z)}{z} = \frac{\overline{BF}-\overline{BC}}{\overline{BF}} = \frac{\overline{CF}}{\overline{BF}} = \frac{\overline{MN}}{\overline{AN}}$$

where ϕ is the cumulative distribution of income. Hence,

$$H.G = \overline{OA} \cdot \frac{\overline{MN}}{\overline{AN}} \cdot = \overline{QM} = \overline{OK}.$$

The measure H.G thus has the following interpretation: If we redistribute all the incomes of the poor so as to bring as many of them as possible up to the poverty line, (thereby changing the income distribution to $\overline{OKMT}$), then H.G would be that percentage of the poor population which is left with no income at all. The measure

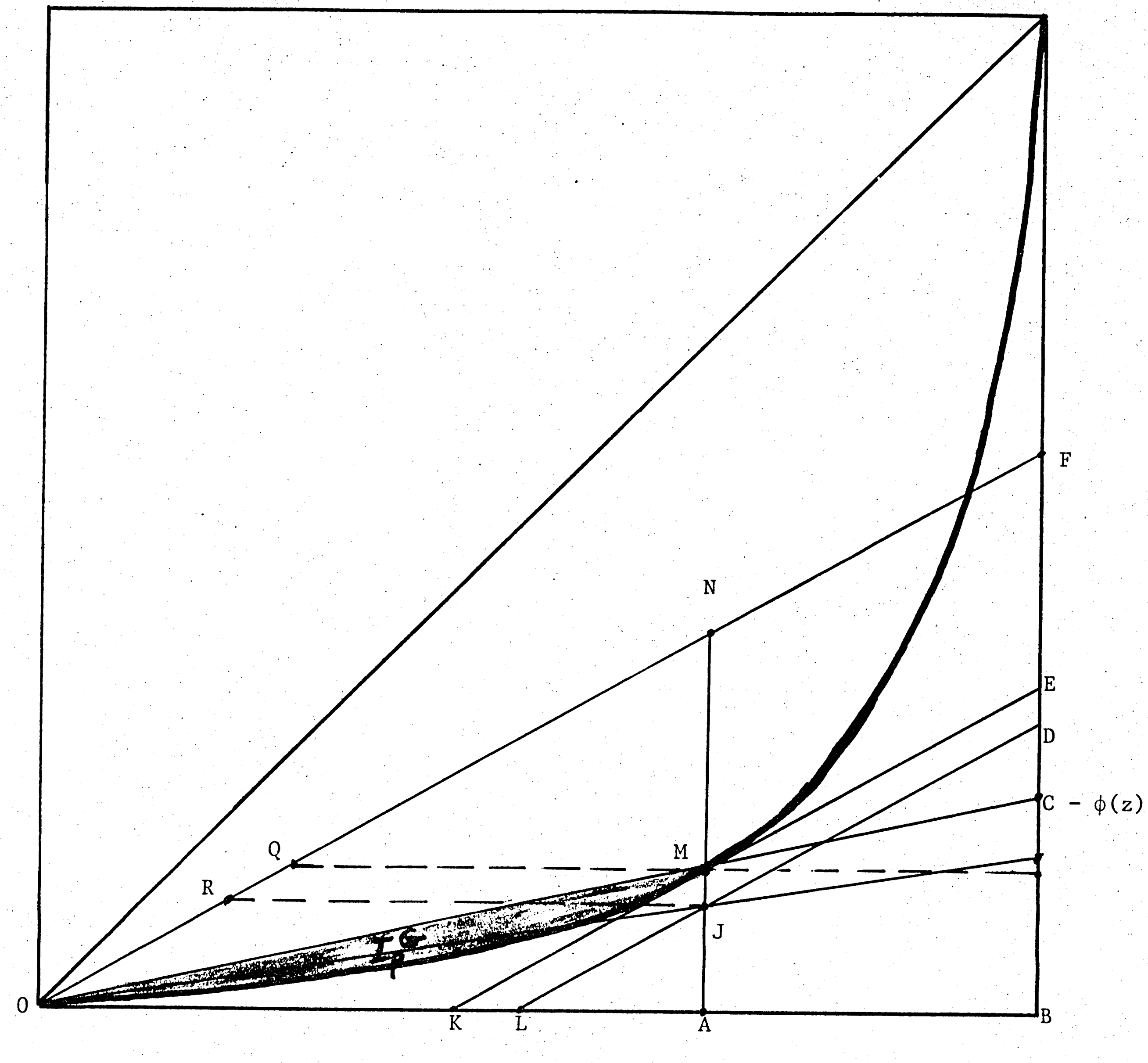

Figure 1: Poverty measures on the Lorenz Curve

$H.G = F(z).[1-\phi(z)/z$, which can be termed "the relative z deviation" (in analogy to the "relative mean deviation") has been advocated by some as an index of inequality (see Elteto and Frigyes (1968)). It was demonstrated by Atkinson (1970), however, that this measure is completely insensitive to transfers between poor individuals. For the same reason this index in itself cannot serve as a poverty index.

The element $[(1-G).H]$, which is given by $\overline{AK}$ in the figure, is that percentage of the poor population which can be brougth up to the poverty line. The inequality losses are measured as a fraciton of that element. To evaluate these losses we determine the representative or the "equally distributed equivalent income" of the poor, y_p^*, from Atkinson's general measure of inequality:

$$y_p^* = \bar{y}_p(1-I_p) \tag{33}$$

If the income inequality is measured by the Gini coefficient, then I_p in the Figure would be the ratio between the shaded area, denoted by I_p^G, and the area of the triangle $\overline{OMA}$. The representative income of the poor, y_p^*, would thus be determined as a fraction, say $\overline{BD}/\overline{BC}$, of their average income $\bar{y}_p$. The inequality measure would thus be given by $\overline{DC}/\overline{BC} = \overline{JM}/\overline{AM}$, and the inequality losses by

$$H.(1-G).I_p = [1-\overline{AK} \cdot \frac{\overline{JM}}{\overline{AM}}] = \overline{KL}.$$

Total welfare losses, as quantified by the poverty measure, are thus given by

$$P(z,\underline{Y}) = \overline{OK} + \overline{KL} = \overline{OL}.$$

We can easily verify the dual approach to the measurement of poverty, i.e.,

$$P(z,\underline{Y}) = H.(\frac{g_p^*}{z}) = \overline{OA}.\frac{\overline{DF}}{\overline{BF}}$$

$$= \overline{OA} \; . \; \frac{\overline{JN}}{\overline{AN}} = \frac{\overline{OA}}{\overline{AN}} \; . \; \overline{JN} = \overline{RJ} = \overline{OL}.$$

thereby illustrating the equivalence between the measure in Eq.(19) and the one in Eq.(23). The general form of the poverty measure in either one of these two equations is not limited, however, to any specific measure of inequality. The choice between alternative possible inequality measures should thus be made so as to reflect the degree of 'inequality aversion' implicit in the social evaluation functions. This is further illustrated in Figure 2.[13] The figure describes a two-person community, both having incomes below the poverty line. The initial income distribution is assumed to be at point P. To derive the total welfare losses we draw an indifference curve from the social evaluation function: $W(U(y_1);U(y_2))$, which crosses point P. If this community is 'inequality averse', the indifference curve will be strictly convex. The "equally distributed equivalnet income" is obtained at point A - at which the indifference curve crosses the (negative) 45^0 line $\overline{OC}$, which represents equal distributions. The (positive) 45^0 line $\overline{PR}$ represents different distributions between the two individuals of the same total

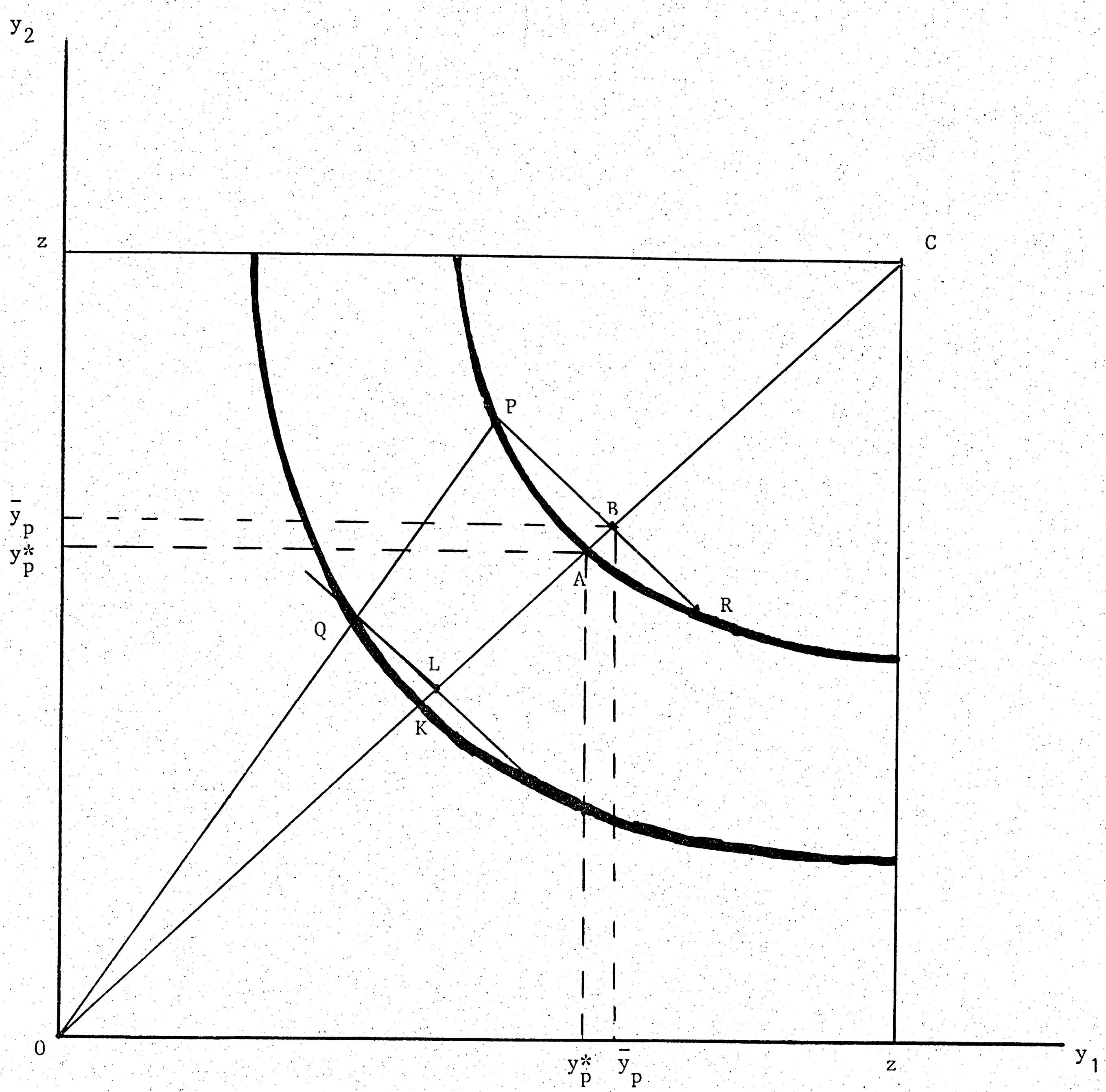

Figure 2: The Welfare Measures of Poverty and Deprivation

quantity available at point P. At point B, the two will receive the average quantity $\bar{y}_p$. The poverty measure at point P is, by definition, given by

$$P(z,\underline{Y}(P)) = 1-\frac{y_p^*}{z} = 1-\frac{\bar{y}_p}{z} \cdot \frac{y_p^*}{\bar{y}_p} = 1 - \frac{\overline{OB}}{\overline{OC}} \cdot \frac{\overline{OA}}{\overline{OB}} = \frac{\overline{AC}}{\overline{OC}}$$

The ratio $[\overline{BC}/\overline{OC}]$ measures the income gap while the ratio $[\overline{AB}/\overline{OB}]$ is the corresponding measure of income inequality. The welfare losses on account of the income inequality depend on the curvature of the indifference curve, i.e., on the degree of 'inequality aversion' implicit in the social evaluation function. The 'dual' poverty measure is given by

$$P(z,\underline{Y}(P)) = \frac{g_p^*}{\bar{g}_p} \cdot \frac{\bar{g}_p}{z} = \frac{\overline{AC}}{\overline{BC}} \cdot \frac{\overline{BC}}{\overline{OC}} = \frac{\overline{AC}}{\overline{OC}} \quad ,$$

and the corresponding measure of the inequality of the income gaps is given by $[\overline{AB}/\overline{BC}]$

Consider now a decrease in the income of the two individuals, represented by a shift along the line $\overline{OP}$ - which leaves the income ratio (y_1/y_2), and thus the relative inequality, unchanged - to the point Q. The poverty measure at that point is given by:

$$P(z,\underline{Y}(Q)) = \frac{\overline{KC}}{\overline{OC}}$$

The corresponding measure of income inequality is $\overline{KL}/\overline{OL}$ and of the inequality of the income gaps is $\overline{KL}/\overline{LC}$. Hence, ceteris paribus the larger the income-gap the larger the income inequality relative to the inequality of the income gaps.

If welfare is a homothetic function of (y_1,y_2), the expansion paths are straight lines out of the origin, and the ratio $y_p^*/\bar{y}_p$ would then be invariant to proportional shifts. In this case the 'equally distributed equivalent income' y_p^* would therefore be homogeneous linear in (y_1,y_2), i.e., doubling the incomes (leaving the poverty line unchanged) doubles also y_p^*. The inequality measure would therefore remain unchanged as the two incomes are doubled while the income gap would be cut in half. In the next section I examine the implications of the homnotheticity assumptions on the form of the poverty measure.

Figure 2 can also serve to illustrate the difference between S-convexity and "ordinary" convexity of the poverty measure. The term S-convexity is related to the degree of 'inequality aversion' of the poverty index and is reflected by the curvature of the indifference curves, e.g., it expresses the desirability of transferring income from one individual to the other. "Ordinary" convexity in this context is related to shifts of the indifference curves along the expansion paths as an effect of changes in <u>both</u> incomes. There is no necessary mathematical relationship between the two terms. Sen's measure, for example, is S-convex but it is not strictly convex in the ordinary sense of this term. Its S-convexity is exhibited in its weighting scheme which gives higher weights the lower the income. At the same time the measure is homogeneous of degree 1 in incomes and thus does not exhibit the decreasing marginal utility of income - as required of strictly convex measures. The two concepts of convexity thus refer to two different mathematical properties of the poverty index, and if both are desired on ethical grounds then one has to specify the correspoinding requirements in two different axioms. This I will do in the next section.

IV. The Form of Poverty Measures and the Form of the Utility Function.

Foster *et.al* proposed a family of poverty measures of the form

$$P_{\alpha}^{F} = \frac{1}{n} \sum_{i=1}^{p} \left(\frac{g_i}{z}\right)^{\alpha} \tag{30}$$

For $\alpha = 1$, this measure is simply G.H. For $\alpha = 2$, the measure takes the normalized gaps (g_i/z) themselves as weights. This measure can be written in the following general form

$$P_{\alpha}^{F} = H.G^{\alpha}[1+(I_g^A)]^{\alpha} \quad :\alpha \geq 1. \tag{31}$$

where I_g^A is the index of inequality of the income gaps of the form proposed by Atkinson, i.e.,

$$I_g^A = \left[\frac{1}{p} \sum_{i=1}^{P} \left(\frac{g_i}{\bar{g}_p}\right)^{\alpha}\right]^{1/\alpha} -1 \tag{32}$$

For $\alpha = 2$, the index I_g^A is simply the coefficient of variation, and the corresponding poverty measure can thus be written as

$$P_2^F = H.[G^2+(1-G)^2.(I_p^{CV})^2]$$

where I_p^{CV} is the coefficient of variation of the poor incomes. For $\alpha = 1$, the index of inequality is $I_g^1 = [(1/P).\Sigma(g_i/\bar{g}_p)]-1$, which is, of course, identically zero. Hence, the Poverty Gap measure corresponds to a welfare funciton which is completely insensitive to the inequality in the distribution of income among the poor.

Earlier I have commented that the poverty measure proposed by Clark <u>et al.</u> is not consistent with their own definition of the individual deprivation functions and the social welfare function. These deprivation and welfare functions suggest, however, a poverty measure of the form proposed by Foster <u>et al.</u> To see this, write the individual deprivation functions proposed by Clark <u>et. al.</u> as:

$$d(g_i) = (g_i/z)^{\alpha} \quad i=1,\ldots,p, \tag{33}$$

and define the poverty measure as the net welfare losses on account of the poverty gaps, given by:

$$P^p(z,\underline{y}^p) = -W(\underline{g},\alpha) = \frac{1}{p}\sum_{i=1}^{p} d(g_i) = \frac{1}{p}\sum_{i=1}^{p}(\frac{g_i}{z})^{\alpha}, \tag{34}$$

Hence, the aggregate poverty measure that comes out of the definition is given by:

$$P(z,\underline{Y}) = H.P^{p}(z,\underline{Y}^{p}) = \frac{1}{n}\sum_{i=1}^{p}(\frac{g_i}{z})^{\alpha} = P^{F}_{\alpha}$$

This poverty measure is thus identical to that proposed by Foster et.al. but is different from the one proposed by Clark et al. This definition of the poverty measure raises however, a question as to how restrictive is the corresponding specification of the individual deprivation functions. Put differently, how general is this specification of the poverty measure. The central proposition of this section states that under certain, rather general conditions on the group evaluation funciton W^p, the welfare approach to and the Dalton-type definition of the poverty measure in Eq. (13), leads to a single family of poverty measures, which has the same general structure as the measure proposed by Foster et al and contains an Atkinson-type measure as the corresponding index of inequality.

The key assumption unerlying this result is that the group welfare function is a homothetic function of $(y_1,...,y_p)$. In this case the inequality index which is implicit in the poverty measure, is invariant with respect to proportional shifts in incomes. This has been observed earlier in reference to Figure 2 by noting that in this case (and in this case only), the expansion paths are straight lines out of the origin along which the income proportions y_1/y_2 do not change.

By referring to results of Pratt (1964) and Arrow (1965), the following theorem shows that homotheticity of the group welfare function (which would then exhibit constant relative 'inequality aversion' - as referred to by Atkinson (p. 251)) implies that the utility function has the form

$$U(y) = ky^{\xi} \qquad :0<\xi<1,\ k>0,$$

where ξ is constrained to the (0,1) interval for monotonicity and concavity of U.

THEOREM 1: Consider the poverty measure defined in Eq.(13), and assume that the gorup welfare function is homothetic. Necessary and sufficient condition for this poverty measure to satisfy axioms (1), (2) and (3) is that the utility funciton has the form.

$$U(y_i) = k.y_i^{\xi} : y_i \leq z;\ \xi>0.$$

PROOF: (i) Necessary - If the utility function has the log-linear, constant relative 'inequality aversion' form [i.e., $\{-u''(y)/u'(y)\}.y$ = const.] then the corresponding poverty measure would be

$$P^p(z,\underline{Y}^p) = 1-(\frac{y_p^*}{z})^{\xi} \quad : \quad \xi>0$$

It is easy to verify that this poverty measure satisfies the three axioms. Non-negativity is assured via the earlier definition of the utility function $U_i = U_i[\mathrm{Min}\{y_i,z\}]$. As a result, y_p^* would be smaller than or equal to the poverty line z, with equality holding iff $y_i \geq z$ for all i. With a homothetic group welfare function, y_p^* is strictly rising with $\underline{Y}^p$, thereby securing the Monotonicity axiom. The Scale Independence axiom follows directly from the definition of the poverty measure in terms of the ratio (y_p^*/z) and because the ratio $y_p^*/\bar{y}_p$ is invariant to proportional shifts.

(ii) Sufficient - Let $P^p(z,\underline{Y}^p)$ be a real valued function defined in Eq.(13) (for a homothetic group welfare function) which satisfies the three axioms. By the Scale Independence Axiom we can write P^p as

$$P^p(z,\underline{Y}^p) = P^p(1,\tfrac{1}{z}\underline{Y}^p)$$

But since y_p^* is homogeneous of degree 1 in $\underline{Y}^p$, we can write the poverty index <u>either</u> as

$$P(z,\underline{Y}^p) = 1- \frac{U(y_p^*)}{U(z)}$$

<u>or</u> as

$$P(1,\tfrac{1}{z}\underline{Y}^p) = 1- \frac{U(y_p^*/z)}{U(1)}$$

Let $U(1) = 1/k \quad : \quad k>0$. Hence,

$$k.U(y_p^*/z) = U(y_p^*)/U(z) \quad : \; y_p^* \geqq 0,\; z>0.$$

To complete the proof we make use of the following Lemma.

LEMMA: Let $f(x)$ be continuously differentiable and strictly positive function for all $x \varepsilon R_+$. Necessary and sufficient condition for the equality: $f(x)/f(y) = g(x,y)$, to hold for all x,y in that domain, is that f has the form $f(x) = x^{\mu}$, and g is proportional to f.

The 'necessary' part of the Lemma is self evident. The sufficient part is proved in the appendix. ||

CORROLARY 6: If (and only if) the poverty measure P^P, defined in Eq.(13) satisfies axioms (1), (2) and (3) then it has the form

$$P^P(z,\underline{y}^P) = 1-(\frac{y_p^*}{z})^{\xi} \quad :\xi>0 \quad ||.$$

CORROLARY 7: If (and only if) the poverty measure defined in Eq.(13) satisfies axioms (1), (2) and (3), then the corresponding group welfare function is the symmetric mean of order ξ and the 'equally distributed equivalent income' is given by

$$y_p^* = [\sum_{i=1}^{p} r_i y_i^{\xi}]^{1/\xi} \quad : \quad \xi>0, \sum_{i=1}^{p} r_i = 1 \quad ||$$

Strict concavity of the individual utility function and thus strict convexity of the corresponding poverty index, implies that $\xi<1$.

Define, as before, $I_p = [1-(y_p^*/\bar{y}_p)]$. Hence,

COROLLARY 8: If (and only if) the poverty measure defined in Eq.(13)

satisfies axioms (1), (2) and (3) and exhibits strict convexity, then it must have the form

$$P(z,\underline{Y}) = H.[1-(1-G)^{\xi}(1-I_p^A)^{\xi}] \quad : \quad 0<\xi<1 \tag{35}$$

where I_p^A is the Atkinson-type measure of the income inequality. ||.

The poverty measure in Eq.(35) is different, however, from that in Eq.(31). Although these two measures are dual to each other, they will be equal if and only if $\alpha = \xi = 1$, in which case they are reduced to PLH poverty measures of the form discussed in Section II. The difference between these two measures is illustrated in the next section. The following theorem provides the motivation for the type of poverty measusre examined in Eq.(31).

THEOREM 2: Consider the poverty measusre defined as

$$P^p(z,\underline{Y}^p) = -W(g) = D(d(g_1),\ldots,d(g_p)) \tag{36}$$

where $d(g_i)$ is the individual deprivation function, and D is the group deprivation function. Assume W to be homothetic in $\underline{g}$. Necessary and sufficient condition for that poverty measure to satisfy axioms (1), (2) and (3) is that the deprivation function has the form

$$d(g_i) = g_i^{\beta} \quad : \quad \beta>0.$$

Furthermore, if (and only if) that poverty measure satisfies axioms (1), (2) and (3) and exhibits strict convexity then it has the form

$$P(z,\underline{Y}) = H.G^{\beta}(1+I_g^A)^{\beta} \quad : \quad \beta>1,$$

where I_g^A is an Atkinson-type measure of inequality of the income gaps, having the form:

$$I_g^A = [\sum_{i=1}^{p} s_i(\frac{g_i}{\bar{g}_p})^{\beta}]^{1/\beta} -1 \quad : \quad \beta>1; \sum_{i=1}^{p} s_i = 1.$$

The proof of the theorem is essentially the same as the proof of Theorem 1.||.

Foster's index is thus a member of that second family.

V. Numerical Illustrations

The different indices that have been proposed in the literature reflect different degrees of 'poverty aversion' implicit in the corresponding individual utility function. The choice between these indices (provided that they meet the requirements specified in the axioms) should be made so as to reflect the "true" aversions of the community concerned. To illustrate the different sensitivities of the indices, I have calculated their values in two numerical examples. The first is for the following vector of incomes: $\underline{Y}$ = (0,1,2,3,4,5,6,7,8,9), and for poverty lines ranging from z = 10 to z = 100. The 10 individuals of that community are therefore always poor. To compare the sensitivity

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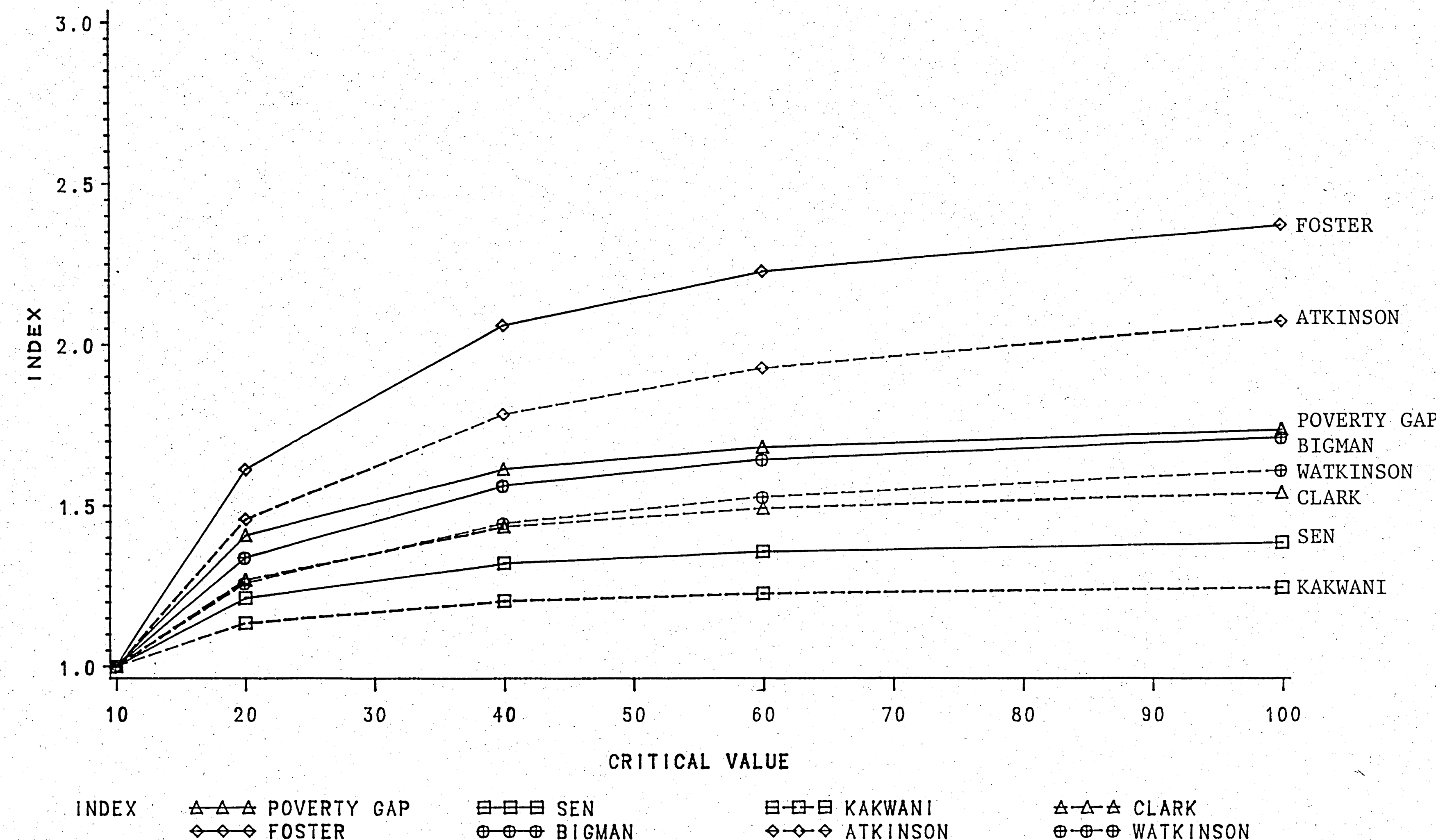

of the different indices to these changes in the poverty line, I have normalized their values to be 1.0 for z = 10. The normalized values of the indices for different levels of the poverty line are summarized in Figure 3. The exact formulae of the indices and the specific parameter values given in appendix 2. In this illustration the index of Foster et al. is revealed to be the most sensitive to changes in the poverty line. Sen's and Kakwani's indices are relatively less sensitive because they put most of the weight on the individuals relative deprivation as measured by his rank vis-a-vis the others. In the present example, however, these ranks do not change. Several indices (e.g., Takayama, the Head-Count) which do not change at all in this example (where all individuals are always poor) are omitted.

The second example is for a much larger sample of incomes drawn from a Normal distribution with mean 100 and variance 20. The poverty line rises from 90 to 200. For a poverty line in these experiments above (approximately) 170 all the incomes are below the poverty line. As before, the values of the indices have been normalized to be 100 at z = 90 in order to allow the comparison. The values of the indices for different poverty lines are presented in Figure 4. Again, the index of Foster et al. proves to be the most sensitive to the changes in the poverty line. The Head Count and Takayama's indices do not change at all as z rises above 170, i.e., when all incomes fall below the poverty line. These two indices will not therefore satisfy the Monotonicity axiom in this case.

The point of these numerical illustrations is not to discredit some indices or credit others but rather to demonstrate that the pluralism in

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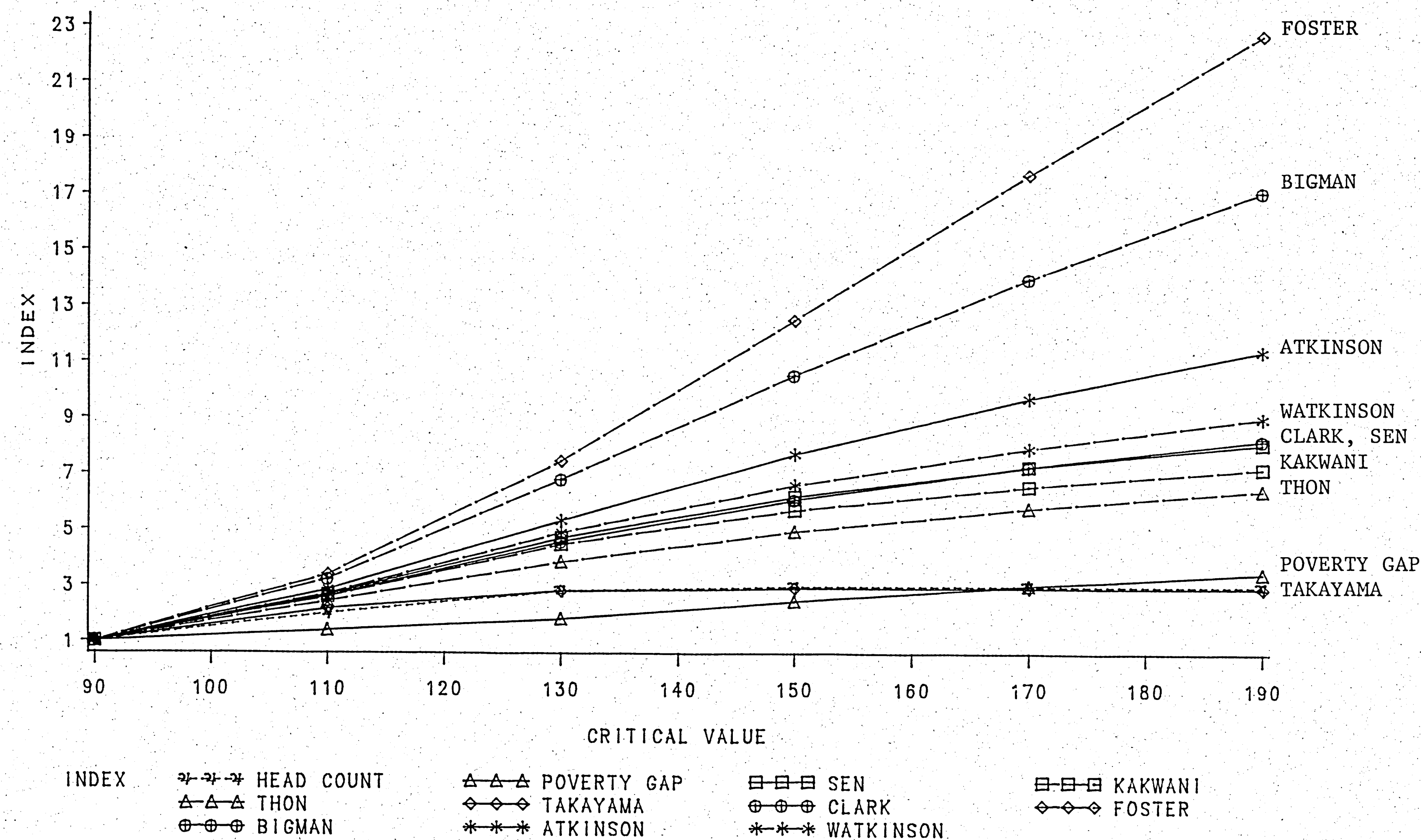

poverty measurement is mostly a reflection of the pluralism in society's ethical preferences, its sensitivity to the basic needs of the poor, its aversion to inequality, and its awareness to people's own perception of poverty and deprivation.

APPENDIX

Lemma: If $f(x)/f(y) = f(x/y)$ for all strictly positive x,y, and if f is continuously differentiable, then the only functional form that f can assume if $f(x) = x^{\alpha}$

PROOF:

It is obvious that $f(1) = 1$, because by definition, $f(x) = f(x/1) = f(x)/f(1)$. Since the equality holds at any point, we must also have:

$$\frac{d}{dx}\left(\frac{f(x)}{f(y)}\right) = \frac{d}{dx}\left(f\left(\frac{x}{y}\right)\right), \text{ for constant } y, \text{ and}$$

$$\frac{d}{dy}\left(\frac{f(x)}{f(y)}\right) = \frac{d}{dy}\left(f\left(\frac{x}{y}\right)\right), \text{ for constant } x.$$

Hence $$\frac{f'(x)}{f(y)} = f'\left(\frac{x}{y}\right) \cdot \frac{1}{y} \tag{A.1}$$

and $$\frac{f(x).f'(y)}{f^2(y)} = f'\left(\frac{x}{y}\right) \cdot \frac{x}{y^2} \tag{A.2}$$

By inserting (A.1) into (A.2) we get

$$\frac{f'(x)}{f(x)} \cdot x = \frac{f'(y)}{f(y)} \cdot y \quad \text{:for all } x,y \in R+.$$

In particular, if we take $y=1$, we get

$$\frac{f'(x)}{f(x)} \cdot x = f'(1) = \alpha$$

Put differently, f is a constatn elasticity function. The solution of this familiar differential equation is

$$f(x) = kx^{\alpha}$$

But since $f(1) = 1$, we must have $k=1$. Another way of proving the lemma, suggested to me by Dr. Lifson, is by transforming the variables into natural logarithm forms and drawing on the familiar result that the only solution to an equation of the form $g(u+v)=g(u)+g(v)$ is $g(u) = \alpha u+\beta$. ||

APPENDIX II

This appendix lists the mathematical formulae of the poverty indices that were examined in the simulation analysis. The notations are those of the paper. To emphasize the effect of the poverty line (which changes in the course of the simulation analysis) on the number of the poor, I will denote their number by p(z).

1. Head Count:

$$P_H = \frac{p(z)}{n}$$

2. Poverty Gap:

$$P_G = \frac{1}{p(z)} \cdot \sum_{i=1}^{p(z)} \left(\frac{g_i}{z}\right)$$

3. Sen:

$$P_S = \frac{2}{n(p(z)+1)} \cdot \sum_{i=1}^{p(z)} r(i) \left(\frac{g_i}{z}\right)$$

where $r(i) = (p(z) + 1-i)$ is the rank of the ith individual among the poor, given by:

4. Kakwani:

$$P_K = \frac{p(z)}{n \, . \, \phi(k)} \cdot \sum_{i=1}^{P(z)} [r(i)]^k \left(\frac{g_i}{z}\right)$$

where

$$\phi(k) = \sum_{i=1}^{p(z)} i^k \qquad : \qquad k \geq 0$$

If $k = 1$ then P_K is simply P_S. In the simulation analysis I have assumed $k = 2$.

5. Thon:

$$P_T = \frac{2}{n(n+1)} \sum_{i=1}^{p(z)} R(i) \left(\frac{g_i}{z}\right)$$

where $R(i) = (n+1-i)$ is the rank of the <u>ith</u> individual among the entire population.

6. Clark <u>et al.</u>:

$$P_C = \frac{p(z)}{n} \cdot \left[\frac{1}{p(z)} \cdot \sum \left(\frac{g_i}{z}\right)^{\gamma}\right]^{1/\gamma}$$

where $\gamma > 1$ for strict convexity. In the simulation analysis I have assumed $\gamma = 2$.

7. Atkinson (Blackorby - Donaldson):

$$P_A = \frac{p(z)}{n} \left\{ 1 - \left[\sum_{i=1}^{p(z)} \frac{1}{p(z)} \cdot \left(\frac{y_i}{z}\right)^{\alpha} \right] \right\}$$

where $\alpha \leq 1$. In the simulation analysis I have assumed $\alpha = 0.5$.

8. "Watkinson": A weighted version of P_A, given by

$$P_W = \frac{p(z)}{n} \{ 1 - [\sum_{i=1}^{p(z)} w(i) \cdot (\frac{y_i}{z})^{\alpha}] \}$$

where $w(i) = \frac{2.[P(z)+1-i]}{p(z).[p(z)+1]}$ and $\alpha = 0.5$

9. Bigman

$$P_B = \frac{p(z)}{n.\phi(k)} \cdot \sum_{i=1}^{p(z)} [r(i)]^k \cdot (\frac{g_i}{z})^{\beta}$$

where $k \geq 1$, $\beta \geq 1$; $\phi(k)$ and $r(i)$ defined as above. In the simulation analysis I have assumed $k = 1$ and $\beta = 2$.

10. Foster _et al._

$$P_F = \frac{1}{n} \sum_{i=1}^{p(z)} (\frac{g_i}{z})^{\beta}$$

In the simulation analysis, I have assumed $\beta=2$. Notice that P_B is essentially a weighted version of P_F. P_F is thus the dual of P_A and P_B is the dual of P_W.

11. Takayama: P_T is the Gini Coefficient of the truncated income vector: $Y^t = (y_1, \ldots y_p, z, \ldots, z)$.

FOOTNOTES

[1] It should be noted that Sen's index does not datisfy the Transfer Axiom as specified above but a weaker version of that axiom that assumes the recepient to be poor and to remain poor also after the transfer. See comment to Corollary 5.

[2] This interpretation is relevant if z has the meaning of a basket of commodities. If, however, z is determined relatiave to the income levels $\underline{Y}$, i.e. if poverty has the meaning of relative deprivation, then this axiom means that doubling all incomes doubles also the poverty line. In other words, z is homogeous of degree 1 in $\underline{Y}$.

[3]Balckorby and Donaldson (1980) have proposed a poverty measure which has a different functional form. They have defined first the "representative income" of the poor and measured the aggregate poverty by the "representative income gap". Chakravarty (1983) has taken much the same approach. The general formulation of their measure allows a wide family of poverty indices, each corresponding to a different measure of inequality. See Section II.

[4] $P(z,\underline{Y})$ is assumed to monotonically strictly decrease with an increase in the income of any poor individual provided that poor individual still remains poor. An increase in his income above the poverty line should not cause any further increase in the poverty index. To assume that, the individual uitlity funcitons in the denominator would have to be

redefined to be $U_i = U_i[\min\{y_i, z\}]$

[5]The Normalization Axiom received considerable attention in the literature. Recently Basu (1985) proposed a decomposition of the Normalization Axiom, showing that this axiom is equivalent to a requirement that P is a first difference preserving transformation of H and G.

[6]See Blackorby and Donaldson (1980).

[7]One example of a PLH social evaluation function is, of course, the additively separable function assumed by Dalton and later on by Atkinson (1970) in the context of inequality measures and more recently by Hagenaars (1984) in the context of poverty measures.

[8]As noted by Atkinson (1970), a linear transformation of the ratio $U(y_p^*)/U^*(z)$ of the form $[Uy_p^*)+c]/[U(z)+c)$ is not independent of c.

[9]Sen's measure can be written as (1976, p. 225)

$$P_S = H.[G+(1-G).I_p.\frac{p}{p+1}]$$

[10]In my opt, cit. paper I have noted, however, that counter examples which show that Sen's or Clark et al. measures do not satisfy the transfer axiom are nothing but illustrations of index number problems inherent in these measures.

[11]If the income of a rich individual rises an argument can be made that poverty should fall because the economy is now better able to handle the poverty problem through e.g. income transfers. Another argument can be made however that poverty should rise because the feeling of relative deprivation of the poor becomes more intense.

[12]Blackorby and Donaldson note that the 'representative' or 'equally distributed equivalent' income is measured by "an arbitrary (homothetic) social evaluation function". (1980, p. 1055). However, if their measure has the general form of Eqs. (13) or (17) then the corresponding social evaluation and individual utility functions must necessarily be linear. This is the result of the requirement they place on the poverty measure to be homogeneous of degree one in $(z-y^*)/z$.

[13] In this exposition I drew on a similar exposition of Hagenaars (1984).

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