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NOTES

THE MATHEMATICAL FORMULATION OF PRODUCTION FUNCTIONS

The applied field of farm management and production economics closely resembles that sector of economic theory that deals with the economy of the firm and the principles of production. These principles provide both simple and complex models which serve as the fundamental hypotheses for research and furnish the schematic framework for establishing the appropriate empirical analyses in solving specific farm production problems. The production economist sets forth in analytical terms, the necessary conditions for the most efficient and rational use of resources to attain a maximum net profit.

The focal point from which the production economist must work in making farm-firm analyses is the various quantitative relationships involved, which are broadly referred to as input-output relationships. These relationships are stated in terms of "production function"—a mathematical equation expressing a given output as a function of certain resource inputs.

Production functions are technical functional relationships between resource inputs and product outputs.[1] Stigler[2] defines a production function as the relationship between inputs of production services per unit of time and outputs of products per unit of time. Production functions are descriptive of techniques or systems of organization of productive services, and are taken from other disciplines. To economic theorists they are mere data of analyses.

Production Possibilities

The conventional production function of economic theory is generally expresssed as an implicit functional relationship between all outputs and all variable inputs, *i.e.*,

$$f\ (Y_1, \ldots, Y_n\ ;\ X_1, \ldots, X_m) = 0,$$

where $Y_1, \ldots, Y_n$ are quantities of outputs and $X_1, \ldots, X_m$ are quantities of inputs. The above equation may generally be solved for any one of the outputs, say Y_1, in terms of all other outputs and the inputs. In this case of one output, Y, in terms of the inputs ;

$$Y = g\ (X_1, \ldots, X_m)$$ [3]

Given explicit or implicit production function for the enterprise and information on the nature of the relationships involved, it is possible to derive functions expressing outputs, costs and derived demands for inputs in terms of given prices of the outputs and inputs.

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1. The first to represent an input-output relationship in symbolic form such as : $P = \phi\ (A, B, C, \ldots)$, where P denotes output per unit of time ; ϕ is a function, and A,B,C,. . . ., etc., are different rates of input per unit of time, was Leon Walras. (See J. G. Stigler: Price Theory, Macmillan, New York, 1947).
2. J. G. Stigler, *Ibid.*, pp. 109-125.
3. M. Nerlove, and K. L. Bachman, "The Analyses of Changes in Agricultural Supply: Problems and Approaches," *Journal of Farm Economics*, Vol. XLII, No. 3, August 1960, pp. 534-535.

Methodological Problems

The problem in resource use in the production of a single enterprise is selection of the mathematical form and the probability distribution of the response function over time. Other related methodological problems are : (*a*) the design of experiments to permit efficient prediction of the response function, and (*b*) the estimating procedure for predicting the surface and optimum use of inputs.

Functional Relationships

In actual fact there are many alternatives which may underlie the production function. The alternatives and their implications for prediction and recommendation are the sole responsibility of the research worker in discovering and making use of them. Consequently, the algebraic equation used for regression and other estimates, will affect the prediction, depending on which of the models is seemingly appropriate.

It seems that the easiest way to state the importance of particular theoretical and technical relationships in production function estimate is to consider a few regression equations and the assumptions inherent in them. Numerous types of functions[4] are available for estimating factor-product relationships.

Data Required in Production Studies

Generally, a production function derived from experimental data is in physical terms alone. But functions can be derived for observations measured in either physical or value units.[5] However, even if the observations are either in

4. (i) The Mitscherlich Function : $Y=M-AR^{x}$
(ii) The Gompertz Curve : $Y=e(M-AR^{x})$
(iii) The Logistic Function : $Y_t = \frac{k}{1+be^{-at}}$
(iv) The Cobb-Douglas Function : $Y=ax^{b}$, or, $\log Y=\log a+b \log x$.
(v) The Polynomial :—
Linear : $Y=B_0+B_1 X$
Quadratic : $Y=B_0+B_1 X+B_{11} X^2$
Square-Root : $Y=B_0+B_1 \sqrt{X}+B_2 X_2$
Cubic : $Y=B_0+B_1 X+B_{11} X^2+B_{111} X^3$
General : $Y=B_0+B_1 X+\ldots+B_{(n-1)}X \ldots (n-1)$

5. Errors may creep in the estimation of a single factor. In the residual process, the market price of each resource, except the one for which the estimates are made, is multiplied by the quantity of the resource. The products are summed and subtracted from the total returns. That is,
$P_{x1} X_1=Y-(P_{x2} X_2+P_{x3} X_3+P_{x4} X_4)$.
This procedure is valid only if the sums of the quantities of resources multiplied by their prices must equal the total value product,
$Y=P_{x1} X_1+P_{x2} X_2+P_{x3} X_3+P_{x4} X_4$.
This condition can be attained under competitive situations only if (a) the $P_{xi}=MVP_{xi}$, and (b) constant returns to scale prevail. By substitution,
$$Y=\left(P_y. \frac{\Delta Y}{\Delta X_1}\right) X_1+\ldots+\left(P_y. \frac{\Delta Y}{\Delta X_4}\right) X_4.$$
$\left(P_y. \frac{\Delta Y}{\Delta X}\right)$'s the marginal value products, are equal to the price of each resource. (See E.O. Heady, "Use and Estimation of Input-Output Relationships or Productivity Coefficients," *Journal of Farm Economics*, Nov. 1952, p. 775).

physical or value units, given the price levels, the production function must correspond with the technical conditions of production, presumably as it is in the purely competitive condition of a farm. Thus the value production function is a transformation from physical units to monetary units. In other words, the algebraic form of the function is the same regardless of the unit of measurement. Hence, the technical requirements for production functions can be discussed regardless of whether the objective is to fit the equations to physical experimental data, physical farm data, or value farm data.

Procedure for Computing the Surface and Optimum Factor Use

The conventional procedure in a production function study is to predict the total output curve as a regression equation. The marginal products for factor inputs can then be predicted singly by computing the derivative of the product Y, with respect to the resource inputs, X_1 and X_2, under consideration.

Given the regression line or production function,

$$Y = a X_1 + 2b X_1^2 + cX_2 - d X_2^2 + e X_1X_2,$$

the marginal physical productivities can be derived by taking the partial derivative of the production function, with respect to X_1 and X_2 inputs :

$$\frac{\partial Y}{\partial X_1} = a - 2b X_1 + e X_2 = MPP_{x1} \qquad (1)$$

and, $$\frac{\partial Y}{\partial X_2} = c - 2d X_2 + e X_1 = MPP_{x2} \qquad (2)$$

By equating both the equations (1) and (2) for marginal products to zero and simultaneously[6] solving for X_1 and X_2, gives the input levels where output Y, may be maximum.[7] Now to derive the maximum total yield, the input levels are substituted in their appropriate places in the original equation.

The Ridge Lines

The ridge lines divide the area on the production surface between the complementarity and substitution. At each point on the ridge lines the marginal product is zero. Therefore, the ridge lines can be derived by equating the two equations for the marginal product to zero and solving for the co-ordinates by substituting arbitrary values for either X_1 or X_2 to determine the corresponding co-ordinate.

The equations for the two ridge lines are :

$$MPP_{x1} = a - 2b X_1 + e X_2 = 0$$

$$MPP_{x2} = c - 2d X_2 + e X_1 = 0$$

6. If there were no interactions, there was no need for a simultaneous set of equations.

7. To make certain that the inputs yield a maximum, the criterion for stability condition used is :

$$d^2Y/dX_1^2 \;.\; \partial^2Y/\partial X_2^2 - \partial^2Y/\partial X_1 \;.\; \partial X_2 > O$$

The Isoquants

The iso-product contours depicting the combination of resources which will produce a specific amount of product, can also be derived from the production function equation or regression equation, by algebraically expressing one category of resource input as a function of another resource input.

In a production process of two variable inputs, the implicit function $Y = f(X_1, X_2)$; expressed in explicit form becomes $X_2 = g(X_1, Y)$, where X_2 is a function of X_1 and Y. The yield Y is a constant, k. The solution lies where X_2 is a function of X_1 at a given yield level. This solution gives points on the isoquant.

In the example above, the regression line as expressed in explicit form becomes,

$$a X_1 - bX_1^2 + cX_2 - d X_2^2 + e X_1 X_2 - Y = 0.$$

The above equation is then arranged in the form of a quadratic in the following manner,

$$(-d) X_2^2 + (c + e X_1) X_2 + (a X_1 - b X_1^2 - Y) = 0,$$

and by using the quadratic formula it is expressed in terms of one variable, say X_2. Substituting arbitrary values for X_1, the corresponding X_2 values, at a given Y, are derived. The co-ordinates trace an iso-product curve at a given level of output.

From this isoquant equation, the marginal rate of substitution between inputs can be computed by taking the derivative of one input with respect to the other. These derivatives then can be (1) equated to the resource-product price ratio to determine the optimum quantity of a particular resource to use ; or (2) equated to resources-resource price ratio to determine the optimum combination of resources for a particular output. If it is desired to know both, that is, the optimum quantity of two or more inputs, and the optimum combination of inputs, the partial derivative for each derivative is equated with the product price ratio.

Isocline

An isocline is a line connecting all points of equal slopes or substitution rates on a family of isoquants. In other words, it connects all input combinations which have the same substitution rates for the various yield levels. There is a different isocline for each possible resource substitution rate.

The isocline is also an expansion path, showing the least cost combination of inputs to use as higher yield levels are attained under a given price ratio for inputs.[8] In the example presented above the general expression for computing isocline is,

$$(-)\frac{a - 2b X_1 + e X_2}{c - 2d X_2 + e X_1} = - \frac{P_{x1}}{P_{x2}} \quad .$$

Use of Studying Production Function

The two main purposes of deriving or studying production functions of an enterprise are : (1) to compute physical input-output ratios to be used for guiding farmers in the use of agricultural practices, for use in budgeting, linear programming, and other types of analyses to indicate optimum farm organization or resource use ; and (2) to provide "benchmarks" or "yardsticks" of how efficiently resources are being used on farms under particular conditions.

The computation of the physical input-output ratios, usually involving physical production functions, can be used for specific farm policy recommendations. The second purpose regarding the provision for a benchmark, usually involving a most common production function of the type such as Cobb-Douglas, provides "diagnostic benchmark."

BRIJ B. KHARE*

DEVELOPMENT OF AGRICULTURE IN CZECHOSLOVAKIA

The development of Czechoslovak agriculture during the years 1949-1960 has received a fillip by the change-over to the socialist system of farming. The socialist sector in Czechoslovak agriculture increased relatively quickly especially in two stages ; the first between 1950-52 and the second between 1957-59. The number of unified agricultural co-operatives increased to an all-time high of 12,560 by the end of 1959. In 1960, 40 per cent of these co-operative farms were joined into larger economic units, so that their number decreased by the end of the year to 10,816. The extent of agricultural land farmed by co-operatives however increased to 12.28 million acres (the areas of arable land to 8.5 million), that is 67.3 per cent of the total agricultural land. About 15 per cent of the cultivated area was under state farms while other enterprises of the socialist sector accounted for only 5 per cent of the land. At the end of 1960 the state farms farmed 2.76 million acres of land.

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8. The general form of isocline equation expressed explicitly in terms of one factor-input is,

$X_2 = f(X_1, P_{x1}, P_{x2})$.

Also, $\frac{dX_2}{dX_1} = \frac{\partial Y/\partial X_1}{\partial Y/\partial X_2} = - \frac{P_{x1}}{P_{x2}} = - \frac{MPP_{x1}}{MPP_{x2}} = MRS_{x1}$ for X_2.

The above marginal product ratio denotes how much of X_2 has to be given up to increase X_1 by one unit and maintain the production at the same level. The relationship, $\frac{MPP_{x1}}{P_{x2}} = \frac{MPP_{x2}}{P_{x2}}$, holds true at all points on an isocline, but at the optimum point,

$MVP_{xi} = P_y . MPP_{x1}/P_{x1} = P_y . MPP_{x2}/P_{x2} = 1$.

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