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SOLUTION OF SPATIAL TRADING SYSTEMS WITH CONCAVE CUBIC PROGRAMMING*

T. G. MacAULAY, R. L. BATTERHAM and B. S. FISHER
Department of Agricultural Economics and Business Management, University of New England, Armidale, NSW 2351, Department of Agricultural Economics, University of Sydney, Sydney, NSW 2006 and Australian Bureau of Agricultural and Resource Economics, Canberra, ACT 2601

Standard spatial equilibrium activity analysis models, as developed by Takayama and Judge (1971), are based on linear supply and demand functions and fixed input–output coefficients. Such models are suitable for multiple-market level trading systems where the fixed input–output coefficients are appropriate. A primal–dual price form of these models is developed in which the assumption of constant per unit costs of transformation is relaxed. In the case when the average cost curves of transformation are quadratic in nature the problem becomes one that will be termed cubic programming (that is, a cubic objective function and linear and/or quadratic constraints) which is solved in a concave region of the solution space. In the paper, the formulation of a simplified spatial equilibrium model with quadratic average costs of transformation is presented and solved. A discussion of possible applications of such a model is also presented.

One of the deficiencies, yet also significant advantages, of the standard spatial equilibrium model is the assumption of linear supply and demand functions and fixed per unit costs of transportation. There have been a number of instances of models where in various ways these two assumptions have been relaxed (King and Ho 1972; MacKinnon 1975; Rowse 1981; Tobin and Friesz 1983; Harker 1984*a*; Pang and Lin 1984; Dafermos and Nagurney 1984; Rathburn and Zwart 1985; Nagurney 1987; Tobin 1988). Generally, specialised solution algorithms have been involved or the approach has required significant expansion of the matrices used in the model through the use of linearisation techniques (Duloy and Norton 1975; McCarl and Tice 1980). With the development of more efficient general non-linear programming computer packages, such as MINOS (Murtagh and Saunders 1987), it now becomes possible to contemplate models with these assumptions relaxed and to solve them for relatively large-scale spatial systems on a standard and widely used solution algorithm. In addition, the behavioural assumptions of recent formulations are being substantially expanded (Harker 1984*b*) and a range of new solution algorithms are being applied to obtain solutions under these new behavioural assumptions, particularly with linear and non-linear complementarity algorithms (Asmuth, Eaves and Peterson 1979; Takayama and Uri 1983; Harker 1988; Tobin 1988).

In this paper, the theoretical basis for a spatial equilibrium model with quadratic average costs of transformation is developed using a

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primal–dual formulation in the price domain. The advantage of the primal–dual formulation is that both price and quantity variables can be simultaneously constrained or modified so as to incorporate various policy interventions. Further, they have the advantage that such models can be extended on the supply side to include representative firm models which are linked in terms of both prices and quantities. Work is in progress on the details of such formulations.

The model has many of the characteristics of the standard spatial activity analysis model as developed by Takayama and Judge (1971). The extension of their model is to allow for the possibility that the costs of transformation from one commodity to another are not based on constant input–output coefficients and therefore the need to have constant unit costs of transformation. The transformation costs can include transportation costs, handling costs or the processing costs of transforming one product into another. The particular modification allows for the direct solution of the types of market problems in which there is transportation of raw materials to a processing site, transformation of the product in some way, and then transportation of processed product. The average costs are assumed to be quadratic and to decline as the volume handled increases and then, as the volume becomes large, to rise again (for an illustration in the case of an abattoir, see Piggott, Dumsday, Small and Wright 1987). There is no theoretical reason why multiple processing levels could not be handled in the model with each level having slightly different cost curves. Grain handling, meat processing, fruit packing and distribution, and milk handling and processing are a few of the areas of analysis that would be suitable for the application of such a model.

Standard Spatial Equilibrium Models

The standard spatial equilibrium model is outlined in detail in Takayama and Judge (1971), and simplified in Martin (1981). Details of the price form of the net social monetary gain model (to be referred to as the 'net revenue' model) are given in MacAulay and Casey (1987). The advantage of the price form of the net revenue model is that it can be substantially reduced in size if the assumption is made that the possibility of the irregular cases occurring, as illustrated by Takayama and Judge (1971), is excluded. This makes no substantial difference to the development of the model and greatly simplifies the algebra. The advantage of the net revenue form of the model over the original social welfare-type objective function proposed by Samuelson (1952) is the greater generality of the model in that asymmetric sets of supply and demand coefficients can be utilised.

To further simplify the initial development of the model it is reasonable to think of the transformation costs as simply transportation costs. The model can then be subsequently modified to cover the more general notion of transformation. The extension to the Samuelson–Takayama–Judge version of the model is then a matter of changing a set of fixed per unit transportation costs to a functional relationship represented as a quadratic average cost per unit of transportation.

The standard spatial equilibrium model and the corresponding price equilibrium is illustrated in the top part of Figure 1 where the excess supply and demand functions are shown as ED_i and ES_i for region i.

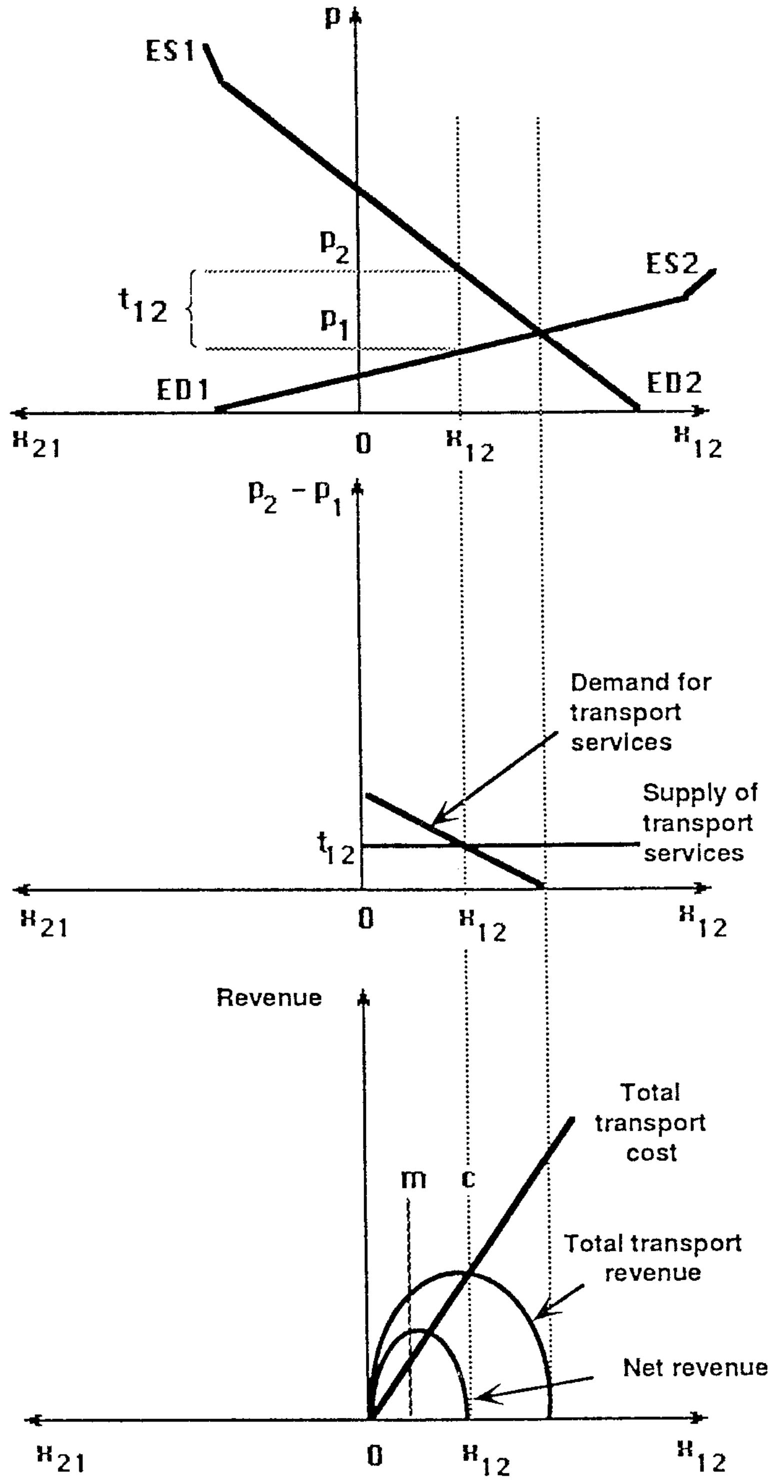

FIGURE 1—Representation of the Net Revenue Objective Function and the Spatial Equilibrium Model with Fixed Unit Transport Costs.

The equilibrium prices after trade takes place are indicated as p_1 and p_2. The fixed per unit transportation cost of shipment from region 1 to region 2 is indicated as t_{12}. The trade from region 1 to region 2 is indicated as x_{12} and is equal to the difference $x_1 - y_1$ or $y_2 - x_2$ in this two-region case where x_i is the quantity supplied and y_i is the quantity demanded for region i.

The basis of the mathematical solution of the standard spatial equilibrium problem and the connection between the prices, trade flows and the objective function are illustrated in Figure 1. If a vertical difference between the excess supply and demand functions is calculated the demand for transport services is obtained. The supply of transport services is assumed to be perfectly elastic and represented by a horizontal line set at the level of the fixed per unit transport cost.

The revenue from the sale of transport services at each volume shipped can be calculated from the demand for transport services and is plotted as a quadratic curve in the lower part of Figure 1. The total cost of transport services is obtained by multiplying the average per unit cost by the volume of trade so as to obtain a linear function for total cost. The difference between the revenue and the transport cost is the net revenue from the system and it is this that is to be maximised in the mathematical programming model formulations. In the case of Figure 1 net revenue is a quadratic function.

The solution to the competitive spatial equilibrium model is indicated in Figure 1 where the net revenue curve intersects the x_{12} axis and is indicated by a small c. The competitive spatial equilibrium has a zero net revenue because arbitragers are in a position to bid any excess profits away by trading in the commodity concerned. The mathematical formulation of the model also generates the same result. The spatial monopolist is assumed to maximise net revenue and thus the equilibrium solution would be at the maximum of the net revenue curve indicated by m. It is also worth noting that the net revenue formulation of the model is essentially a primal–dual formulation in which the primal model is subtracted from the dual model and the constraints from both models are included. For this reason the objective function has a zero value at the optimum since the value of the objective function of the primal part of the problem must equal the value of the dual part at the optimum.

A very useful characteristic of the net revenue formulation is the zero objective function which permits verification that the model has been constructed in a self-dual fashion. Although it is not a guarantee that the problem has been correctly formulated it provides a very useful and practical diagnostic device for helping to ensure the correctness of the formulation.

Non-Linear Transport Cost Function

As indicated above, the standard spatial equilibrium model is based on the assumption of a perfectly elastic supply of transport services. In many cases this may be reasonable when the volume of a particular commodity carried is small in relation to the total volume or when the rates are set according to some government policy and do not change with the volume carried. If, however, the transport rates do change with the volume carried then the standard model needs to be adjusted. Over a reasonable range of volumes a quadratic average cost function could be used to approximate many non-linear functions. Thus, for the purposes of this analysis a quadratic curve seemed appropriate although other functional forms might equally have been used.

In the case of Figure 2 the average cost curve for transport services is presented as quadratic in character. Figure 2 is very similar to Figure 1 except for the average per unit transport cost curve in the middle section of the figure which is quadratic and the consequent cubic-shaped total cost curve in the lower section. With such a cubic total transport cost curve, which must pass through the origin, the net revenue curve also takes on a cubic shape.

The question of whether or not a solution will exist to such a problem is important. Takayama and Judge (1971) have shown that a solution exists to the standard spatial equilibrium model since it is a standard quadratic programming problem for which a unique set of prices exists. Intuitively, from Figure 2, it can be observed that provided the shape of the cubic total cost curve is such that, in the positive quadrant, part of it lies below the total revenue curve then there will be a maximum to the net revenue function and the spatial equilibrium system will have a solution. In fact, the objective function will be concave.

The issue of the concavity of the net revenue objective function can be considered by using a simplified algebraic representation of the spatial equilibrium system. As a starting point the excess supply and demand function will be used to formulate a problem in the quantity domain. Let the excess supply function, measured in terms of the quantity shipped from region 1 to region 2, that is x_{12} in Figure 3 (note in this case $x_{12} = -x_{21}$), be represented as:

$$(1) \quad p_1 = \alpha_1 + \beta_1 x_{12}$$

and let the excess demand function, measured in terms of x_{12}, be

$$(2) \quad p_2 = \alpha_2 - \beta_2 x_{12}$$

and let the quadratic average cost of transportation be

$$(3) \quad t_{12} = \mu - \tau x_{12} + \lambda x_{12}^2$$

The Greek symbols α_1, α_2 and β_1, β_2 represent intercepts and slopes of the excess supply and demand functions and μ, τ and λ are the coefficients of the average cost of transport curve. Since these costs are assumed to initially decline as the volume increases and then increase there are restrictions on the values of these coefficients as follows: $\mu \geq 0$, $\tau \leq 0$, $\lambda \geq 0$. In addition, $\alpha_2 \geq \alpha_1$ and $\beta_1 \geq 0$, $\beta_2 \geq 0$.

The net revenue may be written as:

$$(4) \quad NR = p_2 x_{12} - p_1 x_{12} - t_{12} x_{12}$$

Substituting equations (1) to (3) into (4) and simplifying, the following expression is obtained:

$$(5) \quad NR = (\alpha_2 - \alpha_1 - \mu) x_{12} + (\tau - \beta_1 - \beta_2) x_{12}^2 - \lambda x_{12}^3$$

For this function to be concave the second derivative must be negative and this condition implies that:

$$(6) \quad (\tau - \beta_1 - \beta_2)/3\lambda < x_{12}$$

Thus, as long as $(\beta_2 + \beta_1) < \tau$, then it is possible for x_{12} to have values along the positive axis and the situation is as represented in Figure 3. If

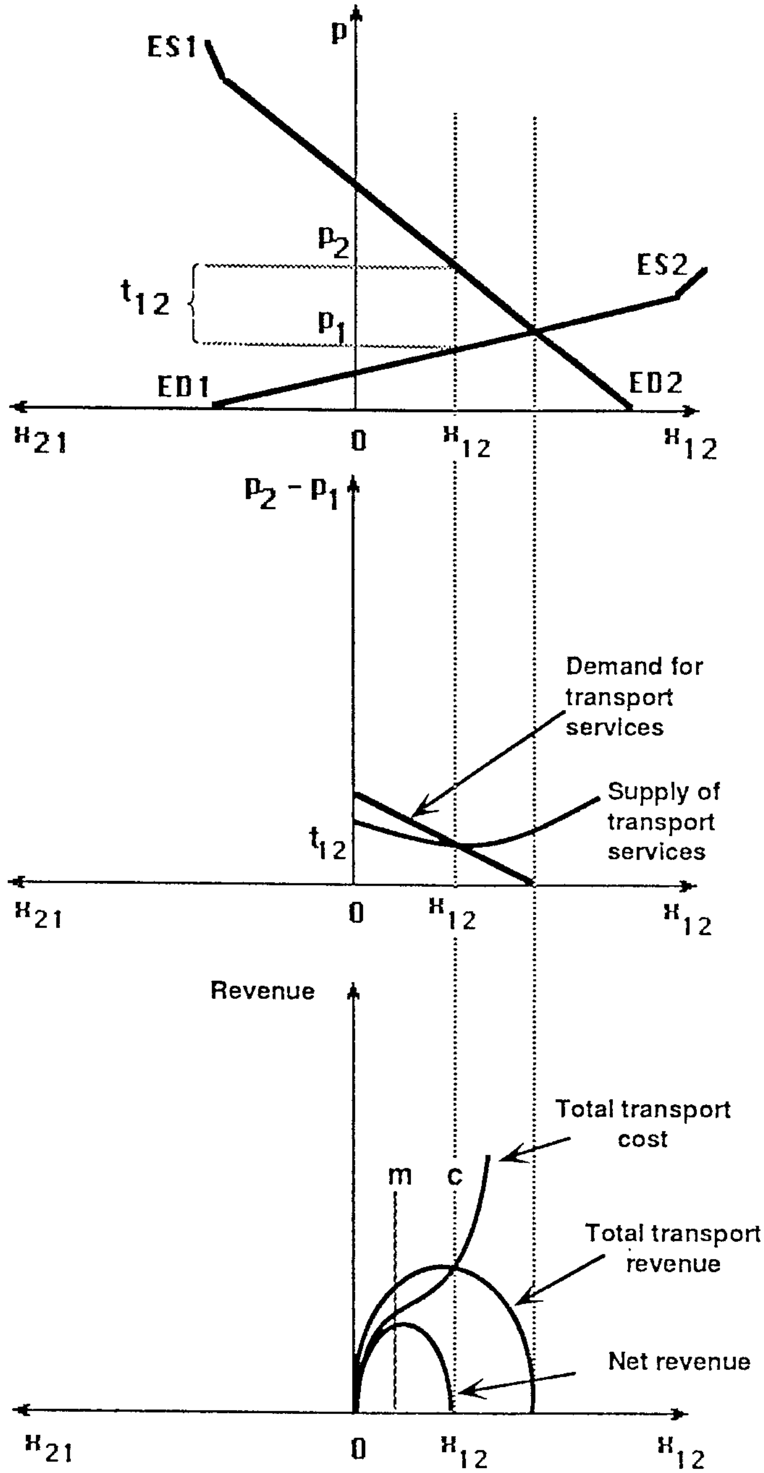

FIGURE 2—Representation of the Net Revenue Objective Function and the Spatial Equilibrium Model with Quadratic Average Unit Transport Costs.

this relationship is not true then the cubic cost function lies above the revenue function as illustrated in Figure 4. This implies that the cost of transportation will always exceed the revenue gains from the process of moving goods from one region to another. Although the above argument does not provide a general proof of the required nature of the transport cost function it provides sufficient rationale to develop a more general model.

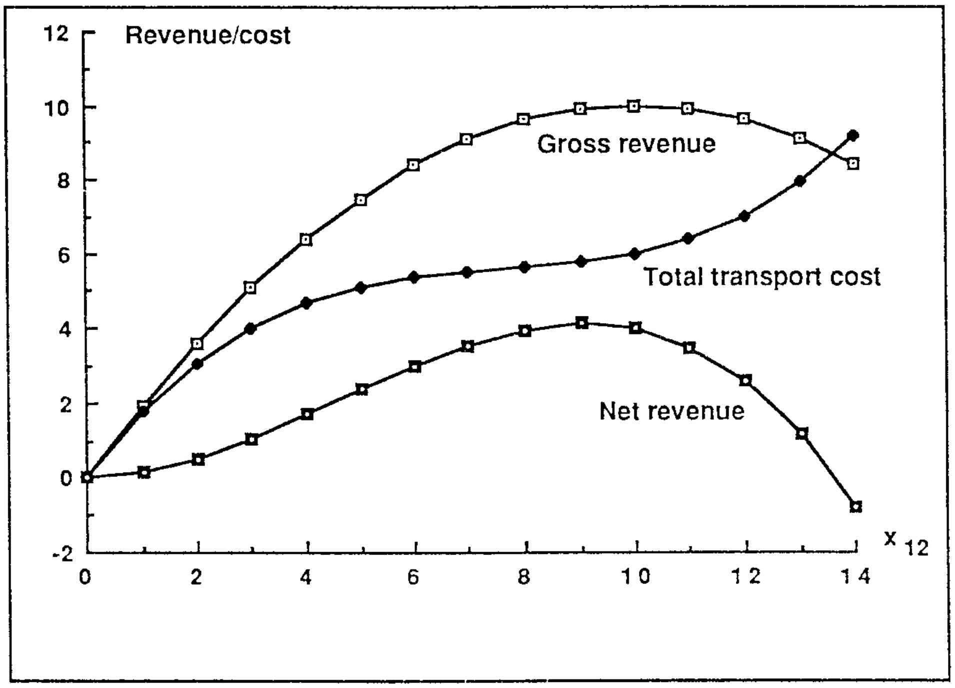

FIGURE 3—Illustration of the Relationship between Total Transport Cost, and Gross and Net Revenues from Trade with a Cubic Total Transport Cost Curve $(\beta_2+\beta_1)=0\cdot1$, $\tau=0\cdot25$. $R=2x_{12}-0\cdot1x_{12}^2$, $t_{12}=2x_{12}-0\cdot25x_{12}^2+0\cdot011x_{12}^3$.

The Concave Cubic Programming Model

To complete the development of the model, it is necessary to modify the standard spatial equilibrium model along the lines of the previous section [for details of the net revenue form of the standard model see MacAulay and Casey (1987) the basic structure of which is followed in this section]. By analogy to a quadratic programming model which has a quadratic objective function and linear constraints, the model with a cubic objective function and quadratic constraints will be referred to as a cubic programming model.

Defining the following notation for n regions (as used by Takayama and Judge 1971), let:

X be a vector of $(n^2\times 1)$ net trade flows, x_{ij}, from region i to region j; ρ be a vector of $(2n\times 1)$ non-negative demand prices in region i, ρ_i, and

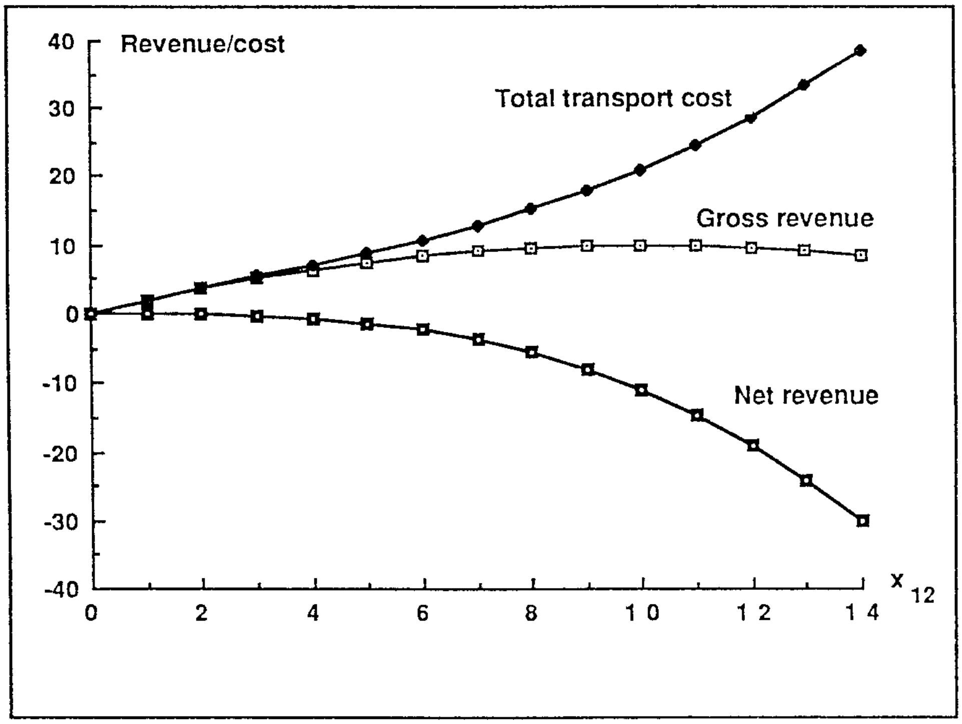

FIGURE 4—Illustration of the Relationship between Total Transport Cost, and Gross and Net Revenues from Trade with a Cubic Total Transport Cost Curve $(\beta_2+\beta_1)=0{\cdot}1$, $\tau=0{\cdot}1$. $R=2x_{12}-0{\cdot}1x_{12}^2$, $t_{12}=2x_{12}-0{\cdot}1x_{12}^2+0{\cdot}011x_{12}^3$.

non-negative supply prices in region j, ρ^j, such that $\rho=[\rho_y\rho_x]'$; T be a vector of $(n^2\times 1)$ transfer costs, t_{ij}, between regions i and j which for the modified model will be a quadratic function of the trade flows X so that $T(X)=\mu-\tau X+\lambda X^2$ where μ, τ and λ are $(n^2\times 1)$ vectors of non-negative coefficients and X^2 an $(n^2\times 1)$ vector of elements $x_{ij}{}^2$; y be a vector of $(n\times 1)$ quantities demanded, y_i, in region i; x be a vector of $(n\times 1)$ quantities supplied, x_i, in region i; p_y be a vector of $(n\times 1)$ demand prices, p_i, in region i; p_x be a vector of $(n\times 1)$ supply prices, p^j, in region j; V be a vector of dimension $(2n\times 1)$ of, w_i, slack variables such that $p_y=\rho_y-w$ where w_i is non-negative and positive so as to ensure $y_i\geq 0$ and, v_i, variables such that $p_x=\rho_x+v$ where v_i is non-negative and positive so as to ensure that $x_i\geq 0$ so that $V=[w\,v]'$. The vectors w and v are used to handle the irregular cases as outlined in Takayama and Judge (1971, p. 156).

The typical demand function will be represented as:

(7) $y_i=\alpha_i-\beta_i p_i, \quad i=1,\ldots,n$

and the typical supply function as:

(8) $x_i=\theta_i+\gamma_i p^i, \quad i=1,\ldots,n$

where α_i and θ_i are the intercepts and β_i and γ_i are the slope coefficients for n regions.

In matrix form the supply and demand functions may be represented as:

$$(9)\quad \begin{aligned} y &= \alpha - \mathrm{B}p_y \\ &= \alpha - \mathrm{B}(\rho_y - w) \end{aligned}$$

$$(10)\quad \begin{aligned} x &= \theta + \Gamma p_x \\ &= \theta + \Gamma(\rho_x + v) \end{aligned}$$

where α and θ are $(n \times 1)$ vectors of demand and supply intercepts respectively, B is an $(n \times n)$ matrix of the demand slope coefficients β, and Γ is an $(n \times n)$ matrix of the supply slope coefficients γ [both may be asymmetric as shown by Takayama and Judge (1971) and Takayama and Uri (1983)].

Net social monetary gain or net social revenue is defined as total social revenue less total social costs less transfer costs as represented in the average cost function $T(X)$, so that:

$$(11)\quad NSR = p_y'y - p_x'x - T(X)'X$$

and by substituting equations (9) and (10) into (11), the following cubic objective function is obtained:

$$(12)\quad NSR = \left[\begin{bmatrix} \alpha \\ -\theta \\ -T(X) \\ -\alpha \\ -\theta \end{bmatrix} - \begin{bmatrix} \mathrm{B} & 0 & G_y & -\mathrm{B} & 0 \\ 0 & \Gamma & G_x & 0 & \Gamma \\ -G_y' & -G_x' & 0 & 0 & 0 \\ -\mathrm{B} & 0 & 0 & \mathrm{B} & 0 \\ 0 & \Gamma & 0 & 0 & \Gamma \end{bmatrix} \begin{bmatrix} \rho_y \\ \rho_x \\ X \\ w \\ v \end{bmatrix} \right]' \begin{bmatrix} \rho_y \\ \rho_x \\ X \\ w \\ v \end{bmatrix}$$

where G_y and G_x are defined below.

A spatial equilibrium solution may be obtained when the objective function (12) is maximised subject to the set of constraints (13) modified to take into account the cost of transport function $T(X)$ [see Takayama and Judge (1971, p. 162) and elsewhere in the text for a detailed justification of the formulation]:

$$(13)\quad \begin{bmatrix} \alpha \\ -\theta \\ -T(X) \\ -\alpha \\ -\theta \end{bmatrix} - \begin{bmatrix} \mathrm{B} & 0 & G_y & -\mathrm{B} & 0 \\ 0 & \Gamma & G_x & 0 & \Gamma \\ -G_y' & -G_x' & 0 & 0 & 0 \\ -\mathrm{B} & 0 & 0 & \mathrm{B} & 0 \\ 0 & \Gamma & 0 & 0 & \Gamma \end{bmatrix} \begin{bmatrix} \rho_y \\ \rho_x \\ X \\ w \\ v \end{bmatrix} \leq 0$$

and

$$(14)\quad (\rho_y' \ \rho_x' \ X' \ w' \ v') \geq 0'$$

The $(n \times n^2)$ matrix G_y is structured so as to sum the shipments into a region and the $(n \times n^2)$ matrix G_x is structured so as to sum the shipments out of a region as follows:

$$G_y = \begin{bmatrix} 1 & & & 1 & & & & 1 & & \\ & 1 & & & 1 & & & & 1 & \\ & & \ddots & & & \ddots & \cdots & & & \ddots \\ & & & 1 & & & 1 & & & & 1 \end{bmatrix}$$

$$G_X = \begin{bmatrix} -1-1\ \dots\ -1 & & & \\ & -1-1\ \dots\ -1 & & \\ & & \ddots & \\ & & & -1-1\ \dots\ -1 \end{bmatrix}$$

To simplify the mathematical notation the vectors w and v will be ignored in the subsequent argument since they make no difference to the essential logic of the formulation and are only included to deal with certain irregular cases as outlined in Takayama and Judge (1971). Thus, in problem 1 the vector ρ does not contain w and v. In presenting the mathematical programming tableau and solution these variables will not be included for the sake of presenting a more compact tableau. A compact form of the problem above can be written as problem 1 by redefining the vectors and matrices.

Problem 1: maximise $d'\rho-\rho'H\rho-T(X)'X$

subject to

$$-d+H\rho+GX\geq 0$$
$$T(X)-G'\rho\geq 0$$

and

$$(\rho'\ X')\geq 0'$$

where G is a combined matrix of G_y and G_x, d is the set of combined intercept vectors, ρ is the set of combined supply and demand price vectors and H is the matrix of demand and supply slope coefficients. The sufficient conditions for a local maximum to this problem are that the objective function be differentiable and concave in the neighbourhood of the maximum, that $T(X)$ also be a concave function in the neighbourhood of the optimum X value, that each constraint is differentiable and convex and that the optimum solution satisfies the following Kuhn–Tucker conditions.

The Lagrangian function for problem 1 and the associated Kuhn–Tucker conditions, evaluated at the optimal values ρ^o, X^o, k_1^o, k_2^o, where k_1 and k_2 are Lagrangian multipliers, are:

$$(15)\qquad \phi=d'\rho-\rho'H\rho-T(X)'X+k_1'[T(X)-G'\rho] + k_2'(-d+H\rho+GX)$$

$$(16)\quad \partial\phi^o/\partial\rho=d-(H+H')\rho^o-Gk_1^o+Hk_2^o\leq 0 \quad \text{and} \quad (\partial\phi^o/\partial\rho)'\rho^o=0$$

$$(17)\quad \partial\phi^o/\partial X=-T(X^o)-[\partial T(X^o)/\partial X]'X^o + [\partial T(X^o)/\partial X]'k_1^o+G'k_2^o\leq 0 \quad \text{and} \quad (\partial\phi^o/\partial X)'X^o=0$$

$$(18)\quad \partial\phi^o/\partial k_1=T(X^o)-G'\rho^o\geq 0 \quad \text{and} \quad (\partial\phi^o/\partial k_1)'k_1^o=0$$

$$(19)\quad \partial\phi^o/\partial k_2=-d+H\rho^o+GX^o\geq 0 \quad \text{and} \quad (\partial\phi^o/\partial k_2)'k_2^o=0$$

$$(20)\quad (\rho^{o'}\ X^{o'}\ k_1^{o'}\ k_2^{o'})\geq 0'$$

An interpretation of these conditions can be found in Martin (1981), except for condition (17) and a slight modification to condition (18). Condition (17) requires the product rule of differentiation and therefore has two additional terms. These cancel out because of the self-dual nature of the problem since at the optimum X^o equals k_1^o. Since the second and third terms cancel out condition (17) then becomes equivalent to condition (18). Condition (18) implies that the price difference between any two regions must be less than or equal to the transport cost $T(X^o)$.

The concavity of the objective function is illustrated in the section above provided that the cubic cost curve falls below the quadratic revenue curve. The constraint qualification can be tested using the Arrow–Enthoven constraint qualification test (Chiang 1984, p. 746). The test will be satisfied if each constraint function is differentiable and quasiconcave and there exists a point in the non-negative orthant such that all constraints are satisfied as strict inequalities at that point, hopefully near $(\rho^{o\prime}\ X^{o\prime})$. Since the constraints in the first set are linear, they satisfy the constraint qualification. Each function in the second set is quadratic and, with the coefficient restrictions as noted above, will be a concave function. Thus, provided there is a point that will satisfy all the constraints as strict inequalities, a local optimum will exist. It is also worth noting again that in the case of the self-dual competitive spatial equilibrium model the value of the objective function at that optimum will be zero.

Transformation Costs and Activity Analysis Extensions

Next consider the problem of transformation costs (essentially a processing margin) rather than the simpler transportation costs. One of the simplest forms of this problem is the case of a short-term storage and handling charge that varies with the volume of the good handled. This form of the problem is simple since the quantity measure of the commodity stays the same (handling losses could, of course, be incorporated). The handling of grain is an illustration of such a case.

One way of envisaging such a model is to consider the shipment of grain from farms to a number of central storage sites and then shipment from these storage sites to a number of consumption points. Thus, there are essentially two commodities, namely, the grain shipped from farm to storage site and the grain shipped out of the storage site. The flows and prices at different points in the system must all be appropriately linked together.

Using the basic structure of equations (12) to (14) and ignoring the vectors w and v (on the assumption that irregular cases will not be encountered) such a model may be expressed as problem 2.

Problem 2: maximise

$$(21)\quad NSR=\left[\begin{bmatrix}\alpha_1\\0\\-\delta\\0\\-\theta_2\\-T_1\\-\mu\\-T_2\end{bmatrix}-\begin{bmatrix}B_1 & & & & & G_{y_1} & & \\ & & & & & G_{x_1} & -I & \\ & & & & & & I & \\ & & & & & & I & G_{y_2}\\ & & & & \Gamma_2 & & & G_{x_2}\\ -G'_{y_1} & -G'_{x_1} & & & & & & \\ & I & -I & -I & & & -g(X_t) & \\ & & & -G'_{y_2} & -G'_{x_2} & & & \end{bmatrix}\begin{bmatrix}\rho_{y_1}\\\rho_{x_1}\\\rho_c\\\rho_{y_2}\\\rho_{x_2}\\X_1\\X_t\\X_2\end{bmatrix}\right]'\begin{bmatrix}\rho_{y_1}\\\rho_{x_1}\\\rho_c\\\rho_{y_2}\\\rho_{x_2}\\X_1\\X_t\\X_2\end{bmatrix}$$

subject to

$$
(22)\qquad
\begin{bmatrix} \alpha_1 \\ 0 \\ -\delta \\ 0 \\ -\theta_2 \\ -T_1 \\ -\mu \\ -T_2 \end{bmatrix}
-
\begin{bmatrix}
\mathbf{B}_1 & & & & & G_{y_1} & & \\
 & & & & & G_{x_1} & -I & \\
 & & & & & & I & \\
 & & & & & & I & G_{y_2} \\
 & & & & \Gamma_2 & & & G_{x_2} \\
-G'_{y_1} & -G'_{x_1} & & & & & & \\
 & I & -I & -I & & & -g(X_t) & \\
 & & & -\mathrm{G}'_{y_2} & -\mathrm{G}'_{x_2} & & &
\end{bmatrix}
\begin{bmatrix} \rho_{y_1} \\ \rho_{x_1} \\ \rho_c \\ \rho_{y_2} \\ \rho_{x_2} \\ X_1 \\ X_t \\ X_2 \end{bmatrix}
\leq 0
$$

and

$$(23)\quad (\rho_{y_1}\ \rho_{x_1}\ \rho_c\ \rho_{y_2}\ \rho_{x_2}\ X_1\ X_t\ X_2) \geq 0$$

In problem 2 a capacity constraint and its associated shadow value have been included where ρ_c is the shadow value on the capacity δ. The shadow value will be non-zero if any of the capacity constraints are effective. The throughput of the storage site or process is indicated by X_t with the subscript t used to indicate throughput. The supply and demand prices and quantities and other coefficients for the two commodities are indicated with subscripts 1 and 2 and the identity matrix is indicated as I. The special character of problem 2 in incorporating the storage and handling costs is in the function $g(X_t)$ and the intercept term μ. The volume-related storage and handling cost function may be written for the quadratic case as:

$$(24)\quad T(X_t) = \mu - \tau X_t + \lambda X_t^2$$

where τ, λ and μ are as previously defined and $g(X_t)$ is the last two terms in this function.

Extension of the model to a full activity analysis type of formulation implies a physical transformation of the traded raw material or the combining of traded raw materials into a traded final product (a number of intermediate products could be included). In this instance the first and third identity matrices may involve transformation coefficients in equations (21) and (22). This alternative will not be described since it is considered in detail by Takayama and Judge (1971).

A Sample Problem

To illustrate the solution of such a model a small-scale sample problem is used with three fixed supply regions for the raw material, each with different transportation costs to the storage and handling facility, two storage and handling facilities and three final demand points, again with different transport costs. The average cost of handling and storage is assumed to be quadratic in terms of the throughput measured as the flow into a facility.

The basic data for the model consist of a set of demand functions, fixed levels of supply from each of the three raw material sites, average storage and processing cost functions and fixed transport rates. The demand functions are assumed to be as follows:

$$y_1 = 200 - 10p_1$$

and

$$y_2 = 100 - 5p_2$$

where y_1 and y_2 are quantities demanded in regions 1 and 2 at prices p_1 and p_2. The raw material supplies are assumed to be $x_1 = 30$, $x_2 = 20$

and $x_3 = 30$. Since a storage capability is assumed it is necessary to specify the carryover stocks of the commodity at the end of the period as $y_3 = 4$ and $y_4 = 5$ (these could be set with minimum and maximum levels if necessary). The average short-run storage and handling costs were both assumed to be the same and as follows:

$$T(X_t)_1 = 7{\cdot}0 - 0{\cdot}19X_{t1} + 0{\cdot}0017X_{t1}^2$$
$$T(X_t)_2 = 7{\cdot}0 - 0{\cdot}19X_{t2} + 0{\cdot}0017X_{t2}^2$$

The transport costs between the various locations in the system are presented in Figure 5 together with an illustration of the various shipment routes and the variables involved. The notation used is slightly different from that of the spatial equilibrium problem formulations above and is similar to the notation used in the tableau for the sample problem. The shipments from raw material sites are indicated as x_{jk}, the shipments of the final product as y_{ki} and the throughput of the storage and handling site as r_k for the j-th raw material site, the k-th storage and processing site and the i-th final demand site.

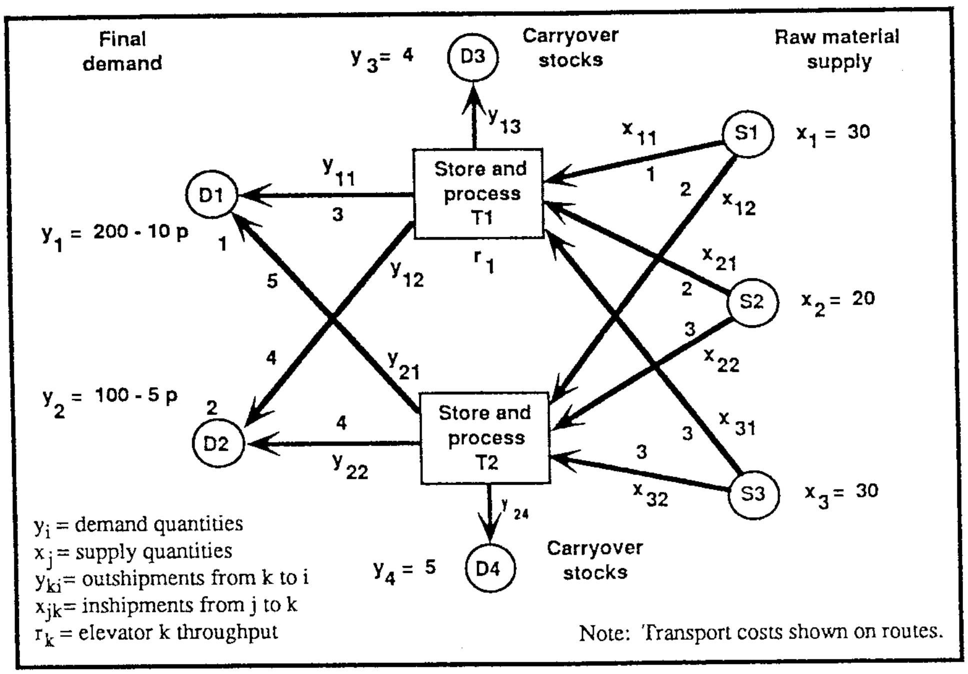

FIGURE 5—Representation of the Shipping, Storage and Processing System.

Storage capacities for each of the storage and handling sites were set at $T1 = 60$ and $T2 = 40$. If these capacities prove to be limiting then a shadow value is generated for the capacity constraint which represents the amount per unit of volume that could be spent profitably on eliminating the capacity constraint. The mathematical programming tableau corresponding to the above set of data is presented as Table 1 and the solution to this system as Table 2.

The problem presented in Table 1 was solved as a minimisation problem using MINOS 5.1 (Murtagh and Saunders 1987) on a Mac-

TABLE 1

Matrix Structure for Sample Cubic Programming Problem[a]

	ADP1	ADP2	ADP3	ADP4	ASP1	ASP2	C1	C2	BDP1	BDP2	BSP1	BSP2	BSP3	Y11	Y12	Y13	Y21	Y22	Y24	R1	R2	X11	X12	X21	X22	X31	X32	RHS
ADP1	10																											
ADP2		5																										
R1																				f(R1)								
R2																					f(R2)							
OBJ	-200	-100	-4	-5	0	0	60	40	0	0	30	30	20	3	4	0	5	4	0	7	7	1	2	2	3	3	1	
RADP1	-10													-1			-1											≤ -200
RADP2		-5													-1			-1										≤ -100
RADP3																-1												≤ -4
RADP4																			-1									≤ -5
RASP1														1	1	1				-1								≤ 0
RASP2																	1	1	1		-1							≤ 0
RC1																				1								≤ 60
RC2																					1							≤ 40
RBDP1																				1		-1		-1		-1		≤ 0
RBDP2																					1		-1		-1		-1	≤ 0
RBSP1																						1	1					≤ 30
RBSP2																								1	1			≤ 20
RBSP3																										1	1	≤ 30
RY11	1				-1																							≤ 3
RY12		1			-1																							≤ 4
RY13			1		-1																							≤ 0
RY21	1					-1																						≤ 5
RY22		1				-1																						≤ 4
RY24				1		-1																						≤ 0
RR1					1		-1		-1											-g(R1)								≤ 7
RR2						1		-1		-1											-g(R2)							≤ 7
RX11									1		-1																	≤ 1
RX12										1	-1																	≤ 2
RX21									1			-1																≤ 2
RX22										1		-1																≤ 3
RX31									1				-1															≤ 3
RX32										1			-1															≤ 1

Note: $f(R1) = -0{\cdot}19R_1^2 + 0{\cdot}0017R_1^3$, $f(R2) = -0{\cdot}19R_2^2 + 0{\cdot}0017R_2^3$, $g(R1) = -0{\cdot}19R_1 + 0{\cdot}0017R_1^2$ and $g(R2) = -0{\cdot}19R_2 + 0{\cdot}0017R_2^2$. The intercept of 7·0 was incorporated into the objective function and the right-hand side.

[a]The solution was obtained using the MINOS program (Murtagh and Saunders 1987). The letter A refers to processed material, B to raw material, DP to demand price, SP to supply price, Y to flows in the processed product, R to throughput of the storage and handling sites, C to the shadow value on any fully utilised capacity (the value of an additional unit of storage) and X to the raw material shipments. The numbers refer to sites and X12 indicates a shipment from raw material site 1 to storage and handling site 2 while Y12 represents a shipment from storage and handling site 1 to demand site 2. The constraint row names have an R as the first letter. The problem is set up as a minimisation problem in the tableau above.

intosh SE microcomputer with a 20 mB hard disk. To use MINOS it was necessary to code in FORTRAN the first order derivatives of the objective function and the non-linear constraint functions. To facilitate the running of the model a routine was developed for the inclusion of the function coefficients at the end of the standard input file for

TABLE 2

Simplified Model: Cubic Programming Solution[a]

Primal variables		Dual variables	
ADP1	15·15	RADP1	15·15
ADP2	15·50	RADP2	15·50
ADP3	12·15	RADP3	12·15
ADP4	11·50	RADP4	11·50
ASP1	12·15	RASP1	12·15
ASP2	11·50	RASP2	11·50
C1	0·0	RC1	0·0
C2	0·0	RC2	0·0
BDP1	10·44	RBDP1	10·44
BDP2	8·44	RBDP2	8·44
BSP1	9·44	RBSP1	9·44
BSP2	8·44	RBSP2	8·44
BSP3	7·44	RBSP3	7·44
Y11	48·5	RY11	48·5
Y12	0·0	RY12	0·0
Y13	4·0	RY13	4·0
Y21	0·0	RY21	0·0
Y22	22·5	RY22	22·5
Y24	5·0	RY24	5·0
R1	52·5	RR1	52·5
R2	27·5	RR2	27·5
X11	30·0	RX11	30·0
X12	0·0	RX12	0·0
X21	20·0	RX21	20·0
X22	0·0	RX22	0·0
X31	2·5	RX31	2·5
X32	27·5	RX32	27·5
Objective function value			0·0
Linear part			−3110·45
Non-linear part			3110·45

[a]The variable names are the same as in Table 1.

cubic programming problems. The time taken, including program overhead and time to enter file names, was 0·97 of a minute. A much larger model (over 480 rows and columns) of a similar structure took about 30 minutes to solve. Much shorter times can be obtained by using the basis restart facilities in MINOS once an initial solution has been obtained.

In examining some of the effects of changes to various parameters in the system it was noted that changing the intercept of the cost functions simply changed the prices for the raw material since the same volumes were traded and processed given the fixed supply [a result reported by Fisher (1981) using graphical methods]. This is an interesting result, since in the case of commodities with inelastic supplies and relatively elastic final demands the major effect of reducing the handling and processing costs will be a rise in the raw materials price with a much smaller rise in the final demand prices. This result might be applicable to some agricultural sectors.

Applications

The technique illustrated above would seem to have wide applicability to situations where there are transformations in time, space and form and costs which vary with the volumes handled. Typically, the analysis of transformation cost behaviour in agricultural economics, when trade is involved, has proved to be difficult because of the inappropriate nature of assuming constant per unit costs of transformation and the uncertainty about the existence of a solution. In addition to transformation costs, the effects of exchange rates, trade restrictions and other domestic policy interventions can be taken into account. Some of the areas in which the approach would seem to be appropriate are in meat processing where live animals are transformed into meat, in the dairy sector where milk transportation and processing is a significant issue, in wool handling and storage where the work of Toft and Cassidy (1985) using transshipment models could be extended to use this framework, and in grains where the efficiency of transportation storage and handling is an important issue. Extension of the model toward optimising over time is also possible so that a greater understanding can be obtained of the interactions between storage policies and trade policy [the standard intertemporal model has been developed by Takayama and Judge (1971)].

Concluding Comment

In the paper, a generalisation of the standard spatial equilibrium model has been presented which allows for the inclusion of non-linear functions for transformation costs. If the average cost function is quadratic in character then a mathematical programming model with a cubic objective function and some quadratic constraints is obtained. It was shown that under a reasonable set of conditions a solution exists to the problem and that the value of the objective function of the cubic competitive spatial equilibrium model will be zero. This is a most useful property in helping to verify the model formulation.

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