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THE PROBABILITY DISTRIBUTION OF THE AVERAGE MARGINAL PRODUCTS OF COBB-DOUGLAS FACTORS WITH APPLICATIONS

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The comparative productivity of inputs is often the focus of applied studies of production. For example, one might wish to compare the marginal product of water in irrigation districts to provide guidelines for capital investment in new water projects or to assist in water allocation decisions.

The purpose of this note is to illustrate a straightforward econometric method for making such comparisons. It is assumed that panel data are available for T periods on N cross-sectional units and that output is produced according to a stochastic Cobb-Douglas production function. The Cobb-Douglas function was chosen because it has a simple formula for the marginal product and has been found to perform well in studies of agricultural production. The statistical results derived in the paper are obtained assuming that the error term of the production function is independently and identically distributed. This assumption can be easily relaxed and the asymptotic results presented in the paper can be generalised for the case where the variance-covariance matrix of the error term is known or can be estimated consistently. An important generalisation is the Zellner (1962) seemingly unrelated regressions framework. Only the simplest case is developed here because of space limitations.

To illustrate the wide applicability of these results a test for risk aversion is developed which can be applied when the production setting satisfies specified assumptions.

Statistical Results

The results of the paper are based on the assumption that output is produced according to the following Cobb-Douglas production function:

$$(1) \qquad Y_{it} = A_{oi} \prod_{j=1}^{k} X_{itj}^{\beta_j} \exp(u_{it}), \; A_{oi}>0, \; X_{itj}>0, \; \beta_j>0$$

where A_{oi} represents fixed factors that are specific to firm i, X_{itj} is the level of factor j used by firm i in period t, and u_{it} is an error term assumed to be distributed independently of all factors.

The A_{oi} represents factors of production which vary across firms but are fixed for each firm, are correlated with the levels of other inputs and are not observed by the econometrician. Examples of such factors are the level of management (Mundlak 1961), the level of technical efficiency (Hoch 1962) and soil quality (Chamberlain 1980). Because these factors

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are correlated with the levels of observed inputs it is important to incorporate them directly in the econometric specification. This can be done by using firm-specific dummy variables and this requires panel data with at least two observations on each firm.

The parameters of equation (1) can be estimated by regressing the logarithm of output on the logarithm of each input and a set of firm-specific dummy variables. The estimated vector of slope coefficients is equivalent to the analysis of covariance estimate of this parameter vector, and each estimated slope coefficient ($\hat{\beta}_j$) estimates the point elasticity of output with respect to a particular factor.

The measure of the average marginal product considered here is calculated by multiplying the estimated output elasticity of the jth factor in the sample and the geometric mean of the average product of the jth factor in the sample.[1]

$$(2) \qquad Z_j = \hat{\beta}_j \Big[\prod_{i=1}^{N} \prod_{t=1}^{T} (Y_{it}/X_{itj}) \Big]^{1/NT}$$

where the available sample is for $t = 1, \ldots, T$ periods ($T \geq 2$) on each of $i = 1, \ldots, N$ firms. This average marginal product may also be written as

$$(3) \qquad Z_j = \hat{\beta}_j \exp(\hat{Y} - \hat{x}_j)$$

where

$$\hat{Y} = \sum_{i=1}^{N} \sum_{t=1}^{T} \ln Y_{it}/NT \text{ and } \hat{x}_j = \sum_{i=1}^{N} \sum_{t=1}^{T} \ln X_{itj}/NT$$

In this note the expected value of Z_j conditional on the values taken by the firm effects (A_{oi}) and the levels of the variable inputs (X_{itj}) in the sample is calculated and interpreted and the asymptotic distribution of Z_j ($j = 1, \ldots, k$) is derived.

Finite Sample Results

Equation (1) can be written as

$$(4) \qquad \ln Y_{it} = \ln A_{oi} + \sum_{j=1}^{k} \beta_j \ln X_{itj} + u_{it}$$

where $i = 1, \ldots, N$ and $t = 1, \ldots, T$. The usual matrix representation of equation (4) is:

$$(5) \qquad Q = K\beta_0 + X\beta + U$$

where Q is an NT vector with elements $\ln Y_{it}$, $K = I_N \otimes e_T$ (where I_N is an $N \times N$ identity matrix and e_T is a T vector of ones) is an $NT \times N$ matrix of firm-specific dummies, β_0 is an N vector with elements $\ln A_{oi}$, X is an $NT \times k$ matrix with elements $\ln X_{itj}$, β is a k vector with elements β_j, and U is an NT vector with elements u_{it}. This is a stochastic regressor model

[1] For applications of this measure of the average marginal product, see Mundlak (1961), Hoch (1962) and Dawson and Lingard (1982).

where $K\beta_0$ and X are stochastic matrices. Following Mundlak (1978) the distribution of U is specified conditional on the firm effects in the sample (β_0) and X. It is assumed that u_{it} is an independent and identically distributed normal random variable with mean zero and variance σ^2, distributed independently of β_0 and X, and that the matrix $[K\ X]$ has rank $N+k$.

Ordinary least squares (OLS) estimation of equation (5) yields estimates of β_0 and β. The latter estimate

(6) $$\hat{\beta}=(X'RX)^{-1}X'RQ$$

(where $R=I_{NT}-I_N \otimes J_T$ and $J_T=e_Te'_T/T$) is the analysis of covariance estimate of β. Under these assumptions, an unbiased (and consistent) estimate of σ_2 is:

(7) $$\hat{\sigma}^2=(Q-K\hat{\beta}_0-X\hat{\beta})'(Q-K\hat{\beta}_0-X\hat{\beta})/(N(T-1)-k)$$

where $\hat{\beta}_0$ is the OLS estimate of β_0.[2]

Since Q is a linear transformation of U, the distribution of Q conditional on β_0 and X is multivariate normal, and since the $k+1$ random vector $(\hat{Y},\hat{\beta}')$ is a linear transformation of Q, it follows that conditional on β_0 and X, the elements of $(\hat{Y},\hat{\beta}')$ have a multivariate normal distribution. It is easy to verify that conditional on β_0 and X, $\hat{Y}$ is uncorrelated with the elements of $\hat{\beta}$ and therefore $\hat{Y}$ and $\hat{\beta}$ are conditionally independent.[3]

The following results concerning the lognormal distribution (Aitchison and Brown 1957) are required to calculate and interpret the conditional expectation of Z_j.

Let ν be a normally distributed random variable with mean μ and variance σ^2, then $R=\exp(\nu)$ is lognormally distributed with mean

(8) $$E(R)=\exp(\mu+\sigma^2/2)$$

and median

(9) $$M(R)=\exp(M(\nu))=\exp(\mu)$$

Application of these results to equation (1) yields

(10) $$E(Y_{it}|A_{0i}, X_{it1}, \ldots, X_{itk})=A_{oi}\prod_{j=1}^{k} X_{itj}^{\beta_j}\exp(\sigma^2/2)$$

and

(11) $$M(Y_{it}|A_{oi}, X_{it1}, \ldots, X_{itk})=A_{oi}\prod_{j=1}^{k} X_{itj}^{\beta_j}$$

Thus, as Goldberger (1968) has noted, the conditional expectation of

[2] The maximum likelihood estimator of σ^2 in this model is inconsistent because it does not correct for the degrees of freedom associated with the firm dummies. For details and further references, see Chamberlain (1980, p. 229).

[3] This result uses some basic properties of the multivariate normal distribution which are discussed in Theil (1971, pp. 70-1 and p. 79).

Y_{it} involves the variance of the random disturbance term in the production function, whereas the conditional median involves only the 'systematic' determinants of output.

Using equations (10) and (11) and the conditional independence of $\hat{Y}$ and $\hat{\beta}$, the conditional expectation of Z_j can be derived:

$$(12) \quad E(Z_j | \beta_0, X) = E(\hat{\beta}_j | \beta_0, X) E(\exp(\hat{Y} - \hat{x}_j) | \beta_0, X)$$
$$= \left[\prod_{i=1}^{N} \prod_{t=1}^{T} \beta_j M(Y_{it} | \cdot)/X_{itj} \right]^{1/NT} \exp(\sigma^2/2NT)$$

where $M(Y_{it} | \cdot) = A_{oi} \prod_{j=1}^{k} X_{itj}^{\beta_j}$, is the conditional median of Y_{it}.

Thus, the conditional expectation of this measure of the average marginal product of a Cobb-Douglas factor is equal to the geometric mean of marginal products of the factor evaluated at median levels of output multiplied by a term containing the variance of the error term of the production function.

When NT is large $E(Z_j | \beta_0, X)$ depends mainly on the systematic determinants of output.

Asymptotic Results[4]

Let $\hat{\gamma}' = (\hat{Y}, \hat{\beta}')$ and $E(\hat{\gamma}' | \beta_0, X) = (E(\hat{Y} | \beta_0, X), \beta')$, then $(NT)^{1/2}(\hat{\gamma} - E(\hat{\gamma} | \beta_0, X))$ has a conditional normal distribution with mean vector zero and covariance matrix Σ_{NT},

$$\Sigma_{NT} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 (X'RX/NT)^{-1} \end{pmatrix}$$

so that providing $(X'RX/NT)$ converges to a positive definite matrix as $(N \to \infty)$, the conditional random variable $(NT)^{1/2}(\hat{\gamma} - E(\hat{\gamma} | \cdot))$ has an asymptotic normal distribution.[5]

Since Z_j is a non-linear function of the elements of $\hat{\gamma}$, providing $\hat{\gamma}$ converges in probability to a constant vector the multivariate δ method can be used to determine the asymptotic distribution of Z_j.[6] It is usual in pooled cross-section and time series studies for N to be large and T small; thus, the asymptotic results will be based on the number of firms, N becoming large. In deriving these results, it is assumed that for fixed T

$$(13) \quad \lim_{N \to \infty} (X'RX/NT) = A$$

where A is a $k \times k$ positive definite matrix, and

[4] Fisk (1966) derives the asymptotic distribution of the marginal product of a Cobb-Douglas factor for a given value of the input vector. The results of this section are different to those of Fisk in that they are derived in a fixed-effects framework and are obtained for geometric means of the estimated marginal products of all factors.

[5] This follows because the characteristic function of $(NT)^{1/2}(\hat{\gamma} - E(\hat{\gamma} | \beta_0, X))$ converges to the characteristic function of a multivariate normal random variable as $NT \to \infty$. See Theil (1971, p. 369) for a discussion of this result.

[6] For a discussion of the multivariate δ method, see Bishop, Feinberg and Holland (1975, pp. 492–4).

(14) $\lim_{N\to\infty} E(\hat{Y}|\beta_0, X) = \bar{\beta}_0 + \sum_{j=1}^{k} \beta_j x_j = m$

where $\bar{\beta}_0 = \lim_{N\to\infty} \sum_{i=1}^{N} \beta_{oi}/N$ *and* $x_j = \lim_{N\to\infty} \hat{x}_j$.

These assumptions are sufficient to guarantee that as $N\to\infty$, $\hat{\beta}$ and $\hat{Y}$ are consistent estimates of β and m respectively. That is,[7]

$$\operatorname*{plim}_{N\to\infty}\hat{\beta} = \beta \text{ and } \operatorname*{plim}_{N\to\infty}\hat{Y} = m$$

Let Z be the k vector with elements Z_j; then, under these assumptions, application of the multivariate δ method yields

(15) $(NT)^{1/2}(Z - E(Z|\beta_0, X)) \xrightarrow[N\to\infty]{D} N(O, H\Sigma H')$

where H is the $k\times(k+1)$ matrix of the elements of $\partial Z'/\partial\gamma$ evaluated at

$$\hat{\gamma}' = (m,\beta'),\ \hat{x}_j = x_j \quad (j = 1, \ldots, k), \quad \Sigma = \begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2 A^{-1}\end{pmatrix}$$

and $N(O,H\Sigma H')$ denotes the multivariate normal distribution with mean vector O and covariance matrix $H\Sigma H'$.

The asymptotic distribution of Z_j can be obtained from equation (15):

(16) $(NT)^{1/2}(Z_j - E(Z_j|\beta_0, X)) \xrightarrow[N\to\infty]{D} N(O, V_j)$

where $V_j = \sigma^2\beta_j^2\exp(2(m - x_j)) + \sigma^2 a_{jj}\exp(2(m - x_j))$

is the variance of the asymptotic distribution of Z_j, and a_{jj} is the jth diagonal element of A^{-1}. In order to apply equation (16), a consistent estimator of V_j is required, and this can be obtained using the following consistent estimators of the components of V_j: $\hat{\sigma}^2$ is a consistent estimator of σ^2, Z_j^2 is a consistent estimator of $\beta_j^2\exp(2(m - x_j))$, $Z_j^2/\hat{\beta}_j^2$ is a consistent estimator of $\exp(2(m - x_j))$ and $NT\hat{\sigma}^2(X'RX)^{-1}_{jj}$ is a consistent estimator of $\sigma^2 a_{jj}$, where $(X'RX)^{-1}_{jj}$ is the jth diagonal element of $(X'RX)^{-1}$. Since $\hat{\sigma}^2(X'RX)^{-1}_{jj} = \hat{V}(\hat{\beta}_j)$ is an estimator of the variance of $\hat{\beta}_j$, the consistent estimator of $\sigma^2 a_{jj}$ can be expressed as $NT\hat{V}(\hat{\beta}_j)$. Thus,

(17) $\hat{V}_j = \hat{\sigma}^2 Z_j^2 + NT\hat{V}(\hat{\beta}_j)Z_j^2/\hat{\beta}_j^2$

is a consistent estimator of V_j. Thus, in large samples the interval

(18) $Z_j \pm \Phi(1 - \frac{\alpha}{2})(\hat{V}_j/NT)^{1/2}$

where $\Phi(1 - \frac{\alpha}{2})$ is the $100(1 - \frac{\alpha}{2})$ percentile of the standard normal distribution, is an approximate $100(1 - \alpha)$ per cent confidence interval for

[7] These results can be proved using the Chebyshev inequality. See Theil (1971, pp. 362–3, Theorem 8.1) for a similar example.

the geometric mean of the marginal products of factor j of firms in the sample evaluated at median levels of output.

Application: Testing for Risk Aversion

The preceding statistical results can be used in a number of applications. For example, they can be used to construct confidence intervals for the geometric means of the expected marginal products of the variable factors of firms in the sample and, under specified conditions, they can be used to estimate the shadow prices of rationed factors. (For details, see Alaouze 1988.)

Another interesting application which is developed below is a test for risk aversion. In developing this application, it is assumed that input and output prices are known with certainty, but similar results can be obtained if the input prices are known with certainty and the output price is independent of output of each firm (Alaouze 1988).

By the nature of the allocation problem, the entrepreneur chooses the levels of inputs which maximise his objective function (expected profit or expected utility, Π_{it}) before the random variable which enters the objective function is realised. Let

$$\Pi_{it} = P_{it} Y_{it} - \sum_{j=1}^{k} P_{itj} X_{itj}$$

(where P_{it} is the price of output from firm i in period t and P_{itj} is the price paid for input j by firm i in period t) be the profit function for firm i in period t. The first order conditions for profit maximisation can be written as

$$\text{(19)} \quad E(\partial Y_{it}/\partial X_{itj}) = P_{itj}/P_{it} \qquad (j=1, \ldots, k)$$

Noting that

$$\text{(20)} \quad \partial Y_{it}/\partial X_{itj} = \beta_j Y_{it}/X_{itj} \qquad (j=1, \ldots, k)$$

and using equations (10), (12) and (20) the geometric mean of the left-hand side of (19) can be written as

$$\text{(21)} \quad \left[\prod_{i=1}^{N} \prod_{t=1}^{T} E(\partial Y_{it}/\partial X_{itj})\right]^{1/NT} = \theta E(Z_j \mid \beta_0, X) \qquad (j=1, \ldots, k)$$

where $\theta = \exp((1 - 1/NT)\sigma^2/2)$.

Letting $\bar{P} = \left[\prod_{i=1}^{N} \prod_{t=1}^{T} P_{itj}/P_{it}\right]^{1/NT}$ and $\hat{\theta} = \exp((1 - 1/NT)\hat{\sigma}^2/2)$ be a consistent estimator of θ, then under these assumptions, the conditional random variable $\hat{\theta} Z_j$ has the following asymptotic distribution:[8]

$$\text{(22)} \quad \underset{N\to\infty}{(NT)^{1/2}}(\hat{\theta} Z_j - \bar{P}) \xrightarrow{D} N(O, \bar{\theta}^2 V_j) \qquad (j=1, \ldots, k)$$

where $\bar{\theta} = \exp(\sigma^2/2)$.

[8] This result follows from proposition (iii) of Theil (1971, p. 371).

If risk neutrality is taken as the null hypothesis (that is, entrepreneurs maximise expected profits), then equation (22) gives the probability distribution of $\hat{\theta}Z_j$ under the null hypothesis.

If entrepreneurs maximise expected utility and the utility function is strictly concave, then

$$(23) \quad E(\partial Y_{it}/\partial X_{itj}) > P_{itj}/P_{it} \qquad (j=1, \ldots, k)$$

This result is proved in the Appendix. Taking risk aversion as the alternative hypothesis, equation (23) implies that:

$$(24) \quad \theta E(Z_j \mid \beta_0, X) > \tilde{P}$$

Thus, under the null hypothesis, and in large samples, the statistic

$$(25) \quad T = (\hat{\theta}Z_j - \tilde{P})/(\hat{\theta}^2 \hat{V}_j/NT)^{1/2}$$

has approximately a standard normal distribution. The null hypothesis is rejected in favour of the alternative hypothesis when T exceeds the critical value of the standard normal distribution corresponding to the chosen level of significance. In order to apply the test, data on the price of output and the price of one input are required. It is also very important that the production function be correctly specified and that the assumptions underlying the test hold. Obviously, a misspecified model or failure of any critical assumptions can lead to a misleading test result.

APPENDIX

In this Appendix a proof of equation (23) is derived. In the following analysis, the assumptions of the last section are maintained except that it is assumed that firms maximise the expected utility of profits. It is also assumed that the expected utility function $U(\Pi)$ is strictly concave in Π $[U''(\Pi) < 0]$ and strictly increasing in Π $[U'(\Pi) > 0]$ where single and double primes represent first and second derivatives respectively. For convenience, the firm and time subscripts are dropped.

The first-order conditions for maximising expected utility can be written as

$$(\text{A1}) \quad E(U'(\Pi)P\partial Y/\partial X_j) = E(U'(\Pi))P_j \qquad (j=1, \ldots, k)$$

and, using equation (20), equation (A1) can be written as

$$(\text{A2}) \quad E(U'(\Pi)PY\beta_j/X_j) = E(U'(\Pi))P_j \qquad (j=1, \ldots, k)$$

In order to analyse equation (A2), a theorem due to Gurland (1967) concerning the correlation between two monotone functions of a single random variable is required.

Theorem: Let f and g be monotone functions of a random variable W and assume that one of these functions is continuous. Then, if f and g are both non-increasing or both non-decreasing,

$$E(f(W))E(g(W)) \leq E(f(W)g(W))$$

If f is non-decreasing and g is non-increasing or vice versa, then

$$E(f(W))E(g(W)) \geq E(f(W)g(W))$$

It is assumed that all the expectations appearing in these inequalities exist. The inequalities are strict if W is not a degenerate random variable.

To apply the theorem, take PY as the random variable and $U'(\Pi)$ and PY as the functions f and g. Since $\partial U'(\Pi)/\partial(PY) = U''(\Pi) < 0$, $U'(\Pi)$ is monotone decreasing in PY for fixed values of all other variables, so that application of the theorem yields the following inequalities between terms on the left-hand side of equation (A2):

$$\text{(A3)} \quad E(U'(\Pi)PY\beta_j/X_j) < E(U'(\Pi))E(PY\beta_j/X_j) \qquad (j = 1, \ldots, k)$$

Replacing the left-hand side of equation (A2) with the right-hand side of inequality (A3) and simplifying yields

$$\text{(A4)} \quad E(PY\beta_j/X_j) > P_j \qquad (j = 1, \ldots, k)$$

Using equation (20), inequality (A4) can be written as

$$\text{(A5)} \quad E(\partial Y/\partial X_j) > P_j/P \qquad (j = 1, \ldots, k)$$

This is equation (23) in the text. Some interesting extensions of these results can be derived. For example, if the output price is random but uncorrelated with output, this approach can be used to establish that

$$\text{(A6)} \quad E(\partial Y/\partial X_j) > P_j/E(P) \qquad (j = 1, \ldots, k)$$

If the output price is random, all input prices are random, PY is independent of all input prices and Y is uncorrelated with P, a generalisation of Gurland's Theorem (Alaouze and Lloyd 1986) can be used to show that

$$\text{(A7)} \quad E(\partial Y/\partial X_j) > E(P_j)/E(P) \qquad (j = 1, \ldots, k)$$

An important requirement in establishing the inequalities (A5), (A6) and (A7) is that the output price is uncorrelated with output, and this requirement is also necessary to derive expressions similar to equation (19) for the risk-neutral case.

In the case where the production function is non-stochastic and only the output price is random, Batra and Ullah (1974) establish that

$$\partial Y/\partial X_j \geq P_j/E(P) \qquad (j = 1, 2)$$

for a general production function with capital and labour as inputs.

It is interesting to note that the assumption that the production function exhibits diminishing returns to each input is not sufficient to guarantee that the risk-averse firm uses less of each input than the risk-neutral firm because (for example) the partial derivatives in inequality (A5) and equation (19) are evaluated at different points (Hartman 1975).

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