PRODUCTION SCHEDULING AND ALLOCATION: A NORMATIVE DECISION MODEL FOR SUGAR MILLING*

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An inter-temporal equilibrium model is developed for the purpose of optimizing the scheduling of production in Australian sugar mills. An application of this quadratic programming model is then discussed, and the procedures used to estimate coefficients are outlined. Two tentative conclusions are that mills tend to commence crushing before the optimal starting date and that in many cases they would have been unable to cover their marginal processing costs when producing No. 2 Pool sugar in recent seasons.

Introduction

Over the past four years the Australian sugar industry has been experiencing a period of economic depression despite record levels of production. The figures in Table 1 show that raw sugar production has virtually doubled since 1961. This increase in output was stimulated initially by a period of high prices on the world free market and was continued despite adverse prices, in the absence of an effective International Sugar Agreement assigning export quotas to all suppliers.[1]

While this expanded production has undoubtedly been a useful bargaining factor in the recent negotiation of quotas for the new International Sugar Agreement, a conclusion which may be drawn from the research reported in this paper is that it has tended to force sugar mills to produce when they were unable to cover their marginal costs. Although this particular difficulty is likely to be avoided in the near future if sugar production is limited to mill peak quantities, the authors' consider it unlikely that the No. 1 Pool sugar price will rise above $95 per ton for the 1969 crop.[2] For this reason it will continue to be important

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[1] The figures for No. 2 Pool payouts in Table 1 give a guide to the average net level of raw sugar prices on the world free market in successive years. The relative stability of the No. 1 Pool payout is brought about by the fact that more than fifty per cent of the sugar in the pool is sold either on the local market or to the U.K. or U.S., at stable prices well above the ruling free market prices of recent years.

[2] This assumes that a free market price in the range £30-£35 sterling per ton, c.i.f. can be maintained on the U.K. market during 1969. The International Sugar Council has already shown that it is prepared to cut supply quotas if necessary, to prevent prices falling below this level. Two factors which might tend to reduce this Pool price, even if the free market price is maintained, are the need to accumulate funds to repay the Commonwealth Government loans used to bolster the Pool price in recent seasons and the need to finance the storage of a minimum buffer stock of 150,000 tons of raw sugar in Australia, in accord with the new International Sugar Agreement.

to the achievement of even modest profits in sugar milling, that millers seek maximum economic efficiency.

One important aspect of this question of economic efficiency involves the attempt to minimize the quantity of cane crushed in achieving mill sugar peak, by crushing cane only in those periods when it has the highest raw sugar content.[3] By this criterion, the crushing periods used in many mills depart substantially from the optimum suggested by the model developed in this paper. The effects of parametric variation of the average costs of transporting cane and raw sugar are also examined, in an attempt to provide some indication of the importance of these controversial elements of overall mill costs.

TABLE 1

Annual Sugar Production in Australia (1960–1968) *and Payouts by the Queensland Sugar Board for No.* 1 *and No.* 2 *Pool Sugar*

Year	Tons of Raw Sugar Produced	No. 1 Pool Payout	No. 2 Pool Payout
		$	$
1960	1,382,611	100.88	61.85
1961	1,382,841	100.24	52.00
1962	1,849,819	106.06	72.20
1963	1,724,253	122.59	145.05
1964	1,950,078	98.20	76.90
1965	1,953,353	86.58	42.50
1966	2,342,776	85.69[a]	35.50
1967	2,334,393	86.00[b]	38.55
1968	2,724,788[c]	89.29	43.65

Source: Reports published in various issues of *The Australian Sugar Journal.*
[a] Includes $8.56 from a loan to the Industry by the Commonwealth Government.
[b] Includes $1.81 from a loan to the Industry by the Commonwealth Government.
[c] Provisional total.

Characteristics of the Production Scheduling Model

The model developed in this paper is designed to establish a constrained optimum crushing period for a typical sugar mill in an Australian context. Attention is directed particularly at the variation in marginal revenue from processing over time and the variation in marginal cost with quantity processed. To effectively establish an economic optimum of a short term partial equilibrium type, it is necessary to suppose that the mill manager seeks to maximize his processing profit and is free to cease processing whenever marginal cost becomes greater than or equal to marginal revenue (see Figure 1.). Apart from this abstraction, an attempt has been made to incorporate other important institutional and technological features of the actual production process in the model.

Two characteristics distinguish this model from the standard deterministic single product production scheduling models commonly employed [13; pp. 746-49]. The first is the explicit introduction of constrained

[3] A number of technical terms, some of which relate to institutional arrangements in the sugar industry, are introduced in the paper. Definitions of these terms and a general description of the industry may be found in the public relations pamphlet, *Notes on the Australian Sugar Industry*, Sydney, The Colonial Sugar Refining Company Limited, January, 1968.

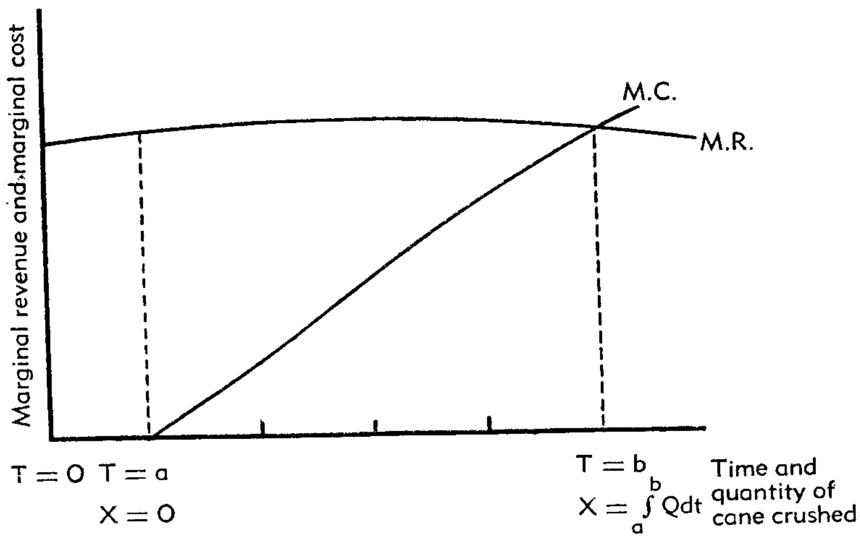

FIG. 1—Marginal conditions for an unconstrained optimum crushing period.

profit maximization in the objective function, while the second is the necessity to determine dates at which to commence and terminate production rather than the rate of production in successive time periods.

The importance of both these characteristics may be indicated by reference to Figure 1, in which both marginal revenue and the rate of processing cane are assumed to be explicit functions of time while marginal cost is assumed to be a linear function of the total quantity processed over a time interval. The economic criterion for an unconstrained optimum crushing period is then, as shown in the figure, that marginal revenue when $T = a$, $X = 0$, must equal marginal revenue when $T = b$, $X = \int_a^b \hat{Q}dt$, while marginal revenue at both points in time must equal marginal cost when the total quantity processed, $X = \int_a^b \hat{Q}dt$.

For a constrained optimum crushing period a further function must be satisfied. If the rate of production of raw sugar is also treated as an explicit function of time then it is necessary that mill sugar peak be greater than or equal to the integral of this function over the crushing interval, i.e. $MSP \geqq \int_a^b \hat{S}dt$. Should this function become a binding constraint on production while marginal cost is still less than marginal revenue, the difference between marginal cost and the marginal revenue earned at each end of the crushing period represents the marginal return to the mill, per ton of cane crushed, from an addition to its sugar peak allocation. The same type of constrained optimum may also occur should cane supply effectively limit production.

The model developed to handle this particular type of production scheduling problem is based upon the 'primal-dual' quadratic programming models originally formulated by Takayama and Judge [8, 11] to handle spatial equilibrium problems. Because of its inter-temporal nature, the fact that it involves the transformation of a primary into an intermediate product, and the fixed price and quantity elements which it contains, the model also resembles in certain ways several other more specific models developed by these authors [9], [10], [12; p. 45-46]. As a quadratic programming model it also relies heavily upon the earlier work of Kuhn and Tucker [5] and Wolfe [15], in this field.

Definitions and Notation

Definitions and notation employed in the present model are as follows:

$'$ denotes the transposition of a column vector;

i denotes the time period in which processing occurs where $i = 1, \ldots, n$; in this instance the time periods are assumed to be of one week in length;

$X = (x_1, \ldots\ldots, x_n)'$ denotes a vector of non-negative quantities of cane processed in each time period;

$T = (t_1, \ldots\ldots, t_n)'$ denotes a vector of n identical non-negative elements each made up of the average cost of transporting a ton of cane from a supply point to the mill, plus the constant element in the price of the raw material;

P_1 denotes a vector having a single non-negative element which is the marginal cost of processing sugar cane for all time periods considered together;

$P_2 = (p_1, \ldots\ldots, p_n)'$ denotes a vector of non-negative prices each of which represents the marginal cost of the restriction on plant processing capacity in the relevant time period;

P_3 denotes a vector having a single non-negative element which is the marginal value of cane to the mill over and above the fixed payment required by institutional arrangements;

P_4 denotes a vector having a single non-negative element which represents the marginal value to the mill of additional demand for raw sugar (i.e. of an increase in sugar peak);

$P_5 = A$ denotes a vector having a single positive element which is the fixed price paid for No. 1 Pool sugar in all time periods;

$P_6 = B$ denotes a vector having a single positive element which is the fixed cost per ton for transporting raw sugar to the bulk terminal;

C_1 denotes a vector having a single non-positive element representing the intercept of an equation which is the inverse of a linear marginal processing cost function;

$C_2 = (c_1, \ldots\ldots, c_n)'$ denotes a vector of non-negative quantities representing the processing capacity of the mill in tons of sugar cane, during each of the n time periods;

C_3 denotes a vector having a single non-negative element which is the total quantity of sugar cane expected to be available for crushing in the mill area during the season;

C_4 denotes a vector having a single non-negative element representing the difference between the mill peak of raw sugar, and the actual quantity of C.C.S. (commercial cane sugar) paid for by the mill in producing mill peak;

$C_5 = C_6$ denotes a vector having the single element zero;

$G_1 = G_3 = (1, \ldots\ldots, 1)$ denotes a row vector of n elements all taking the value unity;

G_2 denotes the identity matrix I_n;

$G_4 = G_5 = (\alpha_1 - \beta_1, \ldots\ldots, \alpha_n - \beta_n)$ denotes a row vector of elements in which the terms α_i represent conversion factors between cane input and raw sugar output in each time period while the terms β_i represent conversion factors between cane input and C.C.S. paid for by the mill in each time period;

$G_6 = (\gamma_1, \ldots\ldots, \gamma_n)$ denotes a row vector made up of the elements α_i defined above, multiplied by a scalar factor representing the difference in purity between standard raw sugar and the actual raw sugar produced;

G_7 denotes a column vector of one element which is unity;

G_8 denotes a column vector of one element which is unity;

Q denotes a matrix of dimensions 1×1, whose single element is the inverse of the marginal processing cost coefficient;

$V = (v_1, \ldots\ldots, v_n)'$ denotes a vector of slack variables representing the difference between the marginal cost of processing cane and the marginal revenue to be earned from processing in each time period;

Y_1 denotes a vector having a single non-negative element representing the quantity of cane it may be necessary to process to ensure that marginal processing cost is non-negative;

$Y_2 = (y_1, \ldots\ldots, y_n)'$ denotes a vector of quantities representing excess processing capacity in each of the n time periods;

Y_3 denotes a vector having a single non-negative element representing the excess supply of cane available to the mill for the season;

Y_4 denotes a vector having a single non-negative element which represents the share of the mill in the quantity of raw sugar, within mill sugar peak, which can only be produced by allowing marginal processing cost to exceed marginal processing revenue;

W_1, W_2 denote vectors each having one non-negative element and representing respectively, the additional quantity of C.C.S. (which the mill receives as its share of raw sugar revenue) that it would be worth producing given the institutional arrangements concerning sugar peaks; and the quantity of raw sugar transported to the bulk terminal;

U_1, U_2 denote vectors each having one non-negative element and representing respectively, the quantity of C.C.S. received by the mill as its share of total raw sugar revenue, and the additional quantity of raw sugar which it would be worth transporting to the bulk terminal at the fixed cost for transportation.

The Production Scheduling Model

In the model, mill processing capacity is limited by a set of relationships which may be written using the notation developed in the previous sub-section as

(1) $C_2 \geqq G_2X$

while cane availability and sugar production within peak are limited respectively, by the relationships

(2) $C_3 \geqq G_3X$

and

(3) $C_4 \geqq G_4X.$

In addition there are fixed price and cost elements involved which may be stated as

(4) $G'_7P_5 = G'_7A$ or $G'_7P_5 \geqq G'_7A$ and $G'_7A \geqq G'_7P_5$

and

(5) $G'_8P_6 = G'_8B$ or $G'_8P_6 \geqq G'_8B$ and $G'_8B \geqq G'_8P_6$

The profitability of production within these constraints depends upon the marginal revenue in each time period and the marginal cost over all time periods. The set of marginal revenue relationships may be written

(6) $M.R. = G'_5P_5$

while the marginal processing cost relationship may be stated as

(7) $P_1 = -Q^{-1}C_1 + Q^{-1}(G_1X + Y_1)$

and other non-processing costs per unit of production may be defined by the relationship

(8) $N.P.C. = G'_6P_6 + T$

The profit maximizing model is obtained by integrating equations (6) and (8) with respect to X and equation (7) with respect to $(G_1X + Y_1)$ which is the total quantity of cane processed[4], then subtracting the costs from revenue. This objective function is subject to the inequality constraints specified in the relationships (1) to (5) and may be stated as follows:

(9) $$\text{Max. } f(P_5, P_6, X, Y_1) = P'_5G_5X - P'_6G_6X - T'X + C'_1Q^{-1}(G_1X + Y_1) - \tfrac{1}{2}(G_1X + Y_1)'Q^{-1}(G_1X + Y_1)$$

subject to (1)—(5) and

(10) $X \geqq 0,\ Y_1 \geqq 0.$

By Theorem 3 of Kuhn and Tucker [5; p. 486] solution of this quadratic maximizing problem is equivalent to solution of the following saddle value problem:

(11) $$\begin{aligned}&\phi(P_5, P_6, X, Y_1, P_2, P_3, P_4, W_1, W_2, U_1, U_2)\\ &= P_5'G_5X - P_6'G_6X - T'X + C_1'Q^{-1}(G_1X + Y_1)\\ &-\tfrac{1}{2}(G_1X + Y_1)'Q^{-1}(G_1X + Y_1) + P_2'(C_2 - G_2X)\\ &+ P_3'(C_3 - G_3X) + P_4'(C_4 - G_4X) + W_1'(G_7'P_5 - G_7'A)\\ &+ W_2'(G_8'P_6 - G_8'B) + U_1'(G_7'A - G_7'P_5) + U_2'(G_8'B - G_8'P_6)\end{aligned}$$

in which the non-negative vectors P_2, P_3, P_4, W_1, W_2, U_1 and U_2 are lagrange multipliers.

The Kuhn-Tucker necessary conditions [5; p. 482–83] for a solution, assuming a feasible saddle value solution exists, may be stated as follows:

(12) $\delta\phi/\delta P_5 = G_5X + G_7W_1 - G_7U_1 \leqq 0$ and $(\delta\phi/\delta P_5)'P_5 \equiv 0;$

(13) $\delta\phi/\delta P_6 = G_6X + G_8W_2 - G_8U_2 \leqq 0$ and $(\delta\phi/\delta P_6)'P_6 \equiv 0;$

(14) $\delta\phi/\delta X = G_5'P_5 - G_6'P_6 - T + G_1'Q^{-1}C_1 - G_1'Q^{-1}(G_1X + Y_1) - G_2'P_2 - G_3'P_3 - G_4'P_4 \leqq 0$ and $(\delta\phi/\delta X)'X \equiv 0;$

(15) $\delta\phi/\delta Y_1 = Q^{-1}C_1 - Q^{-1}(G_1X + Y_1) \leqq 0$ and $(\delta\phi/\delta Y_1)'Y_1 \equiv 0;$

(16) $\delta\phi/\delta P_2 = C_2 - G_2X \geqq 0$ and $(\delta\phi/\delta P_2)'P_2 \equiv 0;$

[4] The formulation which follows includes only the variable costs of the processing activity since only the variable costs are relevant to the optimizing decision. The constants arising from the integration of equations (6) and (8) have similarly been omitted.

(17) $\delta\phi/\delta P_3 = C_3 - G_3 X \geqq 0$ and $(\delta\phi/\delta P_3)' P_3 \equiv 0;$

(18) $\delta\phi/\delta P_4 = C_4 - G_4 X \geqq 0$ and $(\delta\phi/\delta P_4)' P_4 \equiv 0;$

(19) $\delta\phi/\delta W_1 = G_7' P_5 - G_7' A \geqq 0$ and $(\delta\phi/\delta W_1)' W_1 \equiv 0;$

(20) $\delta\phi/\delta W_2 = G_8' P_6 - G_8' B \geqq 0$ and $(\delta\phi/\delta W_2)' W_2 \equiv 0;$

(21) $\delta\phi/\delta U_1 = G_7 A - G_7' P_5 \geqq 0$ and $(\delta\phi/\delta U_1)' U_1 \equiv 0;$

(22) $\delta\phi/\delta U_2 = G_8' B - G_8' P_6 \geqq 0$ and $(\delta\phi/\delta U_2)' U_2 \equiv 0;$

The only feasible solution for each of the pairs of relationships (19a), (21a) and (20a), (22a) is that they hold with equality, hence the variables W_1, W_2, U_1 and U_2 would appear to be unconstrained apart from the non-negativity requirement. A closer examination reveals however that the variables W_1, U_1 and W_2, U_2 are themselves counterpart pairs,[5] since not more than one of the associated pair of relationships will actually be binding at any stage.[6] Because the relationships (19)-(22) require P_5 and P_6 to be positive the relationships given in (12a) and (13a) must necessarily hold with equality. This is consistent with the interpretation placed upon the variables W_1, W_2, U_1 and U_2 when they were defined earlier.

Among the remaining relationships the vector V may be added to (14a) to bring it to equality, with V and X being treated as a counterpart pair, while the vectors Y_2, Y_3 and Y_4 may be subtracted from (16a) (17a) and (18a) respectively to bring them to equality, with Y_2, Y_3, Y_4 and P_2, P_3 and P_4 being treated as counterpart pairs. Reference to equation (7) shows that it would be feasible to replace the terms $G'_1 Q^{-1} C_1 - G'_1 Q^{-1}(G_1 X + Y_1)$ in (14a) with the single term $-G'_1 P_1$ so that it was stated entirely in terms of prices. The addition of the vector P_1 to (15a) brings it to equality, with P_1 and Y_1 being counterparts. If this equation is then multiplied through by Q the inverse of equation (7) is obtained.

A solution to the saddle value problem set out in (11) which satisfies all the conditions specified in the relationships (12)-(22), (if such a feasible solution exists), may be obtained by making use of a modified version of the simplex algorithm for quadratic programming developed by Wolfe [15]. In the simplex tableau for this 'primal-dual' problem [11; pp. 515-19] which has been set out in Table 2, column vectors of artificial variables Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7 and Z_8 of appropriate dimensions have been used to establish an initial basis. Associated with these artificial variables are vectors of arbitrary positive elements, M_1, M_2, M_3, M_4, M_5, M_6, M_7 and M_8, of the same dimensions respectively. In the upper part of the tableau the equality form of the constraints (4) and (5) has been used.

[5] Variables which are 'counterpart pairs' in the present context are non-negative variables which when multiplied together are always exactly equal to zero. For this reason, if one of the pair of variables is known to be positive the other is known necessarily to be zero. The use of 'counterpart variable operations' based on this characteristic, is essential for the successful application of the modified simplex method for quadratic programming. Takayama and Judge used this terminology, to distinguish this relationship between pairs of variables from the primal-dual, price-quantity relationships which they discussed when formulating equivalent spatial equilibrium problems in both the price and quantity domains [11].

[6] In the context of the present problem it is apparent that (21a) which prevents the price of raw sugar from rising above a fixed level and (20a) which prevents the cost of raw sugar transportation from falling below a fixed level are the binding constraints. For this reason the variables W_2 and U_1 will be positive while the variables W_1 and U_2 will be zero.

TABLE 2

The Primal-Dual Quadratic Programming Tableau Used to Solve the Production Scheduling Problem for Sugar Milling

C			0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		P_0	X	W_1	W_2	P_1	P_2	P_3	P_4	P_5	P_6	V	U_1	U_2	Y_1	Y_2	Y_3	Y_4
O	V	T				$-G'_1$	$-G'_2$	$-G'_3$	$-G'_4$	G'_5	$-G'_6$	I_n						
$-M_1$	Z_1	G'_7A								G'_7								
$-M_2$	Z_2	G'_8B									G'_8							
$-M_3$	Z_3	$-C_1$	$-G_1$			Q									$-I_1$			
$-M_4$	Z_4	$-C_2$	$-G_2$													$-I_n$		
$-M_5$	Z_5	$-C_3$	$-G_3$														$-I_1$	
$-M_6$	Z_6	$-C_4$	$-G_4$															$-I_1$
$-M_7$	Z_7	C_5	G_5	G_7									$-G_7$					
$-M_8$	Z_8	C_6	$-G_6$		G_8									$-G_8$				

An Empirical Application of the Model

As Kriebel [4] has pointed out, considerable attention must be paid to problems of coefficient estimation if a quadratic programming model is to be successfully applied to the solution of real world problems. In this instance, because there was a large quantity of data available which related to several mills of comparable size, it was possible to obtain a number of estimates of most coefficients.[7] Median values were usually selected for the model and the resulting empirical example may be regarded as typical of the position of mills in the Central District, able to crush approximately 250 tons of cane per hour.

Estimation of Weekly Crushing Capacities

The quantity of cane which it is feasible to crush during any weekly time period is dependent upon the realized hourly crushing rate of the plant and the number of hours worked per week. It is normal practice to work three shifts per day during the crushing season giving a nominal crushing period of one hundred and twenty hours per week and leaving the weekend period available for major routine maintenance jobs. Crushing time may be lost for a number of reasons such as interruption of cane supplies by heavy rain and mechanical breakdown of mill or tramway equipment. Over the season, after adjusting for statutory holidays, it is normal to find that lost time represents around ten per cent of the nominal working time.[8]

In many cases these time losses are not uniformly distributed through the crushing season and an examination of weekly totals of cane crushed in 1966 suggested that, after adjustments had been made for statutory holidays and major holdups, the weekly quantities of cane crushed followed a distinct parabolic time trend over the crushing season apart from a random factor.[9] The hypothesis that the function

$$(23) \quad Q = \gamma_o + \gamma_1 T + \gamma_2 T^2 + u \qquad (\gamma_o, \gamma_1 > 0, \gamma_2 < 0)$$

where Q = the weekly rate of crushing cane at time t,
T = time in units of one week with origin at noon on 25th May 1966, and
u = a random normal variable,

would provide a satisfactory approximation to the expected variation in the rate of crushing over the season, was tested using data from a number of mills.

[7] We wish to acknowledge the assistance given to our research by the directors of a number of mills, who made available confidential financial data which was used to obtain cost estimates.

[8] Data on lost time as a percentage of available time, averaged over all mills in each of the seasons 1962-1966, is contained in the 63rd to 67th Annual Reports of the Queensland Bureau of Sugar Experiment Stations (Table XII). The figures range from under six to almost fourteen per cent during this five year period but the figure of ten per cent is a reasonable expectation.

[9] Dr C. R. Murry of the Sugar Research Institute states that there is no technical evidence to support such a trend. Theoretically at least, the crushing capacity of mill equipment should be constant throughout the season apart from a random factor. He suggests that reasons for the appearance of such a trend, in addition to the rainfall factor already mentioned, are a reluctance by growers to harvest low C.C.S. cane early in the season, the normal 'settling-in' problems experienced at the start of the season with both mill and harvesting equipment, and the belief by millers that it is cheaper to process high C.C.S. cane, which leads them to make special efforts to achieve large through-puts when it is available. The assumed trend is not critically important to the model and was used simply to smooth out random fluctuations in crushing capacity.

A regression relationship typical of those estimated was

$$(23.1) \quad \hat{Q} = 18327{\cdot}249 + 1261{\cdot}8185T - 37{\cdot}55113T^2 \qquad (R^2 = 0{\cdot}919)$$
$$(t = 31{\cdot}650) \quad (t = 13{\cdot}982) \quad (t = -12{\cdot}338) \qquad (d = 1{\cdot}820)$$

in which the student's t values for the regression coefficients, given in brackets, indicate the high levels of significance attained. Maximum crushing rate occurs about September 20th according to this estimated relationship, and in general the periods with the highest crushing rates, which occur during the months August to October, are those which normally coincide with the periods of lowest monthly rainfall on the Queensland coast north of the Tropic of Capricorn.

Expected weekly quantities of cane crushed in 1966 may be calculated from equation (23.1) by integrating over consecutive time periods from $t - \frac{1}{2}$ to $t + \frac{1}{2}$. Expected weekly quantities of cane which could be crushed in the 1969 season, assuming mill size had not changed, may also be obtained from the equation if the origin of time is first moved to noon on May 28th, 1969 prior to integration. The series so obtained was used to define crushing capacities for the model. These figures, the X and Y_2 vector values given in Table 3, are probably a little low for expected district average crushing capacities in 1969, but an upward adjustment of the figures would serve only to shorten the optimal crushing season.[10]

The total quantity of cane assumed to be available for crushing in 1969 was set at 600,000 tons, a quantity somewhat greater than would normally be required to manufacture the assumed mill peak of 78,000 tons of raw sugar, but considerably less than that grown in any of the mill areas in 1968 when over-peak sugar was manufactured. While growers corporately may be expected to have scaled down production following the statement by the Sugar Board, when the new International Sugar Agreement was finalized, that only mill-peaks would be acquired in 1969, they would need to set out to grow approximately this quantity of cane on their assigned acreage if they were to ensure that mill peak was produced in all seasons.

The mill's share of raw sugar production which was also specified in the model, was not known initially and could only be calculated after a solution had been obtained. An exact value was required for this coefficient as it was used in the model to ensure that exactly the mill peak of sugar was produced. Since the mill usually receives approximately thirty per cent of the revenue from raw sugar a figure of 26,000 tons was chosen for an initial solution and this was adjusted to the required level of 24,786·016 tons for a second solution, by allowing for the quantity of sugar produced in excess of peak.

Estimation of Processing Cost

Processing cost was hypothesized to be a direct function of the quantity of cane processed rather than a function of raw sugar output since the cost of crushing cane is a major factor in variable processing costs. As the production scheduling model specified a linear marginal proces-

[10] The length of crushing season would be reduced by about one working day for each one per cent increase in overall expected crushing capacities.

sing cost function, the functional form chosen to explain variations in total processing cost was

$$(24) \quad C = \alpha_o + \alpha_1 X + \alpha_2 X^2 + u \qquad (\alpha_o, \alpha_1 \text{ and } \alpha_2 \geqq 0)$$

where C = total processing cost in dollars,
X = the number of tons of cane crushed,
X^2 = the square of the number of tons of cane crushed, and
u = a random normal variable.

In the linear marginal cost equation derived from (24) the change in marginal cost per ton of cane processed is constant over all qualities of cane. This implies, as millers' experience would suggest, that the change in marginal processing cost per ton of sugar produced, is lower in periods when high quality cane is available than in periods when only low quality cane is available. If marginal cost had been handled as a linear function of output in the conventional manner this would have implied that the marginal cost of processing a ton of low quality cane was lower than the marginal cost of processing a ton of high quality cane and there is no empirical evidence to support such an assumption.

Because the available data on processing cost and cane crushed were generated over time and the crushing capacities of individual mills were altered by the commissioning of additional plant at various stages, it was necessary to incorporate in the hypothesis set out in equation (24), variables which accounted for these changes. This was done to isolate effectively, for given plant capacities and time periods, the effect of changes in the quantity of cane processed on total processing cost.[11]

Estimation of this extended model, using least squares regression on data available from individual mills, gave the following typical regression equation:

$$(24.1) \quad \hat{C}_t = \underset{(t = 3\cdot074)}{370\cdot11395} + \underset{(t = 2\cdot778)}{30\cdot38596T} + \underset{(t = 1\cdot186)}{2\cdot41217R_t} + \underset{(t = 3\cdot406)}{0\cdot0013154X^2_t}$$

$$(R^2 = 0\cdot973)$$
$$(d = 1\cdot829)$$

where $\hat{C}_t$ = the estimated total cost of processing at the mill in year t, in units of 1,000 dollars, where costs for all seasons have been adjusted to their equivalent 1966 value;
T = a time trend with units of one year, having an origin at 1st October 1946;
R_t = the average number of tons of cane crushed per hour of effective operating time, in year t;
X_t = the total tonnage of cane crushed in year t, in units of 1,000 tons; and
X^2_t = the square of the previous variable.

The variable X_t was omitted from this relationship because of near collinearity between X_t and its square ($r = 0\cdot9825$). When tested with the other variables included in equation (24.1) a *negative* but not significant ($t = -0\cdot419$) regression coefficient was obtained for this variable. The negative sign conflicted with the basic economic theory of the hypo-

[11] For extended discussions of the problems encountered in estimating statistical cost functions the reader is referred to the work of Johnston [3] and Walters [14]. Details of the adjustments required to place the cost series on a constant cost basis over the period considered, are given in another paper at present being prepared for publication by Ryland [6].

thesis set out in equation (24). When X^2_t was excluded and X_t used in its place to derive a relationship similar to equation (24.1) a lower proportion of the variance in cost was explained ($R^2 = 0{\cdot}968$) while the Durbin-Watson statistic ($d = 1{\cdot}590$) gave an inconclusive test at the five per cent level, against positive autocorrelation in the residuals.

These two factors suggest that equation (24.1) more closely approximates the structure of the cost relationship, although an implication of its use is that the intercept of the marginal cost equation used as an estimate of equation (7), is zero. The regression relationships obtained with data from other mills, all of which explained about 97 per cent of the variance in the dependent variable, confirmed these conclusions. In several cases the coefficient of the variable R_t was significant at the five per cent level although it did not reach this level of significance in equation (24.1).

Estimation of Weekly Average C.C.S. Values

The term C.C.S., which is part of the sugar industry's technical jargon, refers to the commercial cane sugar content of cane at the time of processing, as established by laboratory test. The C.C.S. content of cane tends to vary systematically throughout the normal crushing season and a graph plotting weekly average C.C.S. figures [7; p. 32, Fig. 13], indicates that apart from some random variation and weather effects, C.C.S. tends to rise gradually over the first part of the crushing season, to remain fairly steady around its maximum value for some weeks and then to fall off quite rapidly at the end of the season.

To achieve the stated objective of using a minimum amount of cane to produce mill sugar peak, it is first necessary to arrive at typical expected weekly average C.C.S. figures throughout a crushing season. Noting the association between C.C.S. variation and time it was hypothesized that a relationship such as

$$(25) \quad \text{C.C.S.} = \beta_o + \beta_1 T + \beta_2 T^2 + u \qquad (\beta_o, \beta_1 > 0, \beta_2 < 0)$$

where T = time expressed in units of one week, and
u = a random normal variable,

would provide a good approximation to the variation of C.C.S. over the crushing season.

This hypothesis was tested using C.C.S. data from the 1966 season for a number of mills and in all cases the regression coefficients were significant at the one per cent level although the randomness of the deviations from trend was suspect in some instances.[12] A regression relationship typical of those estimated and one which appears to satisfy the usual statistical criteria for least squares regression was

$$(25.1) \quad \text{C.}\hat{\text{C}}\text{.S.} = 11{\cdot}25963 + 0{\cdot}45600T - 0{\cdot}01192T^2. \qquad (R^2 = 0{\cdot}930)$$
$$(t = 46{\cdot}108) \; (t = 11{\cdot}982) \; (t = -9{\cdot}286) \qquad (d = 1{\cdot}795)$$

[12] The incipient autocorrelation which was apparent in some regression relationships was considered to be caused by the exclusion of the relevant variable, weekly rainfall, from the analysis. It could also have been associated with the lack of symmetry in the variation of C.C.S. over the season. This lack of symmetry was judged to be not important to the conclusions of the model but it could have been taken into account if C.C.S. figures representing weekly mean values over a number of mills and seasons, had been used instead of the calculated values derived here.

In this example the exact origin used for the time variable was twelve noon on Wednesday, 25th May 1966. The figures in brackets which are the student's t values for the regression coefficients indicate the very high levels of significance obtained.

Expected weekly average C.C.S. values throughout the 1966 season may be obtained from equation (25.1) by integrating over consecutive time periods from $t - \frac{1}{2}$ to $t + \frac{1}{2}$. Assuming the C.C.S. pattern experienced in 1966, which was a fairly representative year in the Central district, is typical of the likely future experience of the mill, expected weekly average C.C.S. figures for 1969 may be obtained in the same way after adjusting the origin of time to noon on Wednesday, 28th May 1969.

Estimation of Processing Revenues

The No. 1 Pool payout per ton of raw sugar produced by the mill in 1969 was assumed to be \$90.00. This figure was selected as a conservative estimate of anticipated payout but its exact level was not critical to the solution obtained as it was intended to examine the effect of variation in this figure on the optimal solution. Given a particular price of raw sugar the gross revenue earned per ton of cane processed in any given week, t, can be established as

(26) $$P_{r_t} = 0{\cdot}0001(\text{C.W.})(\text{C.C.S.}_t)P_s = \alpha_t P_s$$

where P_{r_t} = the average gross return per ton of cane processed in week t,

P_s = the No. 1 Pool price of raw sugar,

C.W. = the coefficient of work for the mill,[13] and

C.C.S.$_t$ = the average raw sugar content of cane processed during week t.

The average payment to growers from this revenue is based on an agreed formula established by the Central Sugar Cane Prices Board which is

(27) $$P_{c_t} = 0{\cdot}009\,(\text{C.C.S.}_t - 4)P_s + 0{\cdot}3333 = \beta_t P_s + 0{\cdot}3333$$

where P_{c_t} = the average payment per ton of cane processed in week t, and the other terms have the same definitions as in equation (26). The share of the mill in this gross revenue per ton of cane can then be written as

(28) $$P_{r_t} - P_{c_t} = (\alpha_t - \beta_t)P_s - 0{\cdot}3333.$$

Other costs incurred by the mill which must also be deducted from gross processing revenue before a net processing margin can be obtained, are the average cost of transporting a ton of cane to the mill from a supply point and the cost of transporting the raw sugar content of the

[13] This term refers to the extraction efficiency of the manufacturing process as a percentage of the measured raw sugar content of the cane. A figure of 97·5 per cent would represent fairly typical mill performance over a season. This constant figure has been used in the present application although Dr C. R. Murry of the Sugar Research Institute states that some mills, which have rather limited capacity for purifying and drying the juice extracted from the cane, tend to show a negative correlation between C.W. and C.C.S. figures over a crushing season. This could be handled in the present model without any inherent difficulty if an appropriate series of weekly C.W. figures was available.

cane to the local bulk terminal.[14] After analysing the accounting records of a number of mills a figure of 90 cents per ton was selected for the weighted average cane transportation cost while a figure of \$2.50 per ton, which represented something of a compromise between the costs incurred by mills close to a port and those incurred by distant mills, was selected for the cost of transporting raw sugar. In both cases it was planned to examine the effect on the optimal solution of variation in the figure used.

The average level of these non-processing costs incurred by the mill in week t may be defined as

$$(29)\quad P_{\underset{t}{n}} = T_c + 0{\cdot}0001(\text{C.W.})(\text{C.C.S.}_t)(\text{N.T.})T_s = T_c + \gamma_t T_s$$

where $P_{\underset{t}{n}}$ = the average transportation cost incurred per ton of cane processed in week t,

T_c = the average cost of transporting a ton of cane to the mill,

T_s = the average cost incurred in moving a ton of raw sugar from the mill to the bulk terminal, and

N.T. = a ratio indicating the quantity of raw sugar of the average quality produced, which is equivalent to a ton of standard (94 Net Titre) raw sugar.[15]

Discussion of Results

Although the problem as set out in Table 2 exhibits a rather extreme form of semi-definiteness, little difficulty was experienced in obtaining a unique solution because all the elements of the vector G'_5, which determines marginal revenue from processing in each time period, were unique. A total of 28 weekly time periods were included in the model estimated for the 1969 season, beginning from 9th June 1969 and finishing on 19th December 1969. With 63 rows in the simplex tableau a solution to the problem was obtained, after 118 iterations, in less than eight minutes of actual computing time.[16]

The crushing period suggested as optimal in the solution given in Table 3 requires operations to commence on Monday August 4th with crushing being terminated on Friday December 12th. This may be compared with the actual situation in 1968 when most mills in the district commenced crushing before the middle of June and completed their season, which involved the crushing of a large quantity of over-peak cane, by late November. The difference between this production pattern and that given in the solution to the model suggests that the choice of an

[14] For the immediate purpose of the present model, only the weighted average cane transportation cost need be considered since a miller is obliged to accept cane from all his supply points in proportion to the cane assignments held by farmers loading at each supply point.

[15] As the raw sugar currently produced is approximately 97 Net Titre quality, something less than one ton of this sugar is equivalent to the ton of 94 Net Titre sugar which is the basis for Pool payments. A factor of 0·97 was used in the model. The term '94 Net Titre' may be interpreted to mean that approximately 94 tons of refined sugar could be produced from 100 tons of raw sugar of this quality. Similarly 97 tons of refined sugar could be produced from 100 tons of raw sugar of 97 Net Titre quality.

[16] The problem was run on the I.B.M. 360/50 computer at the University of New South Wales using a modified version of 'QP7', a programme specially developed by Takayama and Caine, at the University of Illinois, for the solution of spatio-temporal quadratic programming problems.

TABLE 3

Solution Obtained for an Optimum Crushing Period[a b]

$P_1 = 1\cdot37282$	$X_9 = 27433\cdot2$	$V_1 = 0\cdot10910$
	$X_{10} = 27868\cdot9$	$V_2 = 0\cdot08997$
$P_{2.09} = 0\cdot01049$	$X_{11} = 28229\cdot5$	$V_3 = 0\cdot07203$
$P_{2.10} = 0\cdot02006$	$X_{12} = 28515\cdot0$	$V_4 = 0\cdot05529$
$P_{2.11} = 0\cdot02844$	$X_{13} = 28725\cdot4$	$V_5 = 0\cdot03975$
$P_{2.12} = 0\cdot03562$	$X_{14} = 28860\cdot6$	$V_6 = 0\cdot02540$
$P_{2.12} = 0\cdot03562$	$X_{15} = 28920\cdot8$	$V_7 = 0\cdot01224$
$P_{2.14} = 0\cdot04640$	$X_{16} = 28905\cdot9$	$V_8 = 0\cdot00028$
$P_{2.15} = 0\cdot05000$	$X_{17} = 28815\cdot9$	
$P_{2.16} = 0\cdot05240$	$X_{18} = 28648\cdot7$	$V_{28} = 0\cdot01193$
$P_{2.17} = 0\cdot05361$	$X_{19} = 28412\cdot6$	
$P_{2.18} = 0\cdot05363$	$X_{20} = 28095\cdot2$	$Y_{2.1} = 21244\cdot0$
$P_{2.19} = 0\cdot05245$	$X_{21} = 27704\cdot8$	$Y_{2.2} = 22280\cdot5$
$P_{2.20} = 0\cdot05007$	$X_{22} = 27239\cdot2$	$Y_{2.3} = 23241\cdot9$
$P_{2.21} = 0\cdot04650$	$X_{23} = 26698\cdot6$	$Y_{2.4} = 24128\cdot2$
$P_{2.22} = 0\cdot04174$	$X_{24} = 26082\cdot9$	$Y_{2.5} = 24939\cdot4$
$P_{2.23} = 0\cdot03578$	$X_{25} = 25392\cdot0$	$Y_{2.6} = 25675\cdot5$
$P_{2.24} = 0\cdot02863$	$X_{26} = 24626\cdot1$	$Y_{2.7} = 26336\cdot5$
$P_{2.25} = 0\cdot02028$	$X_{27} = 22649\cdot1$	$Y_{2.8} = 26922\cdot4$
$P_{2.26} = 0\cdot01074$		
	$W_3 = 75660\cdot0$	$Y_{2.27} = 1136\cdot0$
$P_4 = 34\cdot40169$		$Y_{2.28} = 22868\cdot9$
$P_5 = 90\cdot00000$	$U_1 = 24786\cdot0$	
$P_6 = 2\cdot50000$		$Y_3 = 78175\cdot5$

[a] All vector elements omitted took the value zero.
[b] The P and V elements are in units of one dollar, the X and Y elements in units of one ton of cane, and the W and U elements in units of one ton of raw sugar.

optimum production period is of little real practical importance to mill management.

An examination of the solution vector P_2 which gives the intra-marginal value of cane processed in each time period (i.e. the cost of the constraint on crushing capacity in each period) together with the solution vector V, should quickly dispel this idea. If only sugar peak was to be produced and the production period was moved back in time to commence from June 11th instead of August 4th, this would involve a marginal reduction in revenue of 15·6 cents per ton of cane produced ($V_1 + P_{2.21}$), as well as an additional 5·1 cents per ton increase in marginal processing cost, because of the increased quantity of cane to be crushed to achieve sugar peak.[17]

The value of \$34.40 obtained for P_4 in the solution is of considerable economic interest since, as indicated in the definitions, this is the marginal value to the mill of an additional allocation of one ton of sugar-peak, given the assumed No. 1 Pool price of \$90.00. The difference between these two figures which is \$55.60 may be interpreted as the minimum price at which it would be even marginally profitable for the mill to produce No. 2 Pool sugar. This figure suggests that substantial marginal

[17] The marginal cost figures obtained from the solution indicate that a normal mid-week commencing date for crushing such as July 30th could be adopted at a marginal cost in terms of revenue foregone of only 0·03 cents per ton of cane crushed (V_8). If crushing commenced on July 2nd the marginal revenue foregone by the deviation from the optimal solution would rise to 9·1 cents per ton ($V_4 + P_{2.23}$) and marginal processing cost would rise by approximately 1·9 cents because of the additional cane crushed. Crushing would terminate about November 13th.

losses would have been incurred by mills which processed over-peak sugar in the last four seasons.

While the value obtained for P_4 indicates that the mill could profitably produce more sugar at the No. 1 Pool price, were it allowed to do so, no inferences about overall mill profitability can be drawn from this figure because of the high level of overhead costs involved. An *ad hoc* analysis of the solution suggests that after meeting transport costs the return to the miller for processing would average \$3.01 per ton of cane, and that this figure would change by 4·75 cents per ton of cane, for each dollar change in the No. 1 Pool price. While this return would provide an operating profit in conventional financial accounting terms, it is still well below the return necessary to pay market rates of interest on the investment involved, if allowance is made for price level changes over time.[18]

The effect on the average processing margin per ton of cane, of varying raw sugar transport costs, may be calculated from the solution as $\pm 14{\cdot}46$ cents per ton of cane, per dollar change in the freight rate. More interesting perhaps is the effect of changes in transport costs on the break-even price for No. 2 Pool sugar. This may be calculated from the coefficients of the marginal effective price constraint (i.e. that for Week 27) in the model. For a change in raw sugar transport cost, evaluation of the ratio $\gamma_i/(\alpha_i - \beta_i)$ suggests a 29 cent change in the break-even price for a 10 cent change in cost, while for a change in average cane transport cost, evaluation of the ratio $1/(\alpha_i - \beta_i)$ suggests a 32 cent change in the break-even price for a 1·5 cent change in cost.[19]

Summary

In this paper a model has been developed which may be used to determine an optimum crushing period for a sugar mill, assuming mill management is concerned with economic efficiency. Given the increasing awareness in the sugar industry of the potential value of computers and the availability of the necessary data in existing mill records, application of the model to individual mill situations could readily be made, as a guide to management.[20]

While the specific results derived in the present application have limited relevance outside the Central District, they do serve to point up the considerable importance of a correct choice of date for commencing crushing in determining the profitability of mill operations. They also suggest that the break-even price required to make the production of over-peak sugar a profitable venture for the mill is considerably higher than the prices which have prevailed for No. 2 Pool sugar in recent years.

It should be remembered however that the results which have been generated are based on coefficients which either depend upon individual

[18] Such publications as those by Edwards and Bell [1] and Gynther [2] provide a detailed discussion of basic concepts in the relatively new field of accounting for price level changes.

[19] Cost changes of these relative magnitudes would have approximately equal effects on the total costs incurred by the mill. The potential benefits of consolidating cane assignments around a mill and the actual benefits of the recent reduction in raw sugar freight rates, for those mills using rail transport, could be analysed in this fashion.

[20] Two papers relating to the use of computers in the industry were presented at the 1968 Conference of the Queensland Society of Sugar Cane Technologists.

judgement (i.e. the No. 1 Pool price assumed) or are statistical estimates of underlying parameters. The inter-seasonal variance which may occur in C.C.S. patterns for example, is well known [7].[21] This inherent element of variability needs to be recognized whenever tentative policy conclusions are drawn from the solution.

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[21] A reduction of five per cent in the average level of C.C.S. figures used in the model would lengthen the crushing season by six or seven working days. Even allowing for such uncertainties, if only mill peak was to be produced, a Central District miller who began crushing before mid-July would seem to be acting contrary to his own economic interests.