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A QUEUEING MODEL FOR EGG PRICE DETERMINATION

S. M. STRONG and A. M. WOLANOWSKI

University of Queensland, St Lucia, 4067, and Queensland Institute of Technology, Brisbane, Qld. 4001

The determination of the price paid to its suppliers and by its customers is a major task for some marketing authorities. The commodity arrives randomly at the authority's facility and is removed randomly by customers. Between arrival and departure, the commodity awaits processing, is processed (graded, packed), and awaits removal by a customer. It is suggested that this similarity to a queue enables a profit function, dependent on price, to be constructed. Determination of the price maximising this function is seen to be one solution to the price setting problem.

The Problem

Usually in response to seasonal factors, egg marketing authorities in Australia vary egg prices paid to growers and by retailers. Frequently, these price changes are accompanied by adverse publicity. The egg marketing authority in south-east Queensland has not escaped such publicity. The need to balance supply and demand is invoked as the motivation for these price changes. Indeed, an increasing unwillingness to tolerate over-production by supplier members of the authority has been observed.

The Egg Marketing Board (South Queensland) lists two objectives of its operations: 'to ensure a fair return to the producers and to maintain an adequate supply of first quality eggs at reasonable prices to consumers' (Cashel 1981). By concentrating upon the first of these objectives, and making the objective operational by replacing the ambiguous concept of a 'fair' return with that of a maximum return, headway can be made in deriving a method for arriving at the wholesale price to be set by the board. The incorporation of a measure of consumer welfare to balance that of the producers was considered, but not included because it would not have contributed to the essential methodology.

Costs of farm production, transportation and other handling costs, as well as seasonality in production and consumption, are all factors involved in the price setting decision. Without a storage facility, care must be taken to avoid setting prices which would result in large amounts of stock being carried over and lost. The introduction of a storage facility would increase the range of prices the board could charge. If stock behaviour through time in response to price changes were known, the storage facility could be used as a management variable, rather than as a mere insurance against random carryover problems.

A pervasive factor in the price setting problem, and the one analysed here, is the random environment confronting the authority. Its supplies and sales are both random in nature, and it is the difficulty imposed by such randomness that is the specific focus of the method proposed here. Organised markets other than the egg market are characterised with similar randomness, and the treatment of this problem given here could be applied more or less appropriately in these situations. For example, fishing trawlers are often organised into a fleet operating from a single co-operative facility. The trawlers often arrive randomly through time at the plant, where the catch is processed and then sold in an uncertain market.

The Queueing Theory Suggestion

The egg handling authority in south-east Queensland is responsible for the distribution of whole eggs from its member growers to consumers in the region. Growers are obliged to deliver their eggs to the central processing plant, where they are graded, packed and sold to retailers. Deliveries of eggs occur randomly through time; it is postulated that these arrivals obey a Poisson process with λ being the average number of arrivals per unit of time. This process implies that, during any arbitrarily small unit of time, more than one unit cannot arrive and that the time between single unit arrivals obeys the negative-exponential distribution with mean $1/\lambda$. The validity of this supply characterisation is reviewed later.

Upon arrival at the plant, a unit of eggs is either immediately processed and proceeds to await removal by a retailer, or it awaits processing. Suppose that the time taken to process and then sell a unit of eggs is also represented by the negative-exponential distribution, this time with mean $1/\mu$ where μ is the corresponding average number that is processed and sold in a Poisson manner per unit of time. If such a queue discipline operates for a sufficient length of time, the expected number of egg units in the system (waiting to be processed, being processed and awaiting sale), $E(Q)$, is given by:

$$(1) \qquad E(Q) = \lambda/(\mu - \lambda) \text{ for } \mu > \lambda.$$

If $\mu < \lambda$, $E(Q)$ approaches infinity. Equation (1) requires that the system operate uninterrupted for a long period. In reality, nights and weekends interrupt the system and it is suggested, therefore, that the basic unit of time be long enough, say a month or a quarter, that the day-night, week-weekend interruptions are insignificant. That is, the difference between an arrival on a Sunday or a Monday, say, is of little importance when viewed against the large basic unit of time.

The two parameters λ and μ depend on economic forces. Egg producers respond to egg prices and decide how much production should be undertaken per unit of time. At the aggregate level, this behaviour is characterised by letting λ depend on the commodity price, $\lambda = \lambda(p)$, with the functional form assumed to be independent of time.

On the demand side, the price level enters through the term μ, which was previously defined to include processing time. If it is assumed that the processing time is very short, then $\mu = \mu(p)$ is the expected number of units which customers *attempt* to remove from this facility per unit of time. Hence, the time which a unit of eggs waits after processing until an attempt is made to purchase it can be assumed to have a mean of $1/\mu(p)$.

If the cost of storing a unit of eggs for the basic unit of time is s and c is the average cost of producing an egg unit, including all farm and board costs, then the board's profit can be written as:

(2) $$\pi(p) = p.\lambda(p) - s.\lambda(p)/(\mu(p) - \lambda(p)) - c.\lambda(p).$$

The term $p.\lambda(p)$ is the expected total revenue per unit of time. In a steady state, the average number of departures from the system must equal the average number of arrivals. The assumption that the board is engaged primarily in the maximising of its members' profits is emphasised. Provided that $\mu(p)$ and $\lambda(p)$ are differentiable, and that there exists a p such that $\mu(p) > \lambda(p)$, calculus methods can be used to maximise $\pi(p)$ over p. If there is no such p, a situation of permanent glut exists, and the marketing authority would have to change its operations markedly.

Suppose that λ is constant at λ_0, and that:

(3) $$\mu(p) = \alpha - \beta p,$$

where α and β are positive constants.

Then the profit maximising price, $\bar{p}$, is:

(4) $$\bar{p} = (\alpha - \lambda_0)/\beta \pm (s/\beta)^{1/2}.$$

The second order condition requires that the profit maximising value of p be such that $\alpha - \beta p - \lambda$ is greater than zero. The simplicity of this result depends upon producers adhering to some historically determined value, λ_0, and not adjusting their output in response to $\bar{p}$. This is likely in the short run, but as a long-run steady state result is incorporated in equation (2), it is more desirable to view λ as a function of price.

Refinements and Limitations

Some reservations may be expressed about this model. They shall be discussed presently, but it is appropriate that they be examined as they apply to a model more general than that just demonstrated.

Let $A(t)$ be the statistical distribution of the time between single unit arrivals at the plant and $B(t)$ the distribution of the time between consecutive attempts by consumers to purchase a unit of the commodity. It can be shown (Moder and Elmaghraby 1978, p.365) that for these arbitrary distributions, the average time a unit waits to enter the service (selling) stage, $E(W)$, has an upper limit:

(5) $$E(W) \leq \frac{\text{inter-arrival time variance} + \text{service time variance}}{2\ (\text{mean inter-arrival time})(1 - \varrho)}.$$

where $$\varrho = \frac{\text{mean arrival rate}}{\text{mean service rate}},\ \varrho < 1.$$

This powerful result, attributable to Kingman (1962), can be combined with the so-called 'Little's formula' (Moder and Elmaghraby 1978), expressed as:

(6) $$E(Q) = \lambda.E(W) + \lambda.E(V),$$

where $E(Q)$ = expected number of eggs in the system;
$E(V)$ = the expected service time distribution; and
λ = the mean arrival rate per unit of time associated with $A(t)$,

to produce a more general profit function than that shown in equation (2). This function, containing inter-arrival time function parameters and service time parameters all dependent on price, can be maximised over p to obtain a 'maximum' profit. That is, the profit function, now appearing as:

$$\theta(p) = p.\lambda(p) - s.E(Q(p)) - c.\lambda(p), \tag{7}$$

with s and c defined above, can be maximised over p to obtain a level of profit which can at least be obtained. This maximisation process utilises the fact that, in the steady state, the average number of arrivals and actual departures are equal per unit of time. The price dependent parameters characterising the function $A(t)$ and $B(t)$ are amenable to estimation by observing the way the commodity arrives at, and is attempted to be removed from, this facility for varying price levels. This estimation problem is not a major shortcoming of the method; the description of the arrival and service (attempted purchases) times as random time passages between *unitary* arrivals and attempted sales is of greater concern.

Some results from the theory of bulk queues are available. A bulk queue is characterised by either random non-unitary arrivals, or service, through time. A number of such results are presented in Saaty (1961, Ch. 7). Our suggestion is that, rather than attempting to incorporate these complicated results into a profit function, an attempt be made to interpret the arbitrary functions $A(t)$ and $B(t)$ in such a way as to make equation (7) defensible.

Function $A(t)$ is the inter-arrival time distribution. These inter-arrival times must be independent of each other for equation (5) to be established, and so any interpretation of $A(t)$ must take this independence into account. A bulk arrival could be viewed as a sequence of unitary arrivals separated by small random intervals of time. The next bulk arrival, following a sequence of longer random time periods, could be viewed in the same way. Hence, bulk arrival behaviour is seen as resulting from sequences of 'short' random time periods alternating with sequences of 'long' random time periods. This temporal dependence does conflict with the theoretical independence requirement and is identified as the main shortcoming of this model.

Nevertheless, it is proposed that the method suggested here is appropriate, in that it contains an explicit recognition of the fact that supplies and demands occur randomly through time and not merely in random amounts at given points of time. A queueing theory approach does appear proper, despite the appeal of other approaches, such as the theory of dams developed by Moran (1961) and Markov process models.

An Application to the Egg Market

The south-east Queensland egg marketing authority does not engage in storage as part of its marketing operations. It would, therefore, be relevant to apply the method shown here to evaluate the economic gains from the introduction of a storage facility large enough to accommodate the maximum likely stock levels associated with their operations.

Eggs are sold in three grades and demand functions for two of these grades, for a five-week period in April-May 1977, were estimated. These functions represent the weekly consumption of medium and small eggs bought from the Brisbane facility of Sunny Queen Eggs. They relate consumption to egg prices and a meat price index as follows (standard errors in parentheses):

$$\mu_m = \underset{(18.67)}{39.448} + \underset{(0.835)}{1.26p_l} \; - \underset{(0.933)}{1.12p_m} \; - \underset{(2.361)}{2.37p_{mt}}$$

$$\mu_s = \underset{(16.846)}{25.100} + \underset{(0.255)}{0.737p_m} - \underset{(0.252)}{0.749p_s} - \underset{(2.362)}{1.096p_{mt}}$$

where μ_m = medium-egg weekly consumption;
μ_s = small-egg weekly consumption;
p_l, p_m and p_s = prices of large, medium and small eggs; and
p_{mt} = a meat price index.

The method of estimation was a two-stage least squares approach suggested by a series of articles on seasonal estimation procedures by Jorgensen (1967) and Lovell (1963, 1966).

It is observed that the left-hand side sales variables are actual physical sales per week and, as such, do not necessarily represent the *desired* levels of sales required in the model. The data needed to estimate the relationship between desired consumption and price were not collected. This collection would require detailed observations of the timing and magnitude of orders placed, not merely the level of sales per week, and the corresponding price levels. If a storage facility were in place, the average number of units waiting in the system, which depends upon arrival and desired sales rates, could be observed and used to estimate the unknown desired sales rate. However, the discrepancy between orders placed and actual sales can defensibly be assumed to be small, thereby making this qualification a minor one.[1]

The advantages of introducing a storage facility are now studied. The constant mean arrival rates per week of the medium and small grade eggs were derived by taking the mean of the five weeks' deliveries prior to the data generation period, April-May, 1977. A storage cost of 10 cents per dozen per week was chosen, and empirical evidence indicated a production cost of 61.91 cents a dozen.[2]

In summary, $\lambda_m = 12.235$, $\lambda_s = 11.733$, $c = 61.91$, $s = 10.0$, and p_l was set at 102.0 cents, and p_{mt} at 19.55, their values in April 1977. These values are all contained in an expanded version of equation (2). In order to simplify matters in this illustration, and without essentially affecting the simplicity of the technique, it was decided to study the pricing of medium grade eggs separately; this enables the profit function to be maximised over p_m alone. Hence:

$$\mu_m(p_m) = 121.63 - 1.12p_m \text{ and}$$

$$\pi(p_m) = 12.235p_m - \frac{122.35}{109.40 - 1.12p_m} - 757.469.$$

The maximum of $\pi(p_m)$ occurs at $p_m = 94.7$ cents per dozen. This price

[1] This is not the same as saying that the desired sales rate is equal to the observed arrival rate per week, which equality would render the model inoperative, as ϱ must be less than 1.

[2] This is an estimate provided from the work of the Queensland Department of Primary Industries, carried out by Mr N. Byrne.

would create an average level of accumulation of 3.66 dozen units, and a desired average consumption level of 15.57 dozen units per week. This price is lower than that actually set at that time, indicating that increases in sales from the proposed price reductions would have more than compensated for the introduction of storage costs.

Conclusions

The design of queueing systems has received much attention in the operations research literature despite the widespread criticism that the assumptions involved in most queueing models are rarely matched exactly by reality. The model proposed here is not free of this criticism, as discussed above, and so must be applied with caution. Nevertheless, it is commended by its simplicity, in which a single, easily identified and controlled input variable, the price, can be manipulated to vary profit levels. The incorporation of the price variable adds an economic flavour to the queueing model, which has been traditionally used with fixed (or variable with respect to time) parameters.

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