STATIONARY-STATE SOLUTIONS IN MULTI-PERIOD LINEAR PROGRAMMING PROBLEMS

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Multi-period linear programming models are classified initially. The main interest of this paper is in models of cyclical activity. The conditions under which such models lead to stable repetitive optimal solutions are defined. A simple technique is presented by which this stable solution (termed here the "stable core") can be derived without recourse to solving a full extended matrix. Firm-level applications of this model in agriculture are discussed.

1. *Introduction*

This paper deals with those dynamic linear programming (DLP) models in which all elements have subscripts attached, in conventional Hicksian fashion, referring them to discretely defined time periods. The constraint matrices in such models characteristically fall into block-diagonal or block-triangular format.[1] We are concerned with solutions covering the whole of the time horizon studied, as distinct from those decision models in which only the time period nearest to the present is considered relevant for action.[2] Further, we are dealing with models in which no "target" values of variables are specified; this is to distinguish our area of study from, for example, models of accumulation which seek to optimize the time-path of some system from a given initial state to a given final state.[3]

Considering DLP allocation models with block-diagonal constraint matrices, we may classify two categories with respect to each of two criteria. On formal grounds we may categorize these models according to whether or not interaction occurs in their technical structure between activities in period t and resources in period $t + k$ (any $k > 0$), i.e.

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[1] See Dantzig, G. B. *On the status of multi-stage linear programming problems* (RAND P-1028), 1957. Loftsgard, L. D., and Heady, E. O. Application of dynamic programming models for optimum farm and home plans. *J. Farm. Econ.*, 41: 51-62, 1959. Candler, W. Reflections on "dynamic programming models". *Ibid.*, 42. 920-6, 1960. Throsby, C. D. Some notes on "dynamic" linear programming. *Rev. Mark. Ag. Econ.* 30: 119-41, 1962.

[2] As, for instance, considered by Theil, H. A note on certainty equivalence in dynamic planning. *Econometrica* 25: 346-9, 1957.

[3] Such as, for example, the "turnpike" models reviewed by Koopmans, T. C., Economic growth at a maximal rate, *Q. J. Econ.* 78: 355-94, 1964, of which a particular linear programming formulation can be seen in Morishima, M., Proof of a turnpike theorem: the "no-joint-product" case, *Rev. Ec. Studs.* 28 (2): 89-97, 1961. Accumulation models are also treated in R. Dorfman *et al.*, *Linear Programming and Economic Analysis*. N.Y.: McGraw-Hill, 1958, Ch. 12.

according to whether any nonzero coefficients appear in below-diagonal submatrices of the overall constraint matrix. Then, empirically, we may distinguish those models in which technical coefficients differ between periods or groups of periods from those where a cyclically repetitive pattern of numerical elements is assumed to exist. Let us look at problems of solution in each of the four sub-groups.

When no intertemporal interaction between rows and columns exists, the overall constraint matrix will appear as a series of entirely separate diagonal blocks. The overall optimal solution, then, is the totality of the independent suboptima determined by solving each individual block with its associated price (c) and resource endowment (b) subvectors. In the case of cyclical coefficients with a cycle of length r periods, only r sub-problems need to be solved. The simplest case is when the cycle is of one period's duration; here all diagonal submatrices are identical and a solution to only one sub-problem is required. This disposes of two of our sub-groups. Next, when there is interperiod interaction and no repetitive numerical pattern, the only alternative to solving the overall matrix as one "ordinary" linear programming problem is to use one of the short-cut procedures, such as a decomposition algorithm which operates on these specially structured matrices.[4] The final sub-group, in which inter-period interaction and cyclically repetitive coefficients both appear is by no means uncommon, for two reasons. Firstly, in dynamic economic models of production it is unusual to find absolute independence of variables over time unless the model segments time into periods which are relatively very widely spaced. Agriculture provides many examples of situations where, in a multi-stage model, one production process would influence resource use in more than one time period. Secondly, the nature of data availability, particularly for normative planning models in agriculture, frequently leaves no alternative to the assumption that the one given set of unique technical coefficients will obtain at each production cycle, or, more precisely, that variations in coefficients between cycles cannot be forecast or meaningfully hypothesized from the available information.

It is this last important sub-group of DLP models with which we shall be concerned from now on.

2. *Stationary-state Solutions*

Consider a production process whose operation can be divided into regular time cycles. Call one cycle a "period" and denote it by the subscript t. Call each discrete time division within a cycle a "subperiod", denoted by the subscript k. (Subperiods need not be of equal length.) Take as typical of the DLP systems under study the following problem in which the cycle repeats after two subperiods. The argument will subsequently be generalized to cover cycles of any number of subperiods. Define the decision variables as the vectors x_{kt}, ($k = 1, 2$; $t = 1, 2, \ldots, T$). Then using conventional linear programming definitions and notation (transposition signs omitted) we are considering the allocation problem of:

Maximize

$$(1) \qquad z = c_1x_{11} + c_2x_{21} + c_1x_{12} + c_2x_{22} + c_1x_{13} + \cdots$$

[4] Dantzig, G. B., and Wolfe, P. The decomposition algorithm for linear programs. *Econometrica* 29: 767-78, 1961.

subject to

$$
(2)\quad
\begin{array}{llllll}
b_1 \geqslant A_1x_{11} & & & & & \\
b_2 \geqslant A_2x_{11} + A_3x_{21} & & & & & \\
b_1 \geqslant & A_4x_{21} + A_1x_{12} & & & & \\
b_2 \geqslant & & A_2x_{12} + A_3x_{22} & & & \\
b_1 \geqslant & & & A_4x_{22} + A_1x_{13} & & \\
b_2 \geqslant & & & & A_2x_{13} \ldots & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & &
\end{array}
$$

and

$$(3)\qquad x_{kt} \geqslant 0$$

where b_1, b_2, c_1, c_2, are m-, p-, n-, and q-vectors respectively, and A_1, A_2, A_3 and A_4 are $(m \times n)$, $(p \times n)$, $(p \times q)$, and $(m \times q)$ matrices respectively.

Some properties of the optimal solution to the above system can be listed assuming firstly that one exists, secondly that the optimal solution is not infinite, and thirdly that overall capacity constraint(s) are present in the matrix to prevent the optimal x_{kt} from increasing indefinitely with indefinite increase in t. Three phases in any numerical solution to this system can be classified:

(a) an initial phase, in which the system is "getting off the ground";

(b) a stable central phase, in which the optimal solution is the same at each cycle;

(c) a terminal phase, in which the system "closes down", marked for instance, by the rundown of inventories.

Of these it is the solution at the stable phase which is of most interest, since, *under the given production conditions* this represents the repetitive activity pattern which would maximize the objective of the productive system over time.

The optimal values of the decision variables at this stable phase may be determined by extending a given matrix for sufficient periods such that one can be sure the stationary-state solution will emerge. It is not hard to see that this might be extremely cumbersome for any real problem whose submatrices are of even quite modest dimensions. Below we outline a means for rearranging a DLP matrix such that this stationary-state solution may be easily determined. Let us return to the example in (1), (2) and (3) above.

Let $T = \infty$ and suppose that in the optimal infinite-stage solution a stationary-state is reached at period t^* (where t^* is finite). At the optimum

$$(4)\qquad X_{kt^*} = X_{k,t^*+1} = X_{k,t^*+2} = \ldots$$

Looking at the constraint matrix around period $t^* + 1$ we can isolate

$$
(5)\qquad
\begin{array}{l}
b_1 \geqslant A_4x_{2t^*} + A_1x_{1,t^*+1} \\
b_2 \geqslant \qquad\qquad A_2x_{1,t^*+1} + A_3x_{2,t^*+1}.
\end{array}
$$

Now since in the optimal solution $x_{2t^*} = x_{2,t^*+1}$ from (4), we can equivalently set up the constraint set (5) as:

(6) $$b_1 \geqslant A_1 x_{1,t^*+1} + A_4 x_{2,t^*+1}$$
$$b_2 \geqslant A_2 x_{1,t^*+1} + A_3 x_{2,t^*+1}.$$

If this were done for periods $t^* + 1, \ldots, \infty$, we would obtain an overall A matrix consisting of a series of independent diagonal sub-matrices of order $(m + p) \times (n + q)$:

$$\begin{bmatrix} A_1 & A_4 \\ A_2 & A_3 \end{bmatrix}$$

and corresponding to each block would be identical $[b_1, b_2]$ and $[c_1, c_2]$ vectors. As we have noted earlier the optimal solution to such a system is a repetitive application of the optimal solution to any one diagonal block with its associated price and resource endowment vectors.

This result is easily generalizable to the case of cycles with many sub-periods. With the submatrices A_{kk} indexed by resource and activity sub-periods respectively ($k = 1, 2, \ldots, K$), and with b_k and c_k as m_k- and n_k-vectors respectively, the reduced problem to determine the stationary-state optimum is to find the set of K n_k-vectors x_k which maximizes

(7) $$z = c_k x_k$$

subject to

(8) $$b_k \geqslant \begin{bmatrix} A_{11} & A_{K+1,2} & A_{K+1,3} & \cdots & A_{K+1,K} \\ A_{21} & A_{22} & A_{K+2,3} & \cdots & A_{K+2,K} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{K+3,K} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ A_{K1} & A_{K2} & A_{K3} & \cdots & A_{KK} \end{bmatrix} x_k$$

and

(9) $$x_k \geqslant 0.$$

To form the reduced A matrix in (8) we partition off columns $k = 1, \ldots, K$ and rows $k = 1, \ldots, 2K$ of the extended matrix, and superimpose the lower half (i.e. rows $k = K + 1, \ldots, 2K$) on the upper (i.e. rows $k = 1, \ldots, K$).

This procedure provides a compact means of representing the essential components of a cyclical DLP system with inter-period interaction such that the stationary-state optimal solution can be found without recourse to solving an extensive matrix. We call this solution the "stable core".

Let us look briefly at the case where a resource cycle and an activity cycle are of different lengths. If the number of subperiods p in the longer cycle is an exact multiple of that in the smaller, q, then no problems are raised in applying the above reasoning; the cycle for analysis is taken as the number of subperiods in the longer. On the other hand if resource and activity cycles are out of phase, i.e. if p/q is not an integer, the stationary-state optimal solution will oscillate and the only way in which the above technique could be applied would be by considering an overall cycle consisting of r subperiods, where r is the lowest common denominator of p and q.

Most empirical studies using DLP apply discounting to the coefficients of the objective function. Under the above conditions this will not affect the optimal values of the decision variables at the stable phase (although in solving the extended matrix it might affect the time period at which the stable phase begins). Thus the stable core model is independent of the discount rate, except in the unlikely case where for some reason different discount rates are required for different activities.

3. *Application of the Stable Core Model in Agriculture*

We confine ourselves here to firm-level applications, although one might envisage the use of this sort of technique with macro-models now that DLP is finding a place in aggregate analysis.[5]

The periodic nature of agricultural production indicates that for many problems in farm planning, for instance, the use of DLP is likely to be more rewarding than conventional static programming.[6] Further, as noted earlier the cyclical pattern of production and the typical characteristics of data availability suggest that analyses such as the stable core should have some application. One way in which dynamic planning models have been formulated has been to identify one single farm-planning decision each year. In models with a long horizon involving perhaps some activities extending over a number of years (e.g. vines, orchards, some livestock enterprises, etc.) this approach seems reasonable. But a more realistic simulation of the farmer's decision-making environment when the production cycle is an annual one (the most common situation) is provided by the definition of *intra-year* "decision moments" corresponding for any activity to the latest stage at which a decision must be made as to the level of that activity in the forthcoming year. The disaggregation of the overall cycle into sub-periods is then based on the technical characteristics of production of the individual enterprises, and each activity in the matrix is dated by its initial decision moment.

A DLP model formulated with irregular intra-year decision moments can be reduced to manageable size by means of the above techniques. For instance the present author[7] has constructed a six-enterprise model using six unequal intra-year time periods, the beginning of four of which were decision moments; the DLP matrix for this situation could not have been solved in extended form for a horizon of more than about three cycles. When reduced to the stable core formulation a 32×30 matrix was obtained.

The determination of optimal rotation sequences provides a most promising use for this sort of model. To illustrate, suppose a farmer has three feasible rotations, R_1, R_2 and R_3, where R_1 is a two-year rotation, R_2 is one of three years and R_3 one of four years. Assuming

[5] As, for example, in Day, R. H. *Recursive Programming and Production Response*. Amsterdam: Nth. Holland, 1963.

[6] Yet the development of appropriate models has until recently been much neglected. As Burt remarked in 1965, "it is surprising that there have been so few applications of linear programming to temporal resource allocation". See Burt, O. R. Operations research techniques in farm management: potential contribution. *J. Farm Econ.* 47: 1419, 1965.

[7] Throsby, C. D. *Dynamic and stochastic optimising models for farm decision-making*. Univ. of London, London School of Economics and Political Science, unpub. Ph.D. thesis, 1966.

that rotations are measured in acre units and writing down only the land constraints, the *A* matrix for the extended problem will appear as in Table 1. Here we have a four-year resource-use cycle for the longest rotation and a one-year activity cycle (since new rotational sequences may be started in any year). The cycle for analysis is therefore taken as being of four years' duration, for which the stable core land-use matrix is shown in Table 2. Constraints for other resources would be rearranged in similar fashion.

TABLE 1

Rotation Example: Extended Land Constraint Matrix

	Year 1			Year 2			Year 3			Year 4			Year 5			etc.
	R_1	R_2	R_3	R_1	R_2	R_3	R_1	R_2	R_3	R_1	R_2	R_3	R_1	R_2	R_3	
Year 1 land	1	1	1													
Year 2 land	1	1	1	1	1	1										
Year 3 land	0	1	1	1	1	1	1	1	1							
Year 4 land	0	0	1	0	1	1	1	1	1	1	1	1				
Year 5 land				0	0	1	0	1	1	1	1	1	1	1	1	
Year 6 land							0	0	1	0	1	1	1	1	1	
Year 7 land										0	0	1	0	1	1	
Year 8 land													0	0	1	
etc.																etc.

TABLE 2

Rotation Example: Compact Land Constraint Matrix

	Year 1			Year 2			Year 3			Year 4		
	R_1	R_2	R_3	R_1	R_2	R_3	R_1	R_2	R_3	R_1	R_2	R_3
Year 1 land	1	1	1	0	0	1	0	1	1	1	1	1
Year 2 land	1	1	1	1	1	1	0	0	1	0	1	1
Year 3 land	0	1	1	1	1	1	1	1	1	0	0	1
Year 4 land	0	0	1	0	1	1	1	1	1	1	1	1

With careful formulation, a stable core matrix for a farm planning problem will yield an optimal solution which is entirely internally consistent. For instance the supplies of and demands for cash and credit can be simulated in a dynamically realistic fashion by means of standard DLP techniques together with transfer processes. When variable capital available or produced in a given period has some nonzero opportunity cost with respect to some other period, the model will introduce the appropriate capital transfer activity. At the stable optimum, slack capital in any period (having zero on-farm opportunity cost) can be regarded as profit which can be removed by the farmer. The value to a farmer of information as to how much of his gross receipts may be removed as profit and when this may be done needs no stressing. As a check, the numerical consistency of a stable core model with respect to variable capital is verified when the sum of slack capital over all periods of the cycle in the optimal solution equals the maximized value of the criterion in the programme.

In the case of stock resources like hay and straw where erratic supplies to and demands from inventories must be balanced, the stable core model can produce an optimal inventory cycle. Transfer processes may easily be manipulated in order to incorporate many variables affecting inventory decisions since the level of the transfer activity for a given resource for any two periods represents the closing/opening inventories for those periods. Variable costs of storage may be entered for the transfer activities in the capital rows of appropriate periods. Similarly deterioration in stored grains, fodders, etc. may be allowed for, by specifying that a transfer of a unit of resource from period t supplies less than a unit to period $t + 1$. Further, if storage facilities are limited, transfer processes may be directly constrained to given capacity levels. Again, minimum amounts of resources which are required as insurances against uncertainty at specific times of the cycle, or indeed throughout the cycle, may be attached directly to transfer processes.

In terms of realistic farm planning this model might be seen as providing target levels of activities towards which the farm can move. Yet in so far as the model assumes resource supplies to be fixed, it is a short-run planning device. If a recommended stable plan differed very markedly from a farm's present organization, the transition path might be the most pressing short-run problem facing the farmer; in such a case models of adjustment[8] may provide more apposite analytical tools.

The discussion above of *the* stable optimum in connection with farm decision-making indicates that this technique is based on a world of planning under assumptions of perfect knowledge. But some progress in the relaxation of this assumption can be made in the usual directions as follows:

(a) Replacement of coefficients in this model by their expected values does not, of course, alter the formal structure of the model.

(b) Continual revision of technical parameter estimates in the light of accrued information enables re-examination of plans as time proceeds and hence, if necessary, revision of plans. In the context of the decision-moment concept referred to earlier, this replanning might ideally be undertaken at each moment.

(c) Plans produced by this model could be "tested" by simulation, for example after the manner of Renborg,[9] to establish likely results from applying them in practice.

(d) Parametric analyses may be used to indicate the ranges of variation in certain coefficients over which the stable core remains optimal.

(e) If probability distributions of certain critical coefficents are available the expected optimal values of decision variables may be calculated as a natural extension of a parametric analysis.

[8] For example, as constructed by the author in *ibid.*

[9] Renborg, U. *Studies on the Planning Environment of the Agricultural Firm.* Uppsala: Almqvist and Wicksells, 1962.