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PRODUCTION RISK AND EFFICIENT ALLOCATION OF RESOURCES

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Efficient allocation of resources has usually been couched in riskless terms, partly because statistical techniques did not exist for measuring the impact of varying levels of factors of production on risks associated with production. Now that such techniques are available, methods are required for determining efficient allocations. Such models, particularly those exploiting stochastic efficiency analysis, are illustrated here with respect to empirical risk-sensitive, farm-firm production functions.

Introduction

The question of efficient allocation of resources under risk has languished somewhat since the pioneering work of Magnússon (1969). He developed an elegant approach to efficiency analysis, although his work was restrictive in several respects—notably his concentration on restrictive mean-variance utility functions with emphasis on the quadratic utility function, and restrictive empirical exemplification of risky production relationships through power functions featuring multiplicative risk. His approach was taken up by Anderson (1973) and elaborated by Anderson, Dillon and Hardaker (1977, Ch. 6) but, until recently, the challenge of statistically efficient estimation of the underlying risk-response relationships remained as a significant impediment to progress.

The methods for estimation developed by Pope and Just (1977) and Just and Pope (1978) paved the way for new and improved empirical estimation. Anderson and Griffiths (1981) and Griffiths and Anderson (1982) extended their multistage estimation approach for quantifying the impact of selected factors of production on riskiness of production, and attempted to apply it to the Pastoral Zone of the sheep industry of Australia. What has not yet been addressed is the exploitation of such empirical relationships in the analysis of efficient allocation of resources. In this note, we endeavour to rectify this omission through further reference to our work on the Australian Pastoral Zone.

Method

The most general approach to efficient resource allocation under risk is to postulate a utility function, $U(R)$, which has as its argument net financial return, R, defined as:

$$R = p_y Y - \Sigma p_i X_i - F, \qquad (1)$$

where Y = physical output;
X_i = the ith variable factor of production;
F = fixed cost; and
p_y and the p_i = the respective prices which herein, for simplicity, are assumed to be known and nonstochastic.

Output Y, while influenced by the X_i, is taken to be stochastic. Managers must commit scarce resources among the X_i while uncertain as to the final effects on Y. A formal approach to this task is to select levels of X_i to maximise expected utility, that is, max $E[U]$ with respect to the X_i. The first-order conditions for this maximisation are well known. They are relatively transparent and simple for the case where $E[U]$ can be expressed in terms of the mean and variance of R,($E[R]$ and $V[R]$), viz. Anderson et al. (1977, p. 171, eqn 6.25):

$$p_y \partial E[Y]/\partial X_i - redq\, p_y^2 \partial V(Y)/\partial X_i = p_i, \tag{2}$$

where *redq* is the risk evaluation differential quotient that measures the decision maker's optimal trade-off between the mean and variance of returns. As these authors note, solution for the optimal levels of X_i, $i = 1, \ldots, k$, involves simultaneous solution of the k nonlinear equations—a solution that may be made more difficult because of the necessity of accounting for boundary solutions such as non-negativity and maximal limits to levels of factors.

In principle, this could be tackled as a nonlinear programming problem. However, it seems simplest in general to sacrifice elegance for practicality by systematically exploring expected utility computed across the factor space. Such exploration could be done on an arbitrarily fine grid which would guarantee any desired level of precision. This procedure is seemingly quite straightforward when the particular utility function of the decision maker is known. However, a more interesting case to address is where only bounds can be attached to the extent of risk aversion. One general approach then is to invoke the rules and procedures for analysis of stochastic efficiency, using, for example, ordering rules such as for second-degree stochastic dominance (Hadar and Russell 1969; Anderson 1974*a*).[1] More recently, Meyer (1977*a*, *b*) introduced a stronger ordering procedure, termed stochastic dominance, with respect to a function. This more powerful procedure has been implemented in agricultural decision analysis by Kramer and Pope (1981) and King and Robison (1981). A particular application that seems most appealing is to place bounds on the extent of absolute risk aversion that is likely among the class of decision makers faced with the task of allocating resources efficiently. These notions are implemented below as a practical means of narrowing search in the numerical exploration of the expected utility surface in $(k+1)$ dimensional space.

Application

In our study of risk phenomena in the Pastoral Zone sheep-grazing industry in eastern Australia (Anderson and Griffiths 1981), we elected to use power functions to describe both the mean output production relationship and the variance of output relationship. We found that some resources increased riskiness of output as measured by variance, while

[1] A risky prospect is (e.g. second) stochastically dominated by another if it would never be preferred by any member of a defined (e.g. the risk-averse) set of agents with preferences subject to specified constraints (e.g. first and second derivatives of utility positive and negative, respectively). A stochastically efficient set of a given degree consists of those prospects not dominated at that degree by another.

others had a risk-reducing effect.[2] The particular data chosen for illustrative purposes are those estimates reported by Griffiths and Anderson (1982) for Model 1 with all error components (firm, year and other) heteroscedastic.

The forms used for estimation were:

(3) $$E[Y] = \alpha_0 \prod_{i=1}^{k} X_i^{\alpha_i}; \text{ and}$$

(4) $$V[Y] = \beta_0 \prod_{i=1}^{k} X_i^{\beta_i}.$$

We rewrite the estimated equations here using the present notation and in a convenient order as:

(5) $$E[Y] = 28.22\, L^{0.16}\, S^{0.79}\, F^{0.12}\, B^{0.07}\, W^{-0.04}\, P^{-0.007}; \text{ and}$$

(6) $$V(Y) = 6.82\, L^{-0.29}\, S^{1.02}\, F^{-0.18}\, B^{0.68}\, W^{-0.12}\, P^{0.15}.$$

The X_i variables (described fully by Anderson and Griffiths 1981) are:

L = labour;
S = sheep;
F = fencing services;
B = buildings and land services;
W = water services; and
P = plant and machinery services.

Other data pertinent to the following analysis are summarised in Table 1. The ranges and means reported refer to the cross-sectional data for 38 firms, averaged over the 10 years of observation 1964-65 through 1973-74.

The estimated equations are less than ideal, particularly in respect of the anomalous negative α_i coefficients for W and P. However, the equations seem adequate for our illustrative purpose if W and P are fixed, arbitrarily, at their geometric mean levels. Buildings and land services are relatively fixed in the short run, and so the factor B is similarly held at its geometric mean level in what follows.

Unconstrained optimisation of multifactor power functions seldom leads to unequivocal results – the archetypical case being that of expected profit maximisation under constant returns to scale when there is no defined optimum – and, indeed, it may be asking too much of a simplistic empirical production function. What makes rather more common sense, and guarantees more sensible patterns of 'optimal' resource allocation, is maximisation subject to an overall constraint; for example, a fixed outlay or budget constraint set at, say, the geometric-mean total resource use. Duloy (1959) illustrated this approach in his constrained profit maximisation study of resource use in the Australian Pastoral Zone. In the riskless case of profit maximisation, allocation is based on the rule that the share of resources allocated to the ith input is the ratio of the ith partial

[2] The equation here of risk and variance is made in the spirit of the present state of most empirical work. Econometric procedures for measuring induced risk effects beyond those embodied in variance are not well developed, although there have been some attempts to seek more general effects using notions of stochastic efficiency (Hadar and Russell 1969; Rothschild and Stiglitz 1970) in empirical response analysis (Anderson 1974*b*; Anderson et al. 1977).

TABLE 1

Sample and Price Data for the Factors of Production

Variable (short name)	Unit	Price per unit	Range Lower	Range Upper	Geometric mean
		$			
L (labour)	man-weeks	120	61	1395	255
S (sheep)	10^3 sheep equivalents	3000	1.5	31.7	6.7
F (fencing)	\$$10^3$	1050	0.7	5.7	2.4
B (buildings)	\$$10^3$	1050	1.3	48.6	11.8
W (water)	\$$10^3$	1050	0.5	4.4	1.6
P (plant)	\$$10^3$	1050	1.9	34.4	9.7

elasticity of production (α_i) to the total elasticity ($\Sigma_i\alpha_i$), that is $X_i=(\alpha_i/\Sigma\alpha_i)(C/p_i)$, where C is the fixed total outlay.

To proceed, it will be helpful to introduce a specific utility or preference function. The constant absolute risk aversion (negative exponential) function, $U(R)=-exp(-\theta R)$, has been used widely in both theoretical and empirical work, in spite of its restrictiveness in imposing a constant coefficient of absolute risk aversion (θ). One particular advantage in the present context is that, if R is normally distributed as might well follow from the Central Limit Theorem, maximisation of expected utility, $E[U(R)]$, is equivalent to maximising $E[R]-(\theta/2)V[R]$ (Freund 1956).

With this assumption, we can now formulate the maximisation of expected utility subject to a fixed outlay constraint. Writing this as a constrained objective function involving equations (1), (3) and (4) and, eliminating the Lagrange multiplier, the optimal condition is found to be:

$$X_i=((\alpha_i-\phi\beta_i)/(\Sigma_i\alpha_i-\phi\Sigma_i\beta_i))(C/p_i), \tag{7}$$

where $\phi=(\theta/2)p_y V[Y]/E[Y]$ is a risk adjustment coefficient incorporating the effects of risk aversion (θ) and the 'relative' riskiness of production (relative-variance of Y, $V[Y]/E[Y]$).[3] Clearly, in the risk-neutral case where $\theta=0$ or in the risk-absent case where $V[Y]=0$, equation (7) collapses to the standard riskless result noted above. Although equation (7)

[3] The constrained expected utility equation to be maximised is:

$$U(R)=p_yE[Y]-\Sigma_ip_iX_i-(\theta/2)p_y^2V[Y]+\lambda(C-\Sigma_ip_{ii}), \tag{8}$$

where λ is a Lagrange multiplier and setting the partial derivative of (8) with respect to it imposes the outlay constraint:

$$\Sigma_ip_iX_i=C. \tag{9}$$

The partial derivative of (8) with respect to the ith input is found (using (3) and (4)) and set to zero as:

$$\partial U/\partial X_i=p_y\alpha_iE[Y]/X_i-p_i-(\theta/2)p_y^2\beta_iV[Y]/X_i+\lambda p_i=0, \tag{10}$$

which can be arranged as:

$$p_iX_i=(\alpha_i-\phi\beta_i)(p_yE[Y]/(1-\lambda)). \tag{11}$$

Substituting (11) into (9) eliminates λ, and p_y and $E[Y]$, and the result can be rearranged in the convenient form of (7).

is of simple form, the dependence of $V[Y]/E[Y]$ on the levels of the X_i and, in more complex utility specifications, a similar dependence of the risk aversion term, necessitates solution by iterative procedures. Thus, for example, the riskless constrained optimal allocation of $\{L, S, F\}$, namely {66.3, 13.1, 5.7}, can be contrasted with a risk-averse ($\theta = 5 \times 10^{-5}$) allocation based on equation (7) and $p_y = \$1000/t$, namely {67.4, 13.0, 5.8}. The (small) differences between these optimal resource bundles reflect the differing risk inducing or reducing effects captured in equation (6). Both these bundles differ greatly from the geometric means (Table 1), and feature much greater sheep/labour ratios, and substitution of fencing services for labour.

We now take up the suggestion of exploring performance numerically across the k-dimensional factor space and sorting according to stochastic efficiency criteria. For illustrative purposes, W, P, and B are again held fixed and efficient levels of the variable L, S and F are sought. A grid of eight equally spaced levels of each factor spanning the observed sample range (Table 1) is considered, so that there are $8 \times 8 \times 8 = 512$ more-or-less feasible combinations of resources.

The most widely used efficiency criterion is the mean-variance (E-V) criterion. Under the assumption of normally distributed R, this is identical to the more general criterion of second-degree stochastic dominance (Anderson et al. 1977, Ch. 9). The class of utility functions (decision makers) for which E-V efficiency is appraised is wide, ranging from risk neutral to infinite risk aversion. In spite of this, the E-V rule does produce a rather small efficient set of resource combinations (20 of the 512). These can be described most succinctly by reference to triplets of coded levels (e.g. {111} denotes the lowest (of the eight) levels of each of the three variable factors, {13(2-4)} denotes level 1 of factor 1, combined with level 3 of factor 2, with levels 2, 3 and 4 of factor 3). The 20-element E-V-efficient set is {81(2-8), 82(4-8), 83(6-8), 8(4-8)8}.

The idea in stochastic dominance with respect to a function is to focus on a narrower and more relevant class of preferences (decision makers). Finite limits are placed on the extent of absolute risk aversion and the resulting stronger ordering rule presumably compacts further the stochastically efficient set (Drynan 1977; Meyer 1977*b*; King and Robison 1981; Kramer and Pope 1981). In the following simplified application of the idea, the earlier assumption of constant risk aversion is again invoked in an analysis that might be termed stochastic dominance with respect to a constant-risk-aversion utility function. The analysis then proceeds straightforwardly if returns are again assumed to be normal.

The effect of the compaction can be seen in the present illustration by considering application of the procedure to different ranges of absolute risk aversion. Beginning at the risk-neutral end of the range, the range of $\theta = 0$ to 1×10^{-5} produces an efficient set with only one element {888}, as may be expected, the expected profit maximising combination. Examples of sets for more risk-averse ranges are: $\theta = 0.01$ to 0.02, {8(3-8)8}, 6 elements; $\theta = 0.02$ to 0.03, {81(6-8), 8(2-3)8}, 5 elements; $\theta = 0.03$ to 0.05, {818}, 1 element; $\theta = 0.05$ to ∞, {818}, 1 element.

These various risk-efficient sets have much in common, as nearly all feature the highest level of labour and the higher levels of fencing services, reflecting the negative marginal risk of these factors, as well as the assumed expected marginal value productivity. The sheep input (stocking rate) has a strongly positive marginal risk as well as productivity, and thus the level of this input, the second variable factor in the efficient sets, depends crucially on the bounds placed on risk aversion. It ranges from the lowest level at extreme risk aversion to the highest level at slight risk aversion. This result corresponds broadly with the analogous effects on stocking rates in the High Rainfall Zone of Australia explored by McArthur and Dillon (1971). However, the disparate resource bundles involved in the grid considered, and thus in the efficient sets, make precise comparison with the earlier constrained allocations difficult.

Conclusion

Econometric methods are now available for quantifying the effects of factors of production on risk. Decision theory and analysis provide the theory and method for linking these effects to conditionally normative statements about how resources should be combined to maximise the satisfaction of the economic agents concerned. Our approach and illustration serve to bridge the gulf between the emerging econometric practice and the risk-efficient allocation of resources.

What needs to be done now is to progress toward a suite of empirical relationships wherein the effects on risk are estimated appropriately. Such work must be supplemented eventually by enhanced understanding of decision makers' goals and preferences.

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