EXPECTED VALUES OF COSTS AND REVENUES DEPENDENT ON DROUGHT†

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In earlier papers on drought, sequences of wet and dry months were treated as realisations of binomial trials with periodic probabilities of failure (dry months), and the cost—revenue outcomes of alternative drought strategies were handled probabilistically. When costs and revenues may be represented by a set of second degree polynomials over a set of adjacent subranges of the time scale, their expectation may be expressed as a simple closed formula in the twelve probabilities of dry months. As an illustration of the use of these new formulae, means and standard deviations of drought duration calculated in this manner are presented for three typical localities in Queensland. Previously known closed formulae for the mean and the variance in the case where all calendar months have equal probability of being dry emerge as special cases and are verified.

Introduction

Verhagen and Hirst [7] investigated data assembled by Everist and Moule [2] to test the influence of dryness or wetness of preceding months on the probabilities $u_1, u_2, \ldots, u_{12}$, that the calendar months, i.e. 1, 2, . . ., 12 are dry (i.e. have no effective rainfall). Finding little evidence of dependence between rainfall in consecutive months, they proposed a model whereby future patterns of wet and dry months are treated as outcomes of independent binomial trials with periodic probabilities $u_1, \ldots, u_{12}$ of failures (dry months)[1]. On the basis of this model, sufficient estimates of the probabilities $u_1, \ldots, u_{12}$ were obtained, and the distribution of the waiting times, i.e of the number w of (dry) months one may have to wait before the next wet month, was derived. These distributions each have an infinite number of terms, so that costs, revenues, etc., which depend on the waiting time (drought duration) have an expected value which is expressed as an infinite series in the individual terms of the distribution [5]. Dillon and Lloyd [1] achieved approximate evaluations of these expected values by computing partial sums of such series. In the present paper it will be shown that the sum of the infinite series defining the expected values may be reduced to a small finite number of terms, thus enabling rapid and exact evaluation, using directly the 12 probabilities $u_1, \ldots, u_{12}$ from which the distribution was itself derived.

Closed Summation Formulae for the Evaluation of Expected Values

In the Dillon-Lloyd model, the revenue function $f(w)$ is represented by two quadratic functions of the waiting times, one in the range $w \leqslant y$,

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[1] See also Verhagen [6].

i.e. when drought length is covered by fodder reserve, and one in the range $w > y$, i.e. when it is not.[2] The expected revenue, when expressed in the individual terms of the distribution, contains therefore an infinite sum of the form

$$(1) \sum_{w=y}^{\infty} (a + bw + cw^2)P(w)$$

where $P(w)$ is the probability that the waiting time is w. Clearly (1) simplifies to

$$(2)\ aA(y) + bB(y) + cC(y)$$

where $A(y) = \sum_{w=y}^{\infty} P(w)$, $B(y) = \sum_{w=y}^{\infty} wP(w)$ and $C(y) = \sum_{w=y}^{\infty} w^2 P(w)$

and it is therefore proposed to find simple summation formulae $A(y)$, $B(y)$ and $C(y)$ in terms of $u_1, \ldots, u_{12}$.

(i) *Summation formula* $A(y)$ *for* $\sum_{w=y}^{\infty} P(w)$

The individual terms of the distribution of the waiting times, i.e. the probabilities $P(w)$ ($w = 1, 2, \ldots, \infty$) are defined [7] as

$$(3)\ P(o) = (1 - u_2), \qquad (w = o)$$

$$P(w) = \left[\prod_{t=2}^{w+1} u_t\right]\left[1 - u_{(w+2)}\right], \quad (w > o)$$

Thus

$$\begin{aligned}(4)\ P(y) &= [u_2 \ldots u_{(y+1)}] - [u_2 \ldots u_{(y+2)}] \\ P(y+1) &= \qquad + [u_2 \ldots u_{(y+2)}] - [u_2 \ldots u_{(y+3)}] \\ P(y+2) &= \qquad\qquad + [u_2 \ldots u_{(y+3)}] \\ &\qquad\qquad - [u_2 \ldots u_{(y+4)}]\end{aligned}$$

etc.

Vertical summation of the equations (4) gives the required summation formula

$$(5)\ A(y) = \sum_{w=y}^{\infty} P(w) = [u_2 \ldots u_{(y+1)}].$$

(ii) *Summation formula* $B(y)$ *for* $\sum_{w=y}^{\infty} wP(w)$.

The individual terms of the infinite sum $\sum_{w=y}^{\infty} wP(w)$ may be written

$$\begin{aligned}(6)\qquad yP(y) &= y[u_2 \ldots u_{(y+1)}] - y[u_2 \ldots u_{(y+2)}] \\ (y+1)P(y+1) &= \qquad + (y+1)[u_2 \ldots u_{(y+2)}] \\ &\qquad - (y+1)[u_2 \ldots u_{(y+3)}]\end{aligned}$$

etc.

Vertical summation of the equations (6) gives

[2] Segmentation of curves in terms of quadration is discussed by Hudson [4].

$$(7)\ \sum_{w=y}^{\infty} WP(w) = y[u_2 \ldots u_{(y+1)}] + [u_2 \ldots u_{(y+2)}] + [u_2 \ldots u_{(y+3)}] + \ldots$$

which may be rewritten as

$$\begin{aligned}(8)\ \sum_{w=y}^{\infty} wP(w) = {} & y[u_2 \ldots u_{(y+1)}] + \{[u_2 \ldots u_{(y+2)}] + [u_2 \ldots u_{(y+3)}] \\ & + \ldots + [u_2 \ldots u_{(y+13)}]\} \\ & K\{[u_2 \ldots u_{(y+2)}] + [u_2 \ldots u_{(y+3)}] \\ & + \ldots + [u_2 \ldots u_{(y+13)}]\} \\ & K^2\{[u_2 \ldots u_{(y+2)}] + [u_2 \ldots u_{(y+3)}] \\ & + \ldots + [u_2 \ldots u_{(y+13)}]\}\end{aligned}$$

etc.

where K stands for $\prod_{t=1}^{12} u_t$. Vertical summation in (8) gives the required summation formula.

$$(9)\ B(y) = \sum_{w=y}^{\infty} wP(w) = y \prod_{t=2}^{(y+1)} u_t + [1/(1 - K)] \left[\prod_{t=2}^{(y+2)} u_t + \ldots + \prod_{t=2}^{(y+13)} u_t\right].$$

(iii) *Summation formula* $C(y)$ *for* $\sum_{w=y}^{\infty} w^2 P(w)$

It will be convenient first to find a summation formula for $\sum_{w=y}^{\infty} w(w - 1) P(w)$ whose individual terms may be written

$$\begin{aligned}(10)\ y(y - 1)P(y) & = y(y - 1)[u_2 \ldots u_{(y+1)}] \\ & \quad - y(y - 1)[u_2 \ldots u_{(y+2)}] \\ (y + 1)(y)P(y + 1) & = \\ & (y + 1)y[u_2 \ldots u_{(y+2)}] - (y + 1)y[u_2 \ldots u_{(y+3)}]\end{aligned}$$

etc.

Vertical summation of the equation (10) gives

$$\begin{aligned}(11)\ \sum_{w=y}^{\infty} w(w - 1)P(w) = {} & y(y - 1)[u_2 \ldots u_{(y+1)}] + 2y[u_2 \ldots u_{(y+2)}] \\ & + 2(y + 1)[u_2 \ldots u_{(y+3)}] + \ldots\end{aligned}$$

which may be rewritten as

$$\begin{aligned}(12)\ \sum_{w=y}^{\infty} w(w - 1)P(w) = {} & y(y - 1)[u_2 \ldots u_{(y+1)}] \\ & + 2[y + (y + 12)K + (y + 24)K^2 + \ldots][u_2 \ldots u_{(y+2)}] \\ & + 2[(y + 1) + (y + 13)K + (y + 25)K^2 + \ldots][u_2 \ldots u_{(y+3)}] \\ & + \ldots \\ & \cdot \\ & \cdot \\ & \cdot \\ & + 2[(y + 11) + (y + 13)K + (y + 35)K^2 + \ldots][u_2 \ldots u_{(y+13)}].\end{aligned}$$

Repeated use of the equality $\sum_{j=1}^{\infty} jK^j = 1/(1-K)^2$

on the infinite series in each of the brackets in formula (12) gives

$$(13)\ \sum_{w=y}^{\infty} w(w-1)P(w) = y(y-1)\prod_{t=2}^{y+1} u_t$$

$$+ 2[(y \qquad /(1-K) + 12K/(1-K)^2] \prod_{t+2}^{y+2} u_t$$

$$+ 2[(y+1)\ /(1-K) + 12K/(1-K)^2] \prod_{t=2}^{y+3} u_t$$

$$+ \dots$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$+ 2[(y+11)/(1-K) + 12K/(1-k)^2] \prod_{t=2}^{y+13} u_t.$$

Since $C(y)$ may be expressed as the sum of (13) and $B(y)$ it follows that the required summation formula is

$$(14)\ C(y) \equiv \sum_{w=y}^{\infty} w^2P(w) = y^2 \prod_{t=2}^{y+1} u_t$$

$$+ [(2y+1)\ /(1-K) + 24K/(1-K)^2] \prod_{t=2}^{y+2} u_t$$

$$+ [(2y+3)\ /(1-K) + 24K/(1-K)^2] \prod_{t=2}^{+3} u_t$$

$$+ \dots$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$+ [(2y+23)/(1-K) + 24K/(1-K)^2] \prod_{t=2}^{y+13} u_t.$$

The infinite sum $\sum_{w=y}^{\infty} (a + bw + cw^2)P(w)$ may therefore be evaluated exactly in a finite number of operations as $[aA(y) + bB(y) + cC(y)]$ in terms of the summation formulae $A(y)$, $B(y)$ and $C(y)$ derived in this section. Similarly any sum of the form

$$(15)\ \sum_{w=y_1}^{y2} (a + bw + cw^2)P(w)$$

over the subrange (y_1, y_2) may be written in terms of the summation formulae as

$$a[A(y_1) - A(y_2)] + b[B(y_1)] - B(y_2)] + c[C(y_1) - C(y^2)]$$

so that once the subranges and the appropriate constants a, b and c for each subrange are known, computation of expected values becomes a simple routine using the summation formulae, $A(y)$, $B\ (y)$ and $C(y)$.

Expected Values of Some Special Linear and Quadratic Forms

The probabilities $P(w)$ of the waiting time w were waiting times counted after January. The corresponding formulae starting after the rth calendar month are similar.

The mean waiting time $E_1(w)$ after the first month of the year is obtained as $B(o)$ from formula (9) which yields

(16) $E_1(w) =$

$$\sum_{y=0}^{\infty} wP(w) = [1/(1-K][u_2 + u_2u_3 + u_2u_3u_4 + \ldots + u_2u_3u_{12}u_1].$$

Similarly the expected waiting time after the rth calendar month is

(17) $E_r(w) = [1/(1-K)][u_{r+1} + (u_{r+1}u_{r+2}) + \ldots + (u_{r+1} \ldots u_{r+12})]$
$r = 1, \ldots, 12.$

The variance $\mathrm{Var}_1(w)$ of the waiting times after January is $C(o) - E_1(w)^2$, and the variance after the rth month of the year take the form

$$\text{(18) } \mathrm{Var}_r(w) = [\ 1/(1-K) + 24K/(1-K)^2] \prod_{t=r+1}^{y+r+1} u_t$$

$$+ [\ 3/(1-K) + 24K/(1-K)^2] \prod_{t=r+1}^{y+r+2} u_t$$

$$\vdots$$

$$+ [23/(1-K) + 24K/(1-K)^2] \prod_{t=r+1}^{y+r+12} u_t - [E_r(w)]^2$$

$r = 1, \ldots, 12.$

Formulae (17) and (18) were used to compute mean and standard deviation of waiting time after each of the calendar months ($r = 1, \ldots, 12$) of the year for three typical localities in Queensland presented in Tables 2 and 3, from the probabilities $u_1, \ldots u_{12}$ presented in Table 1.

TABLE 1

The Probabilities that the Calendar Months are Dry

Winton	0·53	0·38	0·64	0·90	0·81	0·79	0·74	0·98	0·95	0·93	0·85	0·78
Barcaldine	0·43	0·50	0·64	0·73	0·71	0·55	0·66	0·88	0·85	0·78	0·76	0·55
Roma	0·40	0·45	0·45	0·66	0·59	0·35	0·40	0·67	0·69	0·57	0·50	0·41

TABLE 2

The Mean Waiting Times after Each Calendar Month

Winton	1·8	3·7	4·7	4·3	4·3	4·4	4·9	4·0	3·2	2·4	1·9	1·4
Barcaldine	1·6	2·1	2·3	2·2	2·1	2·7	3·1	2·6	2·0	1·6	1·2	1·1
Roma	0·9	1·0	1·3	1·0	0·7	1·0	1·6	1·4	1·1	0·9	0·7	0·8

TABLE 3

Standard Deviations of the Waiting Times after Each Calendar Month

Winton	3·5	4·3	4·2	4·1	3·9	3·6	2·9	2·8	2·8	2·7	2·7	2·9
Barcaldine	2·4	2·7	2·7	2·7	2·7	2·7	2·3	2·1	2·0	1·9	1·8	2·0
Roma	1·4	1·5	1·4	1·3	1·4	1·7	1·6	1·4	1·3	1·2	1·2	1·3

When the probabilities are all equal to u, the mean $E_r(w)$ and the variance $\mathrm{Var}_r\ (w)$ of the waiting time do not depend on the calendar month of the year and reduce respectively to $u/(1\text{-}u)$ and $u/(1\text{-}u)^2$ the well-known mean and variance of the geometric distribution [3].

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