SIMULTANEOUS-EQUATIONS BIAS AND AGRICULTURAL PRODUCTION FUNCTION ESTIMATION

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Introduction

During the twenty years or so since some early estimations of production functions in agriculture[1] there has been considerable discussion of many of the technical problems associated with such estimation.

Although the simultaneous nature of production decisions, leading to the determination of input levels in the production function, was pointed out early in the piece by Marschak and Andrews,[2] little attention has been paid to this aspect of the problem until comparatively recently. The only systematic treatment of these problems is that of Hoch,[3] although there appeared recently in the *Economic Record* an extended controversy between Konijn and Soper.[4] The purpose of the present paper is to demonstrate that when the profit maximisation conditions are cast in a slightly different form, Hoch's conclusions concerning the existence of simultaneous equations bias cannot be sustained, although there is similarity of assumptions throughout. The treatment initially will be in terms of farms producing a single output. However, the examination of the simultaneous equation problem will be extended to the multiple output case where it will be shown that problems of an entirely different nature arise. The Cobb-Douglas form of production function will be adhered to throughout.

The case of a single output

(a) Profit Maximisation under Certainty

We shall write the production function as:

$$X_0 = \alpha \prod_{i=1}^{m} X_i^{\beta_i} \qquad \ldots\ldots \quad (1)$$

[1] G. Tintner, "A Note on the Derivation of Production Functions from Farm Records", *Econometrica*, Vol. XII, No. 1 (January, 1944), pp. 26-34 and G. Tintner and O. H. Brownlee, "Production Functions Derived from Farm Records", *Journal of Farm Economics*, Vol. XXXVI, No. 3 (August, 1944), pp.566-571.

[2] J. Marschak and W. H. Andrews, "Random Simultaneous Equations and the Theory of Production", *Econometrica*, Vol. XII, No. 3-4 (July-October, 1944), pp. 143-205.

[3] Irving Hoch, "Simultaneous Equations Bias in the Context of the Cobb Douglas Production Function", *Econometrica*, Vol. XXXVI, No. 4 (October, 1958), pp. 566-578.

[4] C. S. Soper, "Production Functions and Cross-section Surveys", *The Economic Record*, Vol. XXXIV, No. 67 (April, 1958), pp. 111-117. Soper discussed some problems of estimation under profit maximisation assumptions. But, as Konijn showed, Soper's models were non-stochastic and hence of limited relevance. See H. S. Konijn, "Estimation of an Average Production Function from Surveys", *The Economic Record*, Vol. XXXV, No. 70 (April, 1959), pp. 118-125.

where farm subscripts are suppressed and

X_0 is the value of output
X_i the value of the input
β_i the elasticity of production of the i-th input
and α is an constant.[5]

To maximise profits, P, where

$$P = X_0 - \sum_{i=1}^{m} X_i \qquad \ldots\ldots (2)$$

the partial derivatives with respect to X_i are set equal to 0.[6]

$$\frac{\partial P}{\partial X_i} = \beta_i \frac{X_0}{X_i} - 1$$
$$= 0$$

or, $$\beta_i \frac{X_0}{X_i} = 1 \qquad \ldots\ldots (3)$$

If the marginal value product of the i-th input exceeds unity, the rational producer, with no capital constraint, increases the use of this input.

We turn now to a consideration of the problem of maximising the value of output (or of profits) when the total quantity of resources is fixed. The situation of constrained profit maximisation seems a more likely model for agriculture, due to the widespread existence of capital rationing, whether internally or externally imposed.[7]

In this case, it is necessary to maximise:

$$X_0 = \alpha \prod_{i=1}^{m} X_i^{\beta_i} \qquad \ldots\ldots (4)$$

subject to the capital constraint,

$$\sum_{i=1}^{m} X_i = \mathrm{T} \qquad \ldots\ldots (5)$$

[5] This definition of the variables entering the production function is adopted for its convenience in later analysis. As a consequence of the competitive nature of agriculture, prices paid for resources and received for output can be expected to be the same for all firms. Hence, this form of the production function does not differ from the "engineering" form except in its numeraire. The β_i, being elasticities, are unchanged by this multiplication by constants.

[6] A second order condition requires that the sum of the elasticities is less than unity. In other words, decreasing returns to scale prevail. (More strictly, $\beta_i > 0$ $\Sigma\beta_i < 1$.)

[7] See, for example, D. Gale Johnson, *Forward Prices for Agriculture,* (Chicago: The University of Chicago Press, 1947), Chs. 4 and 5. Empirical evidence is available from farm surveys. For instance, Iowa farmers in one study suggested that they could profitably expand investment by some 72 per cent were the funds freely available. However, 79 per cent of them stated that they would not borrow more funds even if no interest were payable. See, E. O. Heady and E. R. Swanson, *Resource Productivity in Iowa Farming,* Iowa Agricultural Experiment Station, Research Bulletin 388 (Ames, 1952), pp. 767-773. Similar attitudes were expressed by Australian woolgrowers when interviewed during 1954 and 1955. (K. O. Campbell, personal communication.)

Construct the equation:

$$V = \alpha \prod_{i=1}^{m} X_i^{\beta_i} + \lambda (T - \sum_{i=1}^{m} X_i) \qquad \ldots\ldots (6)$$

where λ is a Lagrange multiplier.

Necessary conditions for a maximum are:

$$\frac{\partial V}{\partial X_i} = \beta_i \frac{X_0}{X_i} - \lambda = 0$$

$$\frac{\partial V}{\partial \lambda} = T - \sum_{i=1}^{m} X_i = 0 \qquad \ldots\ldots (7)$$

Sufficient conditions require also that the β_i are less than 1, for $i = 1, \ldots, m$.[8] However, the sum of the elasticities is not constrained to less than unity.

From (7) by summation,

$$\lambda = \frac{X_0 \sum_{i=1}^{m} \beta_i}{\sum_{i=1}^{m} X_i} \qquad \ldots\ldots (8)$$

Thus, from (7),

$$X_i = \frac{\beta_i T}{\sum_{i=1}^{m} \beta_i} \qquad \ldots\ldots (9)$$

In the case of constrained profit maximisation all marginal products are equal to the constraint, λ, which, by (8) is a function of the volume of funds available. (For unconstrained profit maximisation $\lambda = 1$.) Equation (9) specifies the level of the i-th resource required to maximise output under the constraint. This level of the input will be denoted by X_i^*. This approach to the problem of constrained profit maximisation may readily be extended to the case where, in addition to a general capital constraint, there are additional constraints upon the level of one or more of the particular resources.[9]

Hoch's treatment of the constrained maximisation situation involves a different approach to that detailed above. In his treatment, the conditions for maximisation under some constraint are expressed as:

$$\frac{\beta_i X_o}{X_i} = R_i \qquad (10)$$

where R_i is some constant different from 1.[10] (This may be compared with the condition for unconstrained profit maximisation specified in equation (3) above.)

[8] The sufficient conditions are rather messy to derive, but may be seen by following through the technique of constrained maximisation as described by, e.g. J. M. Henderson and R. E. Quandt, *Micro-economic Theory: A Mathematical Approach* (New York: McGraw Hill Book Company, 1958), pp. 273-274.

[9] See J. H. Duloy, "Resource Allocation and a Fitted Production Function", *Australian Journal of Agricultural Economics,* Vol. 3, No. 2 (December, 1959), pp. 75-85.

[10] I. Hoch, *op. cit.,* p. 568.

(b) Estimation of the Production Function

In his discussion of least-squares bias, Hoch distinguishes two main cases.[11] The first considers the situation where variable inputs are determined for the current period by maximising with respect to anticipated output rather than to current output. The second considers the situation where maximisation occurs with respect to actual (current) output.

Hoch's model of the case of maximisation with respect to anticipated output may be written, in the notation of this paper, as follows:

$$X_0 = \alpha \prod_{i=1}^{m} X_i^{\beta_i} U$$

$$R_i X_i = \beta_i \hat{X}_0 V_i \qquad i = 1, 2, \ldots, m \qquad \ldots\ldots (11)$$

where

$$\hat{X}_0 = \alpha \prod_{i=1}^{m} X_i^{\beta_i} \qquad \ldots\ldots (12)$$

U and V_i are disturbances in the production function and the decision equations respectively. $\hat{X}_0$ is anticipated output as defined by Hoch and R_i is a constant indicating constrained rather than unconstrained maximisation. It should be noted that Hoch's formulation R_i is treated as exogenous, that is, it is regarded as being independent of the disturbances in the production and decision functions. In this case, the X_i are independent of the disturbance, U, in the production function and least-squares estimates of the parameters, β_i, in the production function are consistent. This model has something in common with recursive systems because the X_i, whilst stochastic and a function of disturbances (V_i) are not a function of the disturbances (U) in the equation to be estimated.

It will be noted that equation (12) implies that the anticipated production function is the same for all farmers, there being no disturbance in that equation. This does not appear to be a reasonable assumption. An alternative formulation of Hoch's model of maximisation with respect to anticipated output may be considered, as follows:

$$X_0 = \alpha \prod_{i=1}^{m} X_i^{\beta_i} U$$

$$R_i X_i = \beta_i \hat{X}_0 V_i \qquad i = 1, 2, \ldots, m \qquad \ldots\ldots (13)$$

where

$$\hat{X}_0 = \alpha \prod_{i=1}^{m} X_i^{\beta_i} W \qquad \ldots\ldots (12)$$

In this case, by the inclusion of a disturbance (W) in the equation for anticipated output, we have allowed anticipated output to vary amongst farmers. In this formulation least-squares estimates of the β_i in the production function (13) are unbiased only under the assumption that U, W, are independent random disturbances.

We will turn, however, to the case of maximisation with respect to

[11] *Ibid.*, pp. 568-569.

current output. In Hoch's formulation this model may be written as follows:

$$X_0 = \alpha \prod_{i=1}^{m} X_i^{\beta_i} U$$
$$R_i X_i = \beta_i X_0 V_i \qquad \ldots\ldots (14)$$

It will be noted that the levels of inputs, X_i, are functions of X_0 and hence of the disturbance, U, in the production function. Least-squares estimates of the β_i in the production function are biased.[12]

We shall consider an alternative formulation of this model, in terms of the conditions for a constrained maximisation outlined previously. The model may be written as follows:

$$X_0 = \alpha \prod_{i=1}^{m} X_i^{\beta_i} U$$
$$X_i = \frac{\beta_i T}{\sum_{i=1}^{m} \beta_i} V_i \qquad \ldots\ldots (15)$$

In this model, U, V_i are disturbances as above. Note, however, that the V_i are subject to the constraint $\sum_i (\beta_i V_i) = \Sigma \beta_i$.

In this case it is assumed that the capital constraint expressed in equation (5) above is exogenous. This is in line with Hoch's assumption that the R_i are exogenous. The model is readily extended to the situation where additional constraints are imposed upon one or more of the X_i in addition to the overall capital constraint. It will be noted that in this case the values of the X_i's are functions only of the constraint, parameters of the production function, and disturbances. In this formulation of the constrained maximisation model, least-squares estimates of the parameters of the production function are unbiased. Hence, a case has been developed for the use of ordinary least-squares in the estimation of production functions even in the situation where decision-making occurs in the same time period as the production function. The most important assumption in the model concerns the overall capital constraint. The assumption concerning this constraint seems reasonable in view of the evidence cited in footnote 7 above.

The Case of Multiple Outputs

(a) Profit Maximisation Under Certainty

Where more than one product is produced, more complex criteria for profit maximisation are required than for the single-output case. We shall consider the situation where two products (denoted by Y_1 and Y_2) may be produced in varying proportions, with m inputs (X_i $i = 1, 2, \ldots, m$) subject as before to an overall capital constraint ($\Sigma X_i = T$). We assume as before that inputs and outputs are measured

[12] Hoch demonstrates that the bias of estimates of the β_i can be determined, and an adjustment for it made, in the special case where the covariance matrix of the V_i in the decision equations is diagonal. *Ibid.*, pp. 570-571.

in money terms and also that all of the m inputs enter each production function. We shall write the production function as

$$Y_1 = C_1 \prod_{i=1}^{m} X_i^{a_i}$$
$$Y_2 = C_2 \prod_{i=1}^{m} X_i^{b_i} \qquad \ldots\ldots (16)$$

It is assumed further that proportions, p of T may be allocated to the production of Y_1 and q to Y_2 such that

$$p + q = 1 \qquad \ldots\ldots (17)$$

The maximum output of product Y_1 is obtained when the resources used in its production are allocated amongst the X_i in a least-cost combination such that

$$X_i^* = \frac{a_i p T}{\sum_{i=1}^{m} a_i} \qquad \ldots\ldots (18)$$

and similarly for product Y_2. Following Samuelson, we define the product transformation function as prescribing the maximum of any output, given the production functions, the constraints upon inputs, and the production of other outputs.[13] We may then derive Y_1* and Y_2*, the maximum of each of the outputs from the generalised resource allocated to their production, and from this the transformation function, giving Y_i* as a function of Y_2* for given T.

The transformation function is

$$Y_i^* = T^{\Sigma a_i}\left(1 - \left[\frac{Y_2^*}{T^{\Sigma b_i}\,\Pi\left(\frac{b_i}{\Sigma b_i}\right)^{b_i}}\right]^{1/\Sigma b_i}\right)^{\Sigma a_i} \Pi\left(\frac{a_i}{\Sigma a_i}\right)^{a_i}$$

To maximise revenue (or profits) we set the derivative of this function with respect to Y_2* equal to -1.

An equivalent approach to specifying the maximum profit allocation of limited resources involves equating marginal value products of all resources in both cases.

(b) Estimation of the Production Functions

Most of the production functions estimated for agriculture have applied to multi-enterprise farms.[14] This is not surprising as the production of multiple products is a characteristic of agriculture. However, in spite of the allocation of a considerable volume of research effort to the estimation of production functions, little attention has been paid to the particular problems of such estimation in respect of multi-enterprise farms.

The discussion here assumes that the purpose of the estimation

[13] P. A. Samuelson, *Foundations of Economic Analysis,* (Cambridge: Harvard University Press, 1953), p. 230.

[14] See, for example, the studies reported by E. O. Heady and J. L. Dillon in *Agricultural Production Functions,* (Ames: Iowa State University Press, 1961), pp. 585-643.

of production functions is to derive estimates of the marginal productivities of resources. That is, we are concerned with "structural" estimation. To this end, it is necessary to specify the set of structural equations which determine the level of resource use, allocation of resources between different outputs and the transformation of resources into products. It is necessary to do this so that we can determine how the data were generated and whether it is possible to estimate the production functions. In the case of a single output, such specification was feasible, in the framework of a (stochastic) profit maximisation model. An analogous model is not easily derived in the case where more than one output is produced.

This may be seen by considering the process of profit maximisation in two stages in the two- (or multi-) output case.[15] The first stage concerns the formulation of least-cost combinations of inputs in each production function; the second concerns a choice of different combinations of products. Such was the approach adopted above in specifying the conditions for profit maximisation under certainty. However, it is necessary to consider stochastic models for estimation purposes.

If the decision functions specifying minimum-cost combinations of inputs (equation 18) are stochastic, then it is not possible to specify the transformation function. Thus the transformation function is not purely a technical relationship, as are the production functions, but are the outcome of what Samuelson terms "economic engineering".[16] Where there are departures from the best allocation of resources in the production of each output, the choice of outputs specified by the transformation function is no longer available. The combination of outputs now possible are located somewhere within the "envelope" of the transformation function. One of these combinations may be regarded as a "second-best optimum" in the sense used by Lipsey and Lancaster. Economists, however, are not much assisted by conventional theory of the firm in charting the wastelands which lie within the production possibility frontier. Thus, when disturbances are admitted into the decision functions in the first stage of the profit-maximisation process, it is difficult to specify the form of the second stage of the process.

The fundamental difficulty associated with multi-enterprise farms lies then in the problems of developing a satisfactory stochastic maximisation model. A range of models describing the process by which farmers actually decide upon a combination of outputs may occur to agricultural economists. However, with the present knowledge of decision processes it is likely to be difficult to discriminate among hypotheses concerning what are likely to be rule-of-thumb methods for navigating the uncharted wastelands.

At this stage we are left in doubt, from *a priori* considerations, concerning the validity of estimating production functions on the multi-enterprise farms. However, there are also likely to be difficulties associated with the data. This is particularly so where two products are produced in such an intricate relationship as are wheat and sheep products in Australia. The problem of measurement of input categories is well-nigh insuperable. For instance, improved pastures yield inputs of nitrogen and soil structure to the crop and of grazing to the sheep.

[15] Such an approach is possible where the production functions are of the Cobb-Douglas form (to which this discussion is restricted) because the expansion paths are linear.

[16] P. A. Samuelson, *op. cit.*, p. 230.

It is not surprising, therefore, that difficulties have been encountered in empirical investigations. Hildebrand,[17] estimating production functions for a number of different years and with different resource sets, for the same sample of farms, obtained results "so variable and disconcerting as to defy adequate rationalisation." Hildebrand's procedure involved aggregating linearly over outputs (and thus over production functions!) and obtaining least-squares estimates of coefficients from the resultant data. The difficulties experienced may thus not be so difficult to rationalise, particularly as marked price changes in the outputs occurred during the period of observation.[18] However, his procedures are common practice[19] and there may well be some justification for his suspicion that "throughout the academic world there are many functions with 'unusable' results filed quietly away."[20]

Least-squares Estimates of Farm Production Functions

	Number of Functions	Number of Functions for which $R^2 > 0{\cdot}8$	Total Number of Coefficients	Number of Coefficients not significant at 0·05 level
Single Enterprise Farms	7	5	32	4
Two Enterprise Farms	8	0	36	15

An empirical study by the author suggests conclusions similar to those of Hildebrand, although an attempt was made to estimate the various enterprise functions separately.

Although sample sizes were comparable between the two groups, the results presented in the table above indicate a far greater degree of "difficulty" with multi-enterprise than with single-enterprise farms.

Conclusions

Under acceptable assumptions concerning the form of the capital constraint upon farm operations, it is evident that there is no simultaneous-equations problem associated with the estimation of production functions for single-enterprise farms. No such assurance can be given where more than one output is involved. Indeed, in this case, the appropriate model of the decision process is uncertain. Hence, it is not possible to derive acceptable methods of estimation. In addition, there exist far greater difficulties of measurement of input categories on multi-enterprise in comparison with single-enterprise farms.

Because the multi-product farm is typical of agriculture, these problems are likely to arise in most empirical estimations of production functions. If this is the case, then the technique of estimating production functions in agriculture has only a narrow range of application.

[17] J. R. Hildebrand, "Some Difficulties with Empirical Results from Whole-Farm Cobb-Douglas Type Production Functions", *Journal of Farm Economics,* Vol. XLII, No. 4 (November, 1960), pp. 897-904.

[18] *Ibid.,* p. 903.

[19] For a recent example, see K. Rasmussen and M. M. Sandilands, *Production Function Analyses of British and Irish Farm Accounts,* (Sutton Bonington: Nottingham University School of Agriculture, 1962).

[20] J. H. Hildebrand, *op. cit.,* p. 902.