UCD Department of Agricultural Economics

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24506

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Department of Agricultural Economics

CONJECTURAL VARIATIONS AND PUBLIC ECONOMICS

by

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Working Paper No. 95-2

Conjectural variations and public economics

The private provision of public goods

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Abstract

A debate about the private provision of public goods focuses on four issues: the existence of free-riding, the extent to which contributions are matched equally across individuals, the nature of conjectures consistent with equilibrium, and the allocative inefficiency of alternative regimes. This paper resolves each of these issues, with the following conclusions: A consistent-conjectures equilibrium exists in the private provision of public goods. It is the monopolistic-conjectures equilibrium. Agents acts identically, contributing positive amounts of the public good in an efficient allocation of resources. There is complete matching of contributions among agents, no free-riding, and the allocation is independent of the number of members in the community. Thus the Olson conjecture—that inefficiency is exacerbated by community size—has no foundation in a consistent-conjectures context.

Key words: Public goods, private provision, consistent conjectural variations.

JEL classifications: D43, D61, H41.

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1. Introduction

Bowley's (1924) conjectural variations have been used repeatedly in the *Journal of Public Economics*. Several interesting examples of their application appear in a debate about the private provision of public goods. Following an initial investigation by Cornes and Sandler (1984), subsequent contributions that use conjectural variations have been made, respectively, by Sugden (1985), Cornes and Sandler (1985), Scafuri (1988), and Costrell (1991). The totality of these works identifies three key issues, which are the central focus of the present contribution. These issues are the extent of under provision of the public good from its Pareto-efficient level; whether, under this criterion, the conjectural-variations equilibrium dominates the Nash equilibrium; and the extent of free riding and matching behaviour, if any, which are observed in equilibrium.

This paper makes four contributions. The first is to demonstrate that much of this debate can be resolved deductively. The deduction stems from a feature of the equilibrium, which is engendered by an assumption used in each of the previous contributions. This precludes the need to resort to sophisticated mathematical arguments to resolve the debate. This claim is substantiated further from a comprehensive treatment of conjectural variations applied to model the private provision of public goods, which is the second contribution of the paper. The third is to provide a formal link between the application in this paper, and its foundations in the static theory of homogeneous-product oligopoly. The fourth and final contribution is to elucidate further the key issues in the debate and, where possible, resolve them in the context of a model of consistent-conjectures equilibrium. In doing so, I demonstrate that several unsubstantiated claims about the private provision of public goods are, in fact, correct, although for the wrong reasons.

In section two I present a brief summary of the debate and identify some of its salient features. In section three I present a heuristic argument, drawn from an assumption embedded within the framework, which I substantiate further in section four. Section five concludes the investigation.

2. Background

Cornes and Sandler (1984) present diagrammatic and mathematical treatments of Nash equilibrium in the private provision of public goods. The contentious feature of this model is that the agents make contributions to the public good, ignoring the potential consequences of this action on the corresponding contributions from other agents in the community. This aspect of the model affords conjectural variations an opportunity to extend the theory, in an intuitive way. The corner-stone of the extended model is that it makes specific account of the subjective perception of each agent, about the corresponding contribution from the rest of the community, when agents make contributions to the public good. Although they discuss negative conjectures, Cornes and Sandler focus most of their attention on situations in which agents' contributions are positively correlated, that is, a situation in which a positive contribution from one individual gives rise to a positive contribution from the rest of the community. Within this class of model, they present a particular formulation for the conjectures and use it to demonstrate the 'Olson conjecture.' This is that the amount by which the equilibrium allocation falls short of the Pareto-optimal level is an increasing function of population size. A consistent-conjectures equilibrium, wherein the subjective perceptions of the agents are confirmed, is discussed, but not formalized.

That positive conjectures prevail in equilibrium, prompts critical comment by Sugden (1985). He presents a compelling argument to suggest that negative conjectures are more likely. In this situation, own and rest-of-the-community contributions are inversely related. It is claimed that no equilibrium exists with strictly positive contributions, unless agents hold incorrect conjectures. This leads to the conclusion that contributions may not be made at all. It follows that negative conjectures—reexamined by Cornes and Sandler (1985)—lead to inefficiencies, which are greater under consistent conjectures than in Nash equilibrium.

Scafuri (1988) questions the previous criteria used to characterize consistency. Necessary and sufficient conditions for consistent-conjectures equilibria are proposed.

Costrell (1991) formalizes the existence of interior solutions, extends the analysis to the case of semi-public goods, and characterizes consistent-conjectures, Nash-equilibrium, and Pareto-optimal allocations. The Nash-equilibrium allocation dominates the consistent-conjectures allocation, and it is demonstrated that population growth under consistent conjectures can be immizerizing.

3. Heuristic arguments

Without exception, each of the studies above employs a key assumption. This is that the agents are identical. With this assumption, alone, we can deduce the following, important conclusion:

Lemma (Identical Agents): When agents are identical they act alike.

Proof: Suppose that agents are identical, but do not act alike. A contradiction is implied. Hence, if agents are identical they must act alike.

The proof of this lemma relies on semantics, but its logic should be enough, for the moment, to persuade us of the results that follow. The lemma has some strong implications for models in which strategic interaction is depicted. Conjectural variations are, of course, an example of such a model. When agents conjecture consistently, and are assumed to be identical, a specific pattern of interactions is implied. In the debate about the private provision of public goods, most of the key issues can be resolved in a straight-forward manner, from application of the lemma.

A first contention concerns the extent to which contributions by one individual are matched by contributions from the rest of the community. A related matter is the extent of free-riding, if any, which prevails in equilibrium. A third issue concerns the sign and magnitude of any consistent conjecture, should one exist, and a fourth pertains to the allocative consequences of the consistent-conjectures equilibrium. A fifth and final issue, which we wish to resolve, is whether any misallocation is exacerbated by population size. Applying the lemma, we acknowledge:

Corollary 1 (Matching): Matching is complete.

Corollary 2 (Free-Riding): There is no free riding.

Corollary 3 (Consistency): The consistent conjecture is the monopolistic conjecture.

Corollary 4 (Efficiency): The consistent-conjectures allocation is Pareto efficient.

Corollary 5 (Size): Community size is inconsequential in the consistent-conjectures equilibrium.

The first corollary follows naturally from the lemma, and the second corollary follows naturally from the first. Corollary 3 requires a little more. The intuition can be enhanced by considering an oligopoly. When a firm perceives, correctly, that its rivals are identical to itself, it knows that its adjustment will be matched entirely by a proportionate adjustment from each of its competitors. When this occurs, the firm computes the output level that maximizes industry profits, and selects own output to be one N^{th} of that which is optimal for the industry as a whole. It does this knowing full well that each of its identical rivals will follow suit. In this manner, the industry behaves as though it were a perfect cartel. The output level at equilibrium is the monopolistic level, and Corollary 3 is proven. Corollary 4 is obtained through similar reasoning. Replace the words "industry profits" in the previous argument, with the words "community welfare," and note: Since preferences are identical, cartel contributions to the public good maximize community welfare. It follows that the consistent-conjectures allocation with identical agents is Pareto efficient. Since these results hold independently from the number of agents assumed in equilibrium, Corollary 5 is obtained.

That the conjectural-variations allocation is efficient, requires two assumptions. The first is that the agents are identical. The second is that conjectures are consistent. It is the joint interaction of these two features of the model that engenders efficiency in the equilibrium allocation. In general, when either of these assumptions is relaxed, the allocation will be inefficient.

The second of these assumptions is defensible on two accounts. The first follows from the wide-spread use of models in which agents possess perfect foresight. There can be little debate that this usage stems from the acceptance of rational expectations as an equilibrium concept in stochastic theory. Incompatible with this tenet, is a situation in which agents hold conjectures that are systematically unfulfilled. That conjectures should be consistent, seems quite plausible. The same, however, cannot be said for the other assumption: It is a difficult task to locate a real economy in which the agents are demonstrably similar. A natural question then arises: How robust are the conclusions drawn above to relaxing this second assumption ? In order to answer, a more formal account of equilibrium is required. One is presented below.

4. Formal arguments

For the purpose of comparing the identical-agents equilibrium with its counterpart, the presentation below assumes heterogeneous agents. Homogeneity is invoked as required.

4.1 The model

There are N individuals indexed $\{ i \;\; i=1..N \}$. Each consumes a quantity, $\{ y_i \;\; i=1..N \}$, of a private good, and makes a contribution, $\{ x_i \;\; i=1..N \}$, to a pure, public good, x. The quantity of the public good and the individual contributions are related through the aggregation condition:

$$x = \sum_{i=1}^{N} x_i. \tag{1}$$

Utility $\{ u_i \; i=1..N \}$ is derived from consumption of the private good and from the aggregate stock of the public good:

$$u_i \;=\; U_i(\, y_i\, ,\, x\,), \qquad\qquad i=1..N. \tag{2}$$

About the functions $\{\, U_i(\cdot)\; i=1..N \,\}$, we make the usual assumptions, namely:

ASSUMPTION 1 (Monotonicity): $\partial U_i(\cdot)/\partial y_i > 0$ *and* $\partial U_i(\cdot)/\partial x > 0$, $i=1..N.$

ASSUMPTION 2 (Quasi-Concavity):
$$\begin{vmatrix} 0 & \partial U_i(\cdot)/\partial y_i & \partial U_i(\cdot)/\partial x \\ \partial U_i(\cdot)/\partial y_i & \partial^2 U_i(\cdot)/\partial y_i^2 & \partial^2 U_i(\cdot)/\partial y_i \partial x \\ \partial U_i(\cdot)/\partial x & \partial^2 U_i(\cdot)/\partial y_i \partial x & \partial^2 U_i(\cdot)/\partial y_i^2 \end{vmatrix} > 0, \qquad i=1..N.$$

ASSUMPTION 3 (Essentiality):
$$\lim_{y_i \to 0} \; -\frac{\partial U_i(\cdot)/\partial x}{\partial U_i(\cdot)/\partial y_i} = 0 \quad \text{and} \quad \lim_{x \to 0} \; -\frac{\partial U_i(\cdot)/\partial x}{\partial U_i(\cdot)/\partial y_i} = -\infty, \qquad i=1..N.$$

Assumptions 1 and 2 imply that indifference curves in the y_i-x plane are smooth and convex to the origin, reflecting declining marginal rates of substitution. Assumption 3 implies that, compared to a consumption bundle containing nothing of either of the two goods, agents strictly prefer any alternative bundle containing positive amounts of both goods.

Let $\{\, \sigma_i\; i=1..N \,\}$ denote a set of income endowments, which are strictly positive and finite, and assume that goods are measured such that prices are one. Then,

$$\mathcal{B}_i \;\equiv\; \{\; (\, y_i\, ,\, x_i\,) \;|\; y_i + x_i \le \sigma_i\, ,\; y_i \ge 0\, ,\; x_i \ge 0 \;\}, \qquad\qquad i=1..N, \tag{3}$$

define the budget sets. Assumption 1 implies that agents select bundles in $\mathcal{B}_i$ that lie on their budget constraints. When deciding on their contributions $\{\, x_i\; i=1..N \,\}$, they form *conjectural variations*. These are expressions of the form $\{\, x_j = \chi_{ij}(\, x_i\,)\; j=1..N \,\}$, which depict perceptions about the relationship between own contributions and those of the remaining members of the community. Each agent solves:

$$
\text{Problem 1:} \quad \left\{ \begin{array}{ll} \max_{y_i, x_i}: & u_i = U_i(y_i, x) \\ \text{subject to:} & \\ & \sigma_i = y_i + x_i \\ & x = \sum_{j=1}^{N} x_j \\ & x_j = \chi_{ij}(x_i) \quad j=1..N \end{array} \right. \qquad i=1..N. \tag{4}
$$

To each conjectural variation corresponds a *conjectural elasticity*, or, simply, a *conjecture*, depicting perceptions about rates of response. For notational simplicity I include $\{ \chi_{ii}(x_i) \equiv x_i \; i=1..N \}$ depicting each agent's perception of itself. Let $\{ \hat{\theta}_{ij} \equiv (\partial\chi_{ij}(\cdot)/\partial x_i)(x_i/\chi_{ij}(\cdot)) \; i,j=1..N \}$ denote *i's* perception about *j's* response, when agent *i* adjusts its contribution to the public good. If agents observe contribution levels and know how to count, equation (1) implies a set of correspondences $\{ x = \chi_i(x_i) \equiv \sum_j \chi_{ij}(x_i) \; i=1..N \}$, relating own contributions to the community aggregate, and another set $\{ \hat{\theta}_i \equiv \sum_j (x_j/x)\, \hat{\theta}_{ij} \; i=1..N \}$, relating the inter-agent conjectures, to the set of aggregate ones $\{ \hat{\theta}_i \equiv (\partial\chi_i(\cdot)/\partial x_i)(x_i/\chi_i(\cdot)) \; i=1..N \}$. By assuming that agents know how to count, we avoid some substantial algebra. However, the correspondences above should be kept in mind as we proceed.

Making appropriate substitutions, the constrained optimization in Problem 1 can be reduced to:

$$
\text{Problem 2:} \qquad \max_{x_i}: \quad U_i(\sigma_i - x_i, \chi_i(x_i)), \qquad i=1..N. \tag{5}
$$

The corresponding first-order conditions are:

$$
-\partial U_i(\cdot)/\partial y_i + \partial U_i(\cdot)/\partial x \times \partial\chi_i(\cdot)/\partial x_i \equiv \partial U_i(x_i \mid \sigma_i)/\partial x_i = 0, \qquad i=1..N. \tag{6}
$$

The appearance of the conjectures $\{ \partial\chi_i(\cdot)/\partial x_i \; i=1..N \}$ reflects a departure from the usual Nash rule, which equates the marginal rate of substitution to the price ratio. Unless the conjectures are zero or infinite, Assumptions 1-3 guarantee an interior solution to (5). Applying some standard manipulations, the first-order conditions in (6) can be rewritten:

$$
\hat{\theta}_i \; \frac{x}{x_i} \; \mathrm{MRS}(y_i, x) = 1 \qquad i=1..N, \tag{7}
$$

where $MRS(y_i, x) \equiv \partial U_i(\cdot)/\partial x \div \partial U_i(\cdot)/\partial y_i$.

In this context, the static Nash equilibrium is characterized by the set of conjectures $\{ \hat{\theta}_i = x_i/x \;\; i=1..N \}$, but many other outcomes are feasible. This, of course—if nothing else—is the redeeming feature of conjectural variations, but it also leads to an 'embarrassment of riches.' Which of the possible solutions to this problem will prevail in equilibrium ? In other words, which values for the conjectures $\{ \hat{\theta}_i \;\; i=1..N \}$ and the community shares $\{ x_i/x \;\; i=1..N \}$ are consistent with equilibrium ?

The presentation in this paper parallels the situation in the static theory of oligopoly. There the domains of the conjectures are contained between two familiar extremes. One is the set $\{ \hat{\theta}_i = 0 \;\; i=1..N \}$, which synthesizes competitive behavior. The other, $\{ \hat{\theta}_i = 1 \; i=1..N \}$, depicts monopoly. In the first case the agents conjecture that their behaviour has an imperceptible impact on industry output. In the second, they conjecture that adjustments are matched proportionately by the rest of the community. An intermediate point of interest is the equilibrium with Cournot conjectures, $\{ \hat{\theta}_i = x_i/x \;\; i=1..N \}$, which are consistent with the Nash rule presented above.

Unlike oligopoly, the present application—which is somewhat novel—precludes use of familiar paradigms to restrict the domains of the conjectures. However, from (7), the domains can be bounded by applying some reasonable assumptions. First, negative conjectures can be ruled out by invoking Assumptions 1-3. Monotonicity implies that the marginal rate of substitution is positive. Essentiality implies that, at equilibrium, contribution levels are also strictly positive. They are also finite, since they are bounded by finite endowments of income. It follows that the conjectures cannot be negative at the equilibrium depicted by (7). Neither can they be zero, because convexity of the indifference curves implies that the marginal rate of substitution is finite at the interior optimum. It follows that, in equilibrium, the conjectures must satisfy $\{ \hat{\theta}_i > 0 \;\; i=1..N \}$.

To further restrict the domains of the conjectures, an additional criterion is required. We will use consistency. That is, given a set of actual responses $\{ \theta_i \;\; i=1..N \}$, about which the agents conjecture $\{ \hat{\theta}_i \;\; i=1..N \}$, we will use the conditions $\{ \hat{\theta}_i = \theta_i \;\; i=1..N \}$ to identify equilibrium. Moreover, we will prove it to be unique. Before turning to examine this issue in detail, we should characterize efficiency in the context of (7), and compare the Pareto-efficient allocations with those derived in Nash equilibrium.

4.2 Comparing Allocations

To obtain the set of Pareto-efficient allocations, select { y_i i=1..N } and x to solve:

$$\text{Problem 3:} \quad \begin{cases} \text{max:} \quad u_i = U_i(y_i, x) \\ \text{subject to:} \\ \quad U_j(y_j, x) \geq \bar{u}_j \qquad j=1..N \; j \neq i \\ \quad \sum_{i=1}^{N} y_i + x \leq \sum_{i=1}^{N} \sigma_i \end{cases} \qquad (8)$$

Under Assumptions 1-3, the first-order conditions corresponding to this problem are sufficient for a maximum. The set of efficient allocations is characterized by the condition that the sum of marginal rates of substitution equates with the price ratio. That is,

$$\sum_{i=1}^{N} MRS(y_i, x) = 1, \qquad (9)$$

characterizes the set of efficient allocations. Summing over the N agents in (7), the corresponding conditions in Nash equilibrium are:

$$\sum_{i=1}^{N} MRS(y_i, x) = N. \qquad (10)$$

The corresponding condition under conjectural variations is:

$$\sum_{i=1}^{N} MRS(y_i, x) = \sum_{i=1}^{N} \frac{x_i/x}{\hat{\theta}_i}. \qquad (11)$$

Comparing (9), (10) and (11), an index of economic inefficiency is implied by the extent to which the right-hand sides of these equations depart from the value one. Clearly, the extent of departure in Nash equilibrium increases with the number of agents in the community, which is the Olson conjecture. Under conjectural variations, this need not be the case. In general, the extent of departure depends on the magnitude of each agent's conjecture in relation to its share in community contributions to the public good. An important feature of (11), which does not appear to be recognized previously, is that the conjectural-variations equilibrium may, in fact, lead to an allocation in which *too much* of the public good is supplied.

What is the extent of departure from an efficient allocation under consistent conjectures ? We turn now to examine this issue.

4.3 Equilibrium

Before examining consistency in detail, a few comments about the equilibrium in (7) are in order. The first is that it is static. It is static in both the Nash situation and in the general, conjectural-variations setting. In both cases it is assumed that the first-order conditions in (6) define a strict local maximum on the interior of each agent's budget set, and that the corresponding second-order conditions hold with strict inequality. In Nash equilibrium, Assumptions 1 and 2 are sufficient to guarantee that these conditions hold, but they are insufficient, in general, under conjectural variations. We will assume, momentarily, that they are satisfied and, subsequent to deriving consistent conjectures, ascertain the further restrictions that are required in order for them to hold.

When the second-order conditions are met, an equilibrium is defined by (1) and (6) in the optimal contributions from each of the agents, $\{ x_i^* \; i=1..N \}$, and the aggregate stock of contributions, x^*. That is, given a set of aggregate endowments $\{ \sigma_i \; i=1..N \}$, equations (6) determine the set $\{ x_i^* \; i=1..N \}$, from which equation (1) then determines x^*.

A focus on consistent conjectures, requires that we examine comparative statics. Consequently, the equilibrium is no longer static. There are now two phases to the game. In the first stage, firms form conjectures and select contribution levels, conditional on the responses they expect from their peers. This aspect of the equilibrium is encompassed by (6). An important point to recognize is that, given the information available to them, each individual makes its contribution in an optimal manner. There is, then, no tendency for adjustment until some force, exogenous to equilibrium, displaces the variables in (6) from their initial levels. When adjustments occur, agents observe the responses of their peers and compare these to the ones they conjectured in the initial equilibrium. When the conjectures and the adjustments conform, we say that conjectures are *consistent*. However, the equilibrium concept possesses an additional subtlety. The responses computed around the equilibrium will depend on the values of all parameters in the model. The conjectures { $\hat{\theta}_i$ i=1..N } comprise a subset of these parameters. It follows that the solution to a set of fixed-point equations is implied.

This problem received a good deal of attention in the oligopoly literature in the 1980's. However, both a general characterization of it, and a general solution to it, have proved elusive. The proposed methodology in this literature, is to compute the response between any pair of firms, *ceteris paribus*, by applying the implicit function theorem to one of the firm's first-order conditions. Unfortunately, this procedure is flawed. The reason stems from the definition of the conjectural variations { $\chi_{ij}(x_i)$ j=1..N j≠i }. Since agent *i* has *conjectures* about the contributions of each of its peers, the *contribution levels* { x_j j=1..N j≠i }, no longer appear in agent *i's* objective function. It follows that these contribution levels are absent from the corresponding first-order conditions. Consequently, the traditional methodology cannot be applied because it requires us to perturb contribution levels of two agents, but only one appears. Any approach that perturbs the contribution level of a peer in the first-order conditions is inconsistent with the conjectural-variations paradigm. An alternative approach is required.

The method used below is more akin to traditional equilibrium analysis: Equations (1) and (6) are assumed to determine an equilibrium in the N+1 endogenous variables { x { x_i i=1..N }}, given the N endowments { σ_i i=1..N }. When the endowments are displaced, we compute the changes in contributions which occur, and then proceed to find values for the conjectures that equate the observed adjustments with the ones conjectured in the initial equilibrium.

4.4 Disequilibrium

When displacing the equilibrium, it will prove convenient to express adjustments in proportional-change terms. That is, for some variable, v, let $\tilde{v} \equiv \Delta v/v$ denote its proportional change. Applying this calculus in (1) and (6), the adjustments { $\tilde{x}$, { $\tilde{x}_i$ i=1..N }}, which emanate from the shocks { $\tilde{\sigma}_i$ i=1..N }, are:

$$\tilde{x} = \sum_{i=1}^{N} \frac{x_i}{x} \tilde{x}_i , \qquad (12)$$

$$\upsilon_i \tilde{x}_i + \mu_i \tilde{\sigma}_i = 0, \qquad i=1..N, \qquad (13)$$

where the parameters $\upsilon_i \equiv x_i \partial^2 U_i(\cdot)/\partial x_i^2 \equiv \upsilon_i(\hat{\theta}_i)$ and $\mu_i \equiv \sigma_i \partial^2 U_i(\cdot)/\partial x_i \partial \sigma_i \equiv \mu_i(\hat{\theta}_i)$ depend, implicitly, on the conjectures.

4.5 Consistent conjectures

From (12) and (13) we can compute the ratios { $\tilde{x} / \tilde{x}_i \equiv \theta_i$ i=1..N } and { $\tilde{x}_j / \tilde{x}_i \equiv \theta_{ij}$ i,j=1..N }, which are rates of change in contributions relative to one's own. It is these sets of adjustments about which the agents form their conjectures. For immediate purposes, however, only the aggregate effects are of interest. A set of conjectures are *consistent* if they solve the fixed-point problem that equates the subjective perceptions { $\hat{\theta}_i$ i=1..N } to the set of true values { θ_i i=1..N }. Formally:

Definition (A Consistent-Conjectures Equilibrium): A consistent-conjectures equilibrium is a set of contributions { x^* { x_i^* i=1..N }} that satisfy (1) and (6), a set of adjustments { $\tilde{x}$ { $\tilde{x}_i$ i=1..N }} that satisfy (12) and (13), and a set of solutions { $\hat{\theta}_i$ i=1..N } to the fixed-point problem:

$$\hat{\theta}_i = \frac{\tilde{x}\,(\,\hat{\theta}_1 .. \,\hat{\theta}_N\,)}{\tilde{x}_i(\,\hat{\theta}_i\,)} \qquad i=1..N.$$

4.6 Results

We are now in a position to state formally the conclusions that were drawn in section 3 under the assumption that the agents are identical. The first task is to prove the lemma, which is restated as follows:

Lemma (Identical Agents): When agents are identical they act alike.

Proof: The first-order conditions corresponding to the problem of choosing x_i to maximize { $U_i(\sigma_i - x_i\,, \chi_i(x_i))$ i=1..N } are $\partial U(x_i \mid \sigma)/\partial x_i = 0$ i=1..N. When the functions $U_i(\cdot) \equiv U(\cdot)$ and $\chi_i(x_i) \equiv \chi(x_i)$ are the same for all individuals, and the endowments $\sigma_i \equiv \sigma$ are also the same, these first-order conditions are identical for all agents. When the corresponding second-order conditions hold with strict inequality, these equations define a set of unique solutions: $x_i^* = x_j^*$ $\forall$ i,j=1..N. Since each first-order condition is the same, a small displacement around the equilibrium yields identical adjustments $\tilde{x}_i = \tilde{x}_j$ $\forall$ i,j=1..N. Consequently, when agents are identical they must act alike.

Since $x_i^* = x_j^*$ $\forall$ i,j=1..N, contributions by each individual are completely matched by the contribution from the rest of the community. This proves the first two corollaries, namely:

Corollary 1 (Matching): Matching is complete.

Corollary 2 (Free-Riding): There is no free riding.

Deducing the third corollary requires two steps. The first is to compute the ratios of adjustments in the identical-firms equilibrium. These are, respectively $\{ \tilde{x} \; / \; \tilde{x}_i \equiv \theta_i = 1 \;\; i=1..N \}$. Consequently, if the conjectures are consistent, they are the ones $\{ \hat{\theta}_i = 1 \;\; i=1..N \}$. The second step is to confirm that these values are compatible with strict inequality of the second-order conditions corresponding to (6). We shall demonstrate this subsequently. Since the conjectures $\{ \hat{\theta}_i = 1 \;\; i=1..N \}$ are consistent with monopolistic equilibrium, we have:

Corollary 3 (Consistency): The consistent conjecture is the monopolistic conjecture.

The fourth corollary follows as a simple matter of comparing the allocation given by (9) with that in (11). Since the shares sum to one, observe that (11) conforms with (9) when $\{ \hat{\theta}_i = 1 \;\; i=1..N \}$ are employed in the right-hand side of the former. Consequently, we have:

Corollary 4 (Efficiency): The consistent-conjectures allocation is Pareto efficient.

Finally, observing that N no longer appears in the right side of (11), proves:

Corollary 5 (Size): In the consistent-conjecture equilibrium community size is inconsequential.

This last result must be interpreted with some care, as we shall see below.

4.7 Existence

It remains to examine whether a consistent-conjectures equilibrium actually exists. This is established with reference to the second-order conditions, which appear as part of the comparative statics in (13). The adjustment in each agent's contribution emanating from a change in its endowment can be expressed:

$$\tilde{x}_i \;=\; - \frac{\mu_i(\hat{\theta}_i)}{\upsilon_i(\hat{\theta}_i)} \tilde{\sigma}_i, \qquad i=1..N, \tag{14}$$

and rewritten:

$$\tilde{x}_i = - \frac{-U_i(\cdot)_{yy} + U_i(\cdot)_{xx}\, \hat{\theta}_i x/x_i}{U_i(\cdot)_{yy} - 2\, U_i(\cdot)_{yx}\, \hat{\theta}_i x/x_i + U_i(\cdot)_{xx}\, \hat{\theta}_i x/x_i - U_i(\cdot)_x\, \chi_i(\cdot)_{xx}} \times (\sigma_i / x_i)\, \tilde{\sigma}_i \qquad (15)$$

where $U_i(\cdot)_{yy} \equiv \partial^2 U_i(\cdot)/\partial y_i^2$, $U_i(\cdot)_{yx} \equiv \partial^2 U_i(\cdot)/\partial y_i \partial x$, $U_i(\cdot)_x \equiv \partial U_i(\cdot)/\partial x$, and $\chi_i(\cdot)_{xx} \equiv \partial^2 \chi_i(\cdot)/\partial x_i^2$, i=1..N. The expression in the denominator on the right-hand side is the second-order condition. Its sign depends on three components: the values of the conjectural variations in relation to the market shares, the specific nature of preferences, and the interaction of marginal utility with the curvature of the conjectural variations. It can be shown that in the symmetric, consistent-conjectures equilibrium, the consistent conjectural variations are affine. In this case, the second-order conditions become:

$$U_i(\cdot)_{yy} - 2\, U_i(\cdot)_{yx}\, N + U_i(\cdot)_{xx}\, N \equiv \partial^2 U_i(\, x_i \mid \sigma_i\,)/\partial x_i^2 < 0, \qquad i=1..N. \qquad (16)$$

By Assumption 2, these conditions are satisfied when the equilibrium contains one individual. In this case, the conditions in (16) are identical to those in Nash equilibrium. However, for $N \geq 2$, quasi-concavity alone is insufficient to guarantee satisfaction of (16). Consequently, the restrictions on preferences that ensure existence under consistent conjectures are more stringent than those in Nash equilibrium.

4.8 Discussion

The set-up above is formally equivalent to the one considered in my recent working paper (Holloway, 1995). There I characterize a general solution to the consistent-conjectures problem for a general, N-firm oligopoly, in which the agents are dissimilar. A key step in the solution procedure is to formalize a condition that is necessary for consistency, and which leads, ultimately, to a solution to the problem. This condition is:

$$\sum_{i=1}^{N} \frac{x_i/x}{\hat{\theta}_i} = 1. \qquad (17)$$

This expression appears in equation (11), where we characterize the allocation under conjectural variations. Since this condition is *necessary* for consistency it proves an additional property of the equilibrium, namely:

Corollary 6 (Totality): Every consistent-conjectures allocation is Pareto efficient.

The intuition for this result is the same as that employed in the identical-agents setting: When agents conjecture consistently, they correctly internalize the behaviour of their peers.

In my working paper I derive two sets of Nash strategies, which provide solutions to (17). One is the set of collusive conjectures, $\{ \hat{\theta}_i = 1 \; i=1..N \}$. The other is the set $\{ \hat{\theta}_i = Nx_i/x \;\; i=1..N \}$, which contain conjectures N times the so-called Cournot conjectures. However, in this oligopoly setting, existence is unattainable when the firms are heterogeneous. Consequently, the identical-firms, consistent-conjectures equilibrium is unique.

A similar result can be demonstrated here, with reference to equations (15). The logic follows from observing the conditions $\{ \theta_i \equiv \tilde{x} / \tilde{x}_i = \hat{\theta}_i \;\; i=1..N \}$, which are necessary for consistency. These conditions imply that the set of consistent adjustments must satisfy $\hat{\theta}_i \; \tilde{x}_i = \phi \;\; \forall \; i=1..N$, where ϕ is some common, non-zero scalar. That is, the adjustments must be inversely related to the conjectures, and of the form $\tilde{x}_i = \hat{\theta}_i^{\,-1} \; \phi \;\; \forall \; i=1..N$. From inspection of (15), no conjectures $\{ \hat{\theta}_i = Nx_i/x \;\; i=1..N \}$ satisfy these conditions and, hence, the identical-agents equilibrium is unique. Accordingly, we conclude as follows:

Corollary 7 (Uniqueness): The identical-agents, consistent-conjectures equilibrium is unique.

4. Concluding Comments

We began this investigation by citing four objectives. The first was to demonstrate that much of the debate about the private provision of public goods can be resolved, deductively, from an assumption embedded in the equilibrium. With the exception of the existence conditions, the results above confirm this claim: A great deal can be deduced from exploring the implications that agents are identical. The second objective was to present a comprehensive treatment of conjectural variations applied to the private provision of public goods. We did this for the heterogeneous-agents model, but demonstrated that no equilibrium exists unless individuals are identical. To argue this I invoked some recent results on conjectural variations in oligopoly and, thus, provided a formal link between the use of conjectural variations in the present context, and their foundations in the static theory of oligopoly. This was the third objective. The fourth was to resolve several key issues in the debate about the private provision of public goods. In conclusion, a consistent-conjectures equilibrium exists. It is the monopolistic-conjectures equilibrium. Each of the agents acts identically, contributing positive amounts of the public good in an efficient allocation of resources. There is complete matching of contributions, no free-riding, and the allocation is independent of the number of agents in the community. Thus the Olson conjecture—that inefficiency is exacerbated by community size—is misplaced in a consistent-conjectures context.

More generally, the results of this paper have something to say about models of strategic interaction, in which the agents are identical. Identical agents must act alike. This indicts individuals to a specific interaction, which, like the assumption itself, is quite unreasonable. Moreover, the fact that individuals *are* dissimilar provides the compelling basis for considering strategic interaction in the first place. In the conjectural-variations context, the stringency of the consistency criteria precludes use of the model in this more realistic setting. A good deal of caution is advisable when interpreting results under the identical-agents assumption. Relaxing this condition is surely a fruitful focus for future research.

References

Bowley, A. L., 1924, *The Mathematical Groundwork of Economics*, Oxford University Press.

Cornes, R. and T. Sandler, 1984, The theory of public goods: Non-Nash behaviour, Journal of Public Economics 23, 367-79.

Cornes, R. and T. Sandler, 1985, On the consistency of conjectures with public goods, Journal of Public Economics 27, 125-29.

Costrell, R. M., 1991, Immiserizing growth with semi-public goods under consistent conjectures, Journal of Public Economics 45, 383-89.

Holloway, G. J., 1995, Conjectural variations with fewer apologies, Working paper No. 95-1, University of California-Davis, Davis, California.

Scafuri, A., 1988, On the consistency of conjectures in the private provision of public goods, Journal of Public Economics 37, 395-98.

Sugden, R., 1985, Consistent conjectures and voluntary contributions to public goods: Why the conventional theory does not work, Journal of Public Economics 27, 117-124.