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WP 91-9

UCD Department of Agricultural Economics

WORKING PAPER SERIES

University of California, Davis
Department of Agricultural Economics

POSITIVE MATHEMATICAL PROGRAMMING

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Working Paper No. 91-9

Positive Mathematical Programming

Introduction

This paper is a methodology paper for practitioners rather than theorists. Instead of a new method that requires additional data, the paper takes a new look at an old method, programming models, using a minimal data set in a more flexible manner than the traditional linearly constrained production activities. Sometimes new methodologies are published, but not implemented; over the last seven years positive mathematical programming (PMP) has been implemented on several applied policy models, at the sectoral, regional, and farm level, (Bauer and Kasnakoglu (1988), Hatchett *et al.* (1991), House (1987), Oamek and Johnson (1991), Quinby and Leuck (1988)). but the methodological basis for the approach has not been published. This paper aims to show that the PMP approach, can use the data needed to construct an LP or QP model in a more flexible manner, while generating self calibrating models of agricultural production and resource use that are consistent with micro theory, and prior estimates of demand and supply elasticities.

Programming models are still widely used for agricultural economic policy analysis, despite their relegation to a methodological backwater in the past decade. Their persistency can be attributed to several characteristics. First, they can be constructed from a minimal data set. In many cases, analysts are required to construct models for systems where a respectable time series of data is absent, or inapplicable due to structural changes in a developing or shifting economy. Second, the constraint structure inherent in programming models is well suited to characterizing resource, environmental or policy constraints. In some cases, a set of inequality constraints such as those found in the farm bill commodity program strongly influence crop and resource allocation. Third, the Leontief production technology inherent in most programming models has an intrinsic appeal of input determinism when modelling farm production (Just, Zilberman, and Hochman (1983)). In addition, the concept of fixed proportions of some inputs to the land allocation has been getting increasing empirical support from recent results on the Von Liebig production function (Paris and Knapp (1989), Grimm *et al.* (1987)), and on a behavioral basis from Just *et al* (1990), Wichelns and Howitt (1991)).

The paper opens with a brief review of past approaches to calibrating programming models of farm production and their problems. A quadratic total cost function in land is shown to be a sufficient condition for the observed input allocation. The first order conditions for land allocation are shown to be linked to the dual values on "flexibility" constraints bounding the land allocations under a linear cost specification. The derivation of crop and region specific cost functions from the duals, the first order conditions, and the base data is shown. The following section addresses some problems encountered in empirical production model building, and shows how the PMP specification results in a smooth and continuous response to parameterization of the model. The paper ends with a brief description of a menu driven model generator that greatly simplifies the construction and use of PMP, QP, and LP agricultural policy models.

While the production and cost specification implied by the PMP specification is unconventional, the method works, in that it automatically calibrates models without using "flexibility" constraints. The resulting models are more flexible in their response to policy changes, and priors on the supply elasticities can be specified. With modern algorithms and microcomputers, the resulting quadratic programming problems can be readily solved.

Calibration Problems in Programming Models

In the absence of a data base for estimation, programming models should calibrate against a base year or an average over several years. Policy analysis based on normative models that show a wide divergence between base period model outcomes and actual production patterns is generally unacceptable. But models that are tightly constrained can only produce that subset of normative results that the calibration constraints dictate. The policy conclusions are thus bounded by a set of constraints that are expedient for the base year but often inappropriate under policy changes. This problem is exacerbated when the model is built on a regional basis with very few empirical constraints but a wide diversity of crop production.

Previous researchers such as Day (1961) have attempted to provide added realism by imposing upper and lower bounds to production levels as constraints. McCarl (1982) advocated a decomposition methodology to reconcile sectoral equilibria and farm level plans.

Meister, Chen, and Heady (1978) in their national quadratic programming model, specify 103 producing regions and aggregate the results to ten market regions. Despite this structure, they note the problem of overspecialization:

> If all producing activities are defined by single product activities, as assumed by most theoretical analyses, . . . the tendency of the programming model to produce only one type of commodity in a region or area increases.

The authors suggest the use of rotational constraints to curtail the overspecialization and reflect the agronomic nature of production. However, it is comparatively rare that agronomic practices are fixed at the margin, but more commonly reflect net revenue maximizing trade-offs between yields, costs of production, and externalities between crops. In this latter case, the rotations are themselves a function of relative resource scarcity, output prices, and input costs.

Hazell and Norton (1986) suggest six tests to validate a sectoral model. The capacity test, for over constrained models. The marginal cost test to ensure that the marginal costs of production including the implicit opportunity costs of fixed inputs are equal to the output price. A comparison of the dual on land with actual rental values. Three comparisons of input use, production level and product price tests are also advocated. Hazell and Norton show that the percentage absolute deviation for production and acreage over five sectoral models ranges from 7 percent to 14 percent deviation. The constraint structures needed for this validation are not defined.

In contrast, the PMP approach aims to achieve exact calibration in acreage, production and price. When the PMP approach was applied to one of the models listed by Hazell and Norton, namely TASM, the resulting PMP version of TASM calibrated exactly with the base year, Bauer and Kasnakoglu (1988) and showed consistency in the parameters over the seven years used for calibration.

The calibration problem in farm level, regional, and sectoral models stems from the common condition where the number of binding constraints in the optimal solution are less than the number of nonzero activities observed in the base solution. This is especially prevalent where the constraints represent allocatable inputs, actual rotational limits and policy constraints. Due to the rank condition on the basis matrix, the resulting optimal solution will suffer from overspecialization of production activities.

A root cause of these problems is that linear programming was originally used as a normative farm planning method where full knowledge of the production technology is assumed. Under these conditions any production technology can be represented as linear Leontief, subject to resource and piecewise separable constraints. This normative approach is forced into over simplification of the production and cost technology for more aggregate policy models due to inadequate knowledge of the production and cost technology. In most cases, the only regional production data is an average or "representative" figure for crop yields, and inputs. This common data situation means that the analyst using linear production technology in programming models is attempting to estimate behavioral reactions to policy changes, based on marginal conditions, from average data observations. Only where the policy range is small enough to admit linear technology over the whole range, can the average conditions be assumed to be equal to the marginal conditions.

Two broad approaches have been used to reduce the specialization errors in optimizing models. The demand based methods have used a range of methods to add risk or endogenize prices. These have reduced the problem, but in many models, substantial calibration problems remain.

A common alternative approach is to constrain the crop supply activities by rotational or flexibility constraints or step functions over multiple activities. In regional and sectoral models of farm production, the number of empirically justifiable constraints are comparatively few. Land area and soil type are clearly constraints, as is water in some irrigated regions. Crop contracts and quotas, breeding stock, and perennial crops are others. However, it is rare that some other traditional programming constraints such as labor, machinery, or crop rotations are truly restricting to short-run marginal production decisions. These inputs are limiting, but only in the sense that once the current availability is exceeded, the cost per unit output increases due to overtime, increased probability of disease, or machinery failure. In this situation the analyst has a choice. If the assumption of linear cost (production) technology is retained, the observed output levels infer that additional binding constraints on the optimal solution should be specified. Comprehensive rotational constraints are a common example of this approach. An alternative explanation is that the cost functions are nonlinear in land (scale) for most crops, and the observed crop allocations are a result of a mix of unconstrained and constrained optima. The nonlinear costs, as a function of acreage allocated to a particular crop, can be explained by several causes, but the most common reasons are risk aversion, a nonlinear production function due to heterogeneous land quality, or increasing costs per unit output due to restricted management or machinery capacity.

Since there is a long and exhaustive literature on the addition of risk terms to linear programming models which result in nonlinear costs, we will concentrate on calibrating from the supply side by introducing a nonlinear cost specification for each production activity. This is not to diminish the importance of risk in nonlinear objective functions, but since mean/variance risk specifications have improved, but not completely calibrated LP models, nonlinear cost functions are a useful additional calibration method.

We make the common assumption that farmers are price takers in input and output prices and maximize expected net income. Since we employ a linear-quadratic specification we can invoke the certainty equivalence principle and avoid more complex expectations structures. The revenue is linear in output and thus the concavity of the profit function in land must be contained in the cost function for those crops with interior solutions. The increase in the cost per unit output as additional acres are allocated to a crop may arise from both increased variable inputs per acre, and decreased yields per acre as crops are grown on increasingly less suitable soil types.

This paper is written using cropping activities as examples, but the same procedure can be directly applied to livestock fattening and other activities where the key input is not land but a livestock unit.

Defining the acreage of land allocated to activity i as x_i the traditional linearly constrained Leontief production function specification land and two other inputs is written as

$x_1 \quad \frac{x_2}{\alpha_2} \quad \frac{x_3}{\alpha_3}$

$$y_i = \bar{y}_i \ \mathrm{Min}(x_i,\ \alpha_{2i}x_i,\ \alpha_{3i}x_i) \tag{1}$$

where y_i is the total output for crop i, and $\bar{y}_i$ is the expected yield per acre for activity i and α_{2i}, α_{3i} are the per acre input requirement coefficients for inputs two and three.

If we observe more nonzero activities (n) in the base year than binding constraints (m), but cannot empirically justify additional binding constraints on marginal cropping activities, then it follows directly from the first order conditions that at least (n–m) of the activity profit function are nonlinear in land. The most parsimonious specification change to equation (1) is to define the yield as quadratic in land allocation and Leontief in the other two variable inputs.

$$y_i = \mathrm{Min}(\phi x_i - 1/2\, \Psi x_i^2,\ \hat{\alpha}_{2i}x_i,\ \hat{\alpha}_{3i}x_i) \tag{2}$$

where $\hat{\alpha}_{ji} = \bar{y}_i\ \hat{\alpha}_{ji}$.

Specifying different production technologies for allocatable and variable inputs, is unusual, but there is increasing empirical evidence that farmers allocate some variable inputs in a fixed proportion manner, Just *et al.* (1990). Paris and Knapp (1989). However, allocatable inputs such as land or livestock are heterogenous in quality and are unlikely to yield constant returns to scale. In addition, the specification in equation (2) has the advantage of making full use of the data set usually available for sector and regional models.

The increasing cost per unit output proposed in the PMP specification can be derived from two equivalent but alternative specifications. Using the production function (2) and taking land x as the constraining input, the profit function for activity i is:

(3) $$\pi_i = p_i(\phi\, x_i - 1/2\, \Psi x_i^2) - r_1 x_1 - r_2\alpha_{2i}x_i - r_3\alpha_{3i}x_i$$

Ignoring the opportunity cost of the land restriction for simplicity, the optimal land allocation to activity i is the interior solution where:

(4) $$x_i^* = \frac{p\phi - r_1 - r_2\alpha_{2i} - r_3\alpha_{3i}}{p\Psi}$$

Alternatively, instead of constant production costs per acre (r_1) and a decreasing yield with increasing land, the equivalent first order conditions result from a profit function specification that has constant yields per acre, but requires increasing costs per acre to achieve these yields as the acreage allocated to activity i increases.

(5) $$\pi_i = p\bar{y}x_i - (\alpha_i x_i + 1/2\gamma_i x_i^2) - r_2\alpha_{2i}x_i - r_3\alpha_{3i}x_i$$

The optimal land allocation condition is:

(6) $$x_i^* = \frac{p\bar{y} - \alpha_1 - r_1 - r_2\alpha_{2i} - r_3\alpha_{3i}}{\gamma_i}$$

Since the PMP method uses dual values on base year land allocations to solve for the calibrating parameter values α and γ, we will continue to use the nonlinear cost function specification in equation (5).

The PMP Calibration Approach

From equation (6) we see two additional parameters in the quadratic cost function α_i and γ_i are needed to calibrate the optimal x_i^*. The problem facing the modeler is to calibrate these two parameters knowing the average cost per acre from the normal LP data, and the allocation quantity x_i^* at which the marginal activity revenue is equal to the marginal variable and opportunity cost. The central feature of the approach is to use a two stage approach to calibration in which the first stage, using linear cost specifications, is constrained to be very close to the base year allocations x_i^*. If a particular decoupling procedure is used (Appendix 1), the resulting duals on the calibration constraints yield a second cost equation in α and γ that can be used with the average cost equation to solve for values of α_i and γ_i that precisely calibrate the model.

Diagrammatically, the effect of the PMP specification can be seen by comparing the cost functions on the right hand side of Figures 1 and 2. The PMP derivation tilts the fixed cost specification in Figure 1 to the increasing marginal cost specification in Figure 2. However, the α and γ parameters are calculated so that the average cost, the objective function and the dual on the land constraint are unchanged, but the marginal conditions calibrate to the base year land allocations without constraints.

The PMP method is explained using the simple two crop, one allocatable input example that is shown graphically in Figures 1 and 2. Figure 1 corresponds to the first stage of the method which uses an LP model constrained by inequality calibration constraints. The same approach is used if endogenous prices or risk costs have been specified in the objective function making the stage I problem a quadratic programming problem. In this illustrative example there are two crops, wheat and oats, and one allocatable input, land. Given the gross returns and average costs per acre for each crop, wheat is more profitable than oats, but farmers are observed to grow both wheat and oats in the base year. To calibrate to the base year acreages the problem has to be constrained by calibration constraints and the resulting problem is:

Stage I. L.P. Calibration Model

Given the basic data that (Price) x (Average Yield) of oats and wheat are denoted respectively as P_o and P_w, the average variable cost/acre of growing oats and wheat are $\bar{c}_o$ and $\bar{c}_w$ and the observed crop land allocations are:

$$\tilde{x} = \begin{bmatrix} \tilde{x}_o \\ \tilde{x}_w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The LP model is specified as:

LP Model

$$\text{Max } Z = P_o x_o - \bar{c}_o x_o + P_w x_w - \bar{c}_w x_w$$

$$\text{(7)} \qquad \begin{array}{lllll} \text{Subject to} & x_o & + x_w & \leq 5 & \text{Land.} \\ & x_o & & \leq 2 + \varepsilon & \left.\begin{array}{l} \text{Calibration} \\ \text{Constraints} \end{array}\right. \\ & & x_w & \leq 3 + \varepsilon & \end{array}$$

Without the ε perturbation on the calibration constraints the land resource constraint and both the calibration constraints would bind simultaneously and a degenerate solution would result. The resulting dual values would not be unique. The ε perturbation causes the land constraint to bind before the least profitable calibration constraint is binding. The dual values are therefore unique, but more importantly, the proof in Appendix 1 shows that the ε perturbation decouples the resource constraint set from the calibration constraints. In other words, the dual values on the calibration constraints are functions of the resource constraints, but the resource constraint dual values are not influenced by the calibration constraints. Thus, the opportunity costs of resources are used in the calibration process, but are not changed by it.

In Figure 1 the position of the resource constraint and two calibration constraints are shown by dotted vertical lines. It can be seen that the wheat calibration constraint and the land constraint will become binding first. The average return from oats (λ_1) sets the opportunity cost of land. λ_2, the marginal value on the calibration constraint for wheat is the opportunity cost of constraining wheat to three acres, given the linear costs and returns. λ_2 is equal to the difference in the marginal returns to wheat and oats under the LP specification.

want marginal return to set opportunity cost

Due to the ε perturbation, the calibration constraint for oats is slack and degeneracy is avoided.

Stage II - Derivation of the PMP Cost Functions

Since we know that the marginal cost of growing wheat must be greater than the average cost at $\tilde{x}_w$, given that the marginal net returns to wheat and oats are equal at $\tilde{x}_w$ and $\tilde{x}_0$, a quadratic cost function for wheat growing is specified. This is the simplest specification that can explain the observed behavior.

The calibration constraints are removed, and the model becomes:

(8) $$\text{Max } J = P_o x_o - \bar{c}_o x_o + P_w x_w - \alpha_w x_w - 1/2\, \gamma_w x_w^2$$

$$\text{Subject to} \qquad x_o + x_w \leq 5.$$

In Figure 2, $\alpha_w x_w - 1/2\, \gamma_w x_w^2$ is a quadratic total cost function which is derived from the dual values on the binding calibration constraints.
The term $\bar{c}_o x_o$ is the LP linear cost function which is retained for simplicity in this stage, and yields the opportunity cost for the binding land resource.

The unknown parameters α_w and γ_w can be calculated from the optimal solution of the LP problem in stage I. Since the first order conditions for allocatable inputs require that at the optimal solution, the marginal net returns to land are equal across outputs, Figure 1 shows that λ_2, the calibration dual is the difference between marginal and average cost at output level $\tilde{x}_w$.

The derivation of the two types of dual value λ_1 and λ_2, can be shown for the general case using the appendix. The stage I problem can be written in general as

(9)
$$\text{Max } f(x)$$
$$\text{Subject to} \quad Ax \leq b$$
$$Ix \leq \tilde{x} + \varepsilon$$

Partitioning A into an mxm basis matrix B and an mx(k–m) matrix N of nonbasic activities, the first partition of equation (A15) in the appendix for λ_1 is:

(10)
$$\lambda_1^* = B'^{-1} \nabla_{x_B} f(x^*)$$

where $\nabla_{x_B} f(x^*)$ is the gradient of VMPs of the vector x_B at the optimum value.

The elements of vector x_B are the acreages produced in the constrained crop group, and λ_1 is associated with the set of mx1 binding resource constraints b. Equation (10) states that the value of marginal product of the constraining resources is a function of the revenues from the constrained crops. The more profitable crops (x_N) do not influence the dual value of the resources. This is consistent with the principle of opportunity cost in which the marginal net return from the least profitable use of the resource determines its opportunity cost.

Yes, compare p. 13

The second partition of equation (A15) determines the dual values on the upper bound calibration constraints on the crops.

(11) $$\lambda_2 = -N'B'^{-1}\nabla_{x_B}f(x^*) + I\,\nabla_{x_N}f(x^*)$$
$$= \nabla_{x_N}f(x^*) - N'\lambda_1^*$$

The dual values for the binding calibration constraints are equal to the difference between the marginal revenues for these crops and the marginal opportunity cost of resources used in production of the constrained crops.

Equation (11) substantiates the dual values shown in Figure 1, where the duals for constraint set II (λ_2) in the stage I problem are equal to the divergence between the ' P average value product per acre and the sum of average cost and opportunity cost per acre. For the problem in (7) and Figure 1, the objective function does not have an increasing cost term, therefore, $\nabla_{x_N}f(x^*)$ is the average value product of land for the calibrated crop (x_w in this case). Since the opportunity cost of land is λ_1^*, and the marginal input requirement coefficients for calibrated crops, under the specification of problem P2 in the appendix is N, it follows that the term $N'\lambda_1^*$ is the value of marginal product of land.

From primary data collection we know that the average cost of wheat production is $\bar{c}_w$. The PMP objective function in equation (8) yields

(12) $$\text{Marginal Cost of Wheat} = \alpha_w + \gamma_w x_w$$
$$\text{Average Cost of Wheat} = \alpha_w + 1/2\,\gamma_w x_w$$

therefore

(13) $$\lambda_{2w} = MC_w - AC_w = \alpha + \gamma_w \tilde{x}_w - \alpha - 1/2\gamma_w \tilde{x}_w = 1/2\gamma_w \tilde{x}_w$$

therefore, $$\gamma_w = \frac{2\lambda_{2w}}{\tilde{x}_w}$$

The average cost expression is:

(14) $\bar{c}_w = \alpha + 1/2\,\gamma_w \tilde{x}_w,$ substituting in from (13)

$\alpha_w = \bar{c}_w - \lambda_{2w}.$

Using the calibration constraint dual values from stage I, we can solve equations (13) and (14) uniquely for the intercept and slope parameters that result in a quadratic optimization program that equilibriates at the base period acreage.

This approach that solves for a new cost function differs from the method developed in the working paper Howitt and Méan (1986). In this earlier paper the calibration dual values were used to add an additional nonlinear cost to the empirical average cost. As a result, the objective function values, the resource duals and the average costs of production were inconsistent with the empirical values. With the current PMP approach these values are consistent with the basic farm data.

Figure 2 shows how the LP problem in Stage I and Figure 1 is modified by substituting equations (13) and (14) into (8), and tilting the cost function so that the model is self-calibrating at the base level values, but unconstrained in its ability to respond to cost, price, or resource changes. In Figure 2 the quadratic cost function coefficients for wheat are

(15) $$\gamma_w = \frac{2\lambda_{2w}}{\tilde{x}_w + \varepsilon} \quad \text{and} \quad \alpha_w = \bar{c}_w - \lambda_{2w}$$

The key point that bears reiteration is that in Figure 2, the profit maximizing solution will allocate three acres to wheat production ($\tilde{x}_w$). At values greater than this allocation the marginal net return to land is greater in oat production, so in this example the remaining land will be allocated to oats. Note also that at $\tilde{x}_w$, the average cost of growing wheat calibrates with the observed average value of $\bar{c}_w$. At the optimal calibrated values of $\tilde{x}_w$ and $\tilde{x}_o$, the necessary condition for allocatable inputs holds, in that the marginal net return per acre for wheat is equal to the marginal net return from growing oats, and hence the opportunity cost of land.

The fundamental PMP procedure can be solved in three stages. First formulate and solve the problem as an LP (or QP) constrained by perturbed calibration constraints as in equation (7). Second, use the data on average costs of production, and the dual values for the binding calibration constraints to solve equations (13) and (14) for the nonlinear cost parameters. Those activities whose calibration constraints are not binding will be constrained by the resource constraints. Third, solve the PMP problem specified in equation (8) using the values of α and γ from the previous stage. For activities without calibration dual values, α is set equal to the average cost $\bar{c}$ and γ is set equal to zero at this point.

Extensions Using Elasticity Priors

In sectoral and regional QP models, the linear demand functions are often calibrated to a particular base year market price and quantity using a prior estimate of the elasticity of demand for the product, obtained from econometric estimates. In the same way, priors on the aggregate or regional supply elasticity can be used to augment or bound the basic PMP procedure outlined in the previous section.

There are three empirical situations in which prior knowledge of the supply elasticity can be used by a PMP model. The first case uses an elasticity value to calibrate a quadratic cost function for the lower profitability activities that provide the resource duals in stage I of the calibration. This enables the PMP model to have a quadratic cost function for all activities.

The second case is when the cost function coefficients calculated in stage two imply an unreasonably high supply elasticity. The PMP procedure enables the model builder to specify parameters that satisfy the upper bound for the elasticity.

The third case of nominal negative net returns is often encountered when using empirical farm production data. These cases can be identified and calibrated, using a prior elasticity of supply.

Case 1. Marginally Profitable Crops

In stage II of the previous example, the cost technology for the least profitable crop, oats, which sets the opportunity cost for land, remains as a linear specification. Since this marginal crop is constrained by land, we know that the condition that equates marginal revenue to the sum of marginal production and opportunity cost for the unconstrained crops, does not hold. From the observed land allocations and empirical average cost data, there simply is not enough information on these marginal crops to calibrate a quadratic cost function. Two alternatives face the modeler, leave the marginal crops with a linear cost technology, or use exogenous prior information to calibrate a quadratic total cost function for the marginal crops.

If the marginal crops are left with a linear cost technology, the model requires no prior information to calibrate exactly. However, a number of problems arise. The first difficulty is to justify the difference in cost specifications between the marginal and mainstream crops. Why should the mainstream crops have a quadratic cost technology and marginal crops have linear costs? In addition to this conceptual problem, the linear costs on the marginal crops can lead to some strange changes in marginal crop acreages for some extreme cases of parameterization.

If a prior value on the marginal crop supply elasticity is available, a quadratic cost function can be calibrated for the marginal crops as follows:

Given the fixed yields per acre, the elasticity of supply can be written in terms of acreage, marginal cost and the slope of the cost function. The quadratic total cost function is:

(16)
$$TC = \alpha x + 1/2\gamma x^2$$
$$MC = \alpha + \gamma x$$

Supply elasticity $\eta = \frac{dq}{d(MC)} \frac{MC}{q}$ can be rewritten (dropping the yield ($\bar{y}$) for simplicity) as:

(17)
$$\eta = \frac{1}{\gamma} \frac{\alpha + \gamma x}{x}$$

Using the elasticity η and the average cost $\bar{c}$ we get the two equations

(18)
$$\eta \gamma x = \alpha + \gamma x$$
$$\bar{c} = \alpha + 1/2\gamma x$$

solving for γ yields

(19)
$$\gamma = \frac{\bar{c}}{(\eta x - 1/2x)} \qquad \text{and} \qquad \alpha = \bar{c} - 1/2\gamma x$$

Thus, the quadratic cost function can be solved for α, and γ in terms of η, $\bar{c}$ and x.

Figure 3 shows that since the average cost of x* calibrates with $\bar{c}$, the empirical average cost, the marginal cost, and hence the dual value on land will be lower than in the calibration LP (Figure 1). Thus, the resulting PMP model will reach an optimum solution with the wheat acreage slightly above the base acreage, and the oats acreage slightly below. The amount that these acreages diverge from the base is proportional to the prior elasticity assigned to the marginal crop.

A three step calibration approach can be used to ensure precise acreage calibration with any specified marginal crop elasticity if the acreage deviation from the base value is excessive.

Case 2. Upper Bounds for Supply Elasticities

Substituting the values in equation (15) into equation (17), we see that a very small calibration dual (λ_2) can lead to a highly elastic supply specification. For crops whose net return per acre is only slightly above the opportunity cost of land, the calibration dual will be relatively small and the supply elasticity correspondingly large. In this case, the model builder can substitute a previously specified upper bound supply elasticity for the calibration dual and use equations (18) to solve for the supply intercept and slope coefficients. This procedure was first implemented by House *et al.* (1987) in the USMP model.

Case 3. Activities with Negative Nominal Net Returns

In agricultural data bases, gross returns minus allocated cash costs often show negative net returns to land and management in some regions or years. When this occurs, the yields, prices, and costs that generate these negative revenues should be examined closely. However, the negative net returns frequently persist. There are three aspects of farm production that would result in negative net cash returns to a crop in a particular region or year: Revenue expectations, rotational externalities, and overestimated costs. In the first case, farmers may include a crop with highly variable revenue in their output, in the expectation of positive net revenues over a longer planning horizon. Alternatively, if a relatively low revenue crop is part of an observed rotation, it may be because the crop produces positive yield effects on subsequent crops. This positive externality is not incorporated in the nominal revenues, which consequently undervalue the output from the rotational crop. Negative net returns may be due to overestimated costs. In many models variable costs are allocated by model builders across crops on a per acre basis. Labor and machinery operating costs are typical examples. However, some crops require these inputs at a time of year when there is excess capacity, and thus a lower opportunity cost. These crops are sometimes termed "filler" crops, since they may be short season crops that fill in between the more profitable crops. Under a standard method of allocating of operating costs by acres, the costs assigned to filler crops will be higher than those in the farmer's decision calculation, and the crops may be grown despite nominal negative returns.

Focusing on rotational activities, the negative nominal returns require that the calibration procedure is modified. The basic microeconomic assumption that land is allocated, so that the marginal expected net revenue equals the marginal land cost, is assumed to hold. The marginal land cost is usually composed of both cash costs and the opportunity cost for land. If a set of lower bound calibration constraints are added to the linear program, the dual values on the binding constraints will be equal to the marginal rotational benefit from the crop. The value of the benefit from cost savings or positive externalities is added to the rotational crop by shifting its average cost down so that the value of marginal product of land in "rotational" crops equals the least profitable of the crops with positive revenues

The second model assumption of increasing marginal costs with increased acreage allocated to a crop is maintained. There is no reason to suppose that rotational crops are not subject to the same effects of heterogeneous land types, risk, and fixed management inputs, that lead to increasing cost functions for normal crop activities.

In short, we assume that the cost function for these crops remains upward-sloping with quadratic total cost function, but there is a unrecorded positive externality or cost reduction associated with the crop, that makes it at least as profitable as the crop with the lowest positive return in the rotation. That is to say we assume the farmer equates his expected return from these rotational crops with the opportunity cost of land in production.

The lower bound calibration dual is negative for rotational crops and is used to derive the upward-sloping cost function. A constant correction factor k is added to the total cost function which exactly offsets the externality benefit in the objective function at the calibration acreage, and "prior" supply elasticity values are used to complete the calibration.

Using a simplified example shown in Figure 4, the "rotational" crop is a legume with an observed acreage of $\tilde{x}_L$. The nominal average cost per acre is $\bar{c}_L$ and P_L is the nominal revenue per acre. Since c_L exceeds P_L, the legume crop appears to generate negative net revenues. Assuming similarly to Figures 2 and 3, that the lowest positive revenue crop is oats, which sets the opportunity cost of land at λ_1, then the dual on the lower bound calibration constraint, λ_2 in Figure 4, will have two components. The nominal negative net revenue $(P_L - \bar{c}_L)$ and the opportunity cost of land λ_1.

(20) $$\lambda_2 = (P_L - \bar{c}_L) - \lambda_1$$

Given the quadratic total cost function for the rotational crop x_L

(21) $$TC = k + \alpha x_L + 1/2\, \gamma\, x_L^2$$

The Marginal cost per acre is:

(22) $$MC = \alpha + \gamma\, x_L$$

Three conditions characterize the cost function for the rotational crops. First, the marginal cost at the calibration acreage must equal the nominal cost minus the dual value on the lower bound calibration constraint. Note that the value of the dual is negat. . e.

(23) $$\alpha + \gamma\, \tilde{x} = \bar{c} + \lambda_2$$

Second, the supply elasticity at the calibration acreage is equal to the specified prior value η, which implies that

(24) $$\eta\gamma\, \tilde{x} = \alpha + \gamma\, x$$

Third, the calibrated quadratic total cost function is revenue-neutral at the calibration acreage $\tilde{x}$, implying

(25) $$\bar{c}\tilde{x} = k + \alpha\tilde{x} + 1/2\gamma\, \tilde{x}^2$$

Equating the marginal cost condition and the elasticity condition at the calibration acreage $\tilde{x}$ we obtain:

(26) $$\eta\gamma\, \tilde{x} = c + \lambda_2, \text{ solving for } \gamma \text{ we get}$$

(27) $$\gamma = \frac{c + \lambda_2}{\eta\tilde{x}}$$

Substituting this expression for γ into the marginal condition results in

(28) $$c + \lambda_2 = \alpha + \frac{(c + \lambda_2)}{\eta\tilde{x}}\tilde{x}$$ and rearranging yields:

$$\alpha = c + \lambda_2 - \frac{c + \lambda_2}{\eta}$$

(29) $$\alpha = (1 - 1/\eta)(c + \lambda_2)$$

From the net revenue condition, the constant term in the total cost function is solved as:

(30) $$k = (c - \alpha - 1/2\gamma\,\tilde{x})\tilde{x}$$

This calibrated quadratic total cost function results in a precise, unconstrained, and revenue-neutral calibration at the observed level and prior supply elasticity.

Policy Analysis Properties of PMP Models

In the previous sections the calibration of a positive programming model with endogenous supply costs on all activity acreages was described. The purpose of such models is to analyze the impact of quantitative policy scenarios which take the form of changes in prices, technology, or constraints on the system. The policy response of the model can be characterized by its response to sensitivity analysis and changes in constraints.

The primal PMP problem can be written in general as:

(31) $$\text{Max } J = p'x - \alpha'x - 1/2\, x'Gx$$

$$\text{subject to} \quad Ax \leq b$$

$$x \geq 0$$

where p, α, and x are nx1 vectors

G is an nxn diagonal matrix

A is mxn and b is mx1

The revenue vector p is the product of the price and average yield as in equations (7) and (8). The properties of the dual values under parametric changes to the model can be seen from the dual specification. The dual specification of the PMP model in equation (31) is:

$$\text{(32)} \qquad \text{Min} \quad \lambda'b + 1/2\, x'Gx$$

$$\text{(33)} \qquad \text{subject to} \quad A'\lambda \geq p - \alpha - Gx$$

$$\lambda \geq 0$$

Briefly interpreted, the PMP dual problem minimizes the sum of resource quasi rent ($\lambda'b$) and producer surplus ($1/2\, x'Gx$), subject to the constraint (33) that the opportunity cost of resources used to produce each product cannot be less than the marginal net revenue from that product.

Defining the optimal basis matrix in A to be of rank m, from the initial PMP conditions, the number of nonzero activities is k (k>m). It follows that k of the n rows in (33) are equalities and can be written as:

$$\text{(34)} \qquad E\lambda = \tilde{p} - \tilde{\alpha} - \tilde{G}\tilde{x}$$

where E is a kxm submatrix of A, and $\tilde{P}, \tilde{\alpha}, \tilde{x}$ are kx1 subvectors, and $\tilde{G}$ is a kxk diagonal matrix.

Defining the generalized inverse of E as E^{+}, the dual values are defined as:

$$\text{(35)} \qquad \lambda = E^{+}(\tilde{p} - \tilde{\alpha} - \tilde{G}\,\tilde{x}^{*}).$$

Equation (35) shows that the dual values are linear combinations of the price and cost parameters and the level of the nonzero activities. It follows that parameterization of the PMP problem will result in smooth continuous changes in all the optimal values of activity levels and dual values. This is in contrast to LP or step-wise problems, where the dual values, and sometimes the optimal solution are unchanged by parameterization until there is a discrete change in basis, when they jump discontinuously to a new level.

The ability to represent policies by constraint structures is important. The PMP formulation has the property that the nonlinear calibration can take place at any level of aggregation. That is, one can nest an LP subcomponent within the quadratic objective function and obtain the optimum solution to the full problem. An example of this is used in technology selection. Suppose a given regional commodity can be produced by a combination of five alternative linear technologies, whose aggregate output has a common supply function. The PMP can calibrate the supply function while a nested LP problem selects the set of linear technology levels that make up the aggregate supply (Hatchett *et al.* 1991).

Since the intersection of the convex sets of constraints for the main problem and the nested subproblem is itself convex (Marlow 1978) then the optimal solution to the nested LP subproblem will be unchanged when the main problem is calibrated by replacing the calibration constraints with quadratic PMP cost functions. The calibrating functions can thus be introduced at any level of the linear model. In some cases, the available data on base year values will dictate the calibration level. Ideally, the level of calibration would be determined by the properties of the cost functions, as in the example of linear irrigation technology selection. The PMP approach does not replace all linear cost functions with equivalent quadratic specifications, but only replaces those that data or theory suggest are best modeled as nonlinear.

Conclusions

Programming models still have a strong role to play in agricultural policy analysis, particularly for problems where time series data is absent, or the shifts in market institutions or constraints have changed substantially over time. The problem of calibrating programming models without excessive constraints is addressed in this paper. The solution proposed by the PMP approach is based on the derivation of nonlinear activity cost functions from the base year data and prior supply elasticities. The derivation is achieved by a simple two step procedure.

An analyst who is interested in direct applications can skip over the derivations and calibration steps by using a menu driven program "AgMod" (Howitt and Vayssiéres 1990). AgMod generates a GAMS program for the model specified, and automatically runs the self calibrating models, using the GAMS/Minos optimization package. The AgMod program is available from the authors.

The PMP approach is shown to satisfy the main criteria for calibrating sectoral and regional models. Using PMP, the model calibrates precisely to output and input quantities, the objective function value, and dual constraint values and output prices. In addition, the PMP approach incorporates priors on aggregate demand and supply elasticities.

The PMP method has been successfully used to calibrate a range of optimization models of different size and complexity over the past eight years. This paper has attempted to explain the economic and optimization basis for the method, and thus broaden the discussion and exposure of the approach among applied agricultural policy analysts.

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Appendix I

Proof of Constraint Decoupling

Given the degenerate problem

Problem P1

(A1) Maximize $f(x)$

subject to $\bar{A}\,x = \bar{b}$ (I)

$\hat{A}\,x < \hat{b}$

$I\;\;x = \tilde{x}$ (II)

$\bar{A} = m{\rm x}k \quad \hat{A} = (l-m){\rm x}k \quad \tilde{x} = k{\rm x}l \quad k>m \quad \bar{b} = m{\rm x}l \quad \hat{b} = (l-m){\rm x}l.$

Where f(x) is monotonically increasing in x with first and second derivatives at all points, and A is bounded and nondegenerate.

Proposition. There exists a perturbation ε of the values $\tilde{x}$ such that:

(a) The constraint set (I) in equation (A1) is decoupled from the constraint set (II) in the sense that the dual values associated with constraint set I do not depend on constraint set II.

(b) The number of binding constraints in constraint set II is reduced so that the problem is no longer degenerate.

(c) The binding constraint set I remains unchanged.

Proof. Define the perturbed problem.

Problem P2

(A2) Maximize $f(x)$

subject to $\bar{A}\,x \le \bar{b}$ (I)

$\hat{A}\,x \le \hat{b}$

$I\;\;x \le \tilde{x} + \varepsilon$ (II)

Any row of the nonbinding constraints $\hat{A}\,x < \hat{b}$ in problem P1 can be written

(A3) $$\sum_{j=1}^{k} \hat{a}_{ij}\,\tilde{x}_j < \hat{b}_i \qquad i=1,\ldots,(l-m)$$

and a constraint i will not become binding under the perturbation ε if

(A4) $$\sum_{j=1}^{k} \hat{a}_{ij}\,\varepsilon_j < \left[b_i - \sum_{j=1}^{k} \hat{a}_{ij}\,\tilde{x}_j \right]$$

Select the constraint $i = 1,\ldots,(l-m)$ such that $b_i - \sum_{j=1}^{k} \hat{a}_{ij}\,\tilde{x}_j$ is minimized. If $\varepsilon_j > 0$ $j = 1,\ldots,k$ are selected such that

(A5) $$\sum_{j=1}^{k} \hat{a}_{ij}\,\varepsilon_j < \left[b_i - \sum_{j=1}^{k} \hat{a}_{ij}\,\tilde{x}_j \right]$$

then no additional constraints in the set $A\,x \le b$ will become binding under the perturbation ε.

The invariance of the binding resource constraints for the perturbation ε can be shown using the reduced gradient approach (Luenberger 1973). Using (A5) we can write problem P2 using only constraint sets I and II.

(A6) $$\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & \bar{A}\,x \le \bar{b} \\ & I\,x \le \tilde{x} + \varepsilon \end{array}$$

where $\bar{A}$ (mxk), and I = kxk. Invoking the nondegeneracy assumption for $\bar{A}$ and starting with the solution for problem P1 $\tilde{x}$, the constraints can be partitioned

(A7) $$\begin{bmatrix} B & N \\ I_1 & \\ & I_2 \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} \begin{array}{l} = \bar{b} \\ \le \tilde{x}_B + \varepsilon \\ \le \tilde{x}_N + \varepsilon \end{array}$$

where $\bar{A} = [B : N]$, $B = m \times m$, $N = m \times (k-m)$. For brevity, we assume that the partition of $\bar{A}$ has been made so that the (k–m) activities associated with N have the highest value of marginal products for the constraining resources.

at equil all net marginal products equal. only can choose base on linear model

From P1, the resource constraints can be written

$$\text{(A8)} \qquad B\,\tilde{x}_B + N\,\tilde{x}_N = \bar{b} \qquad \text{thus,}$$

$$\tilde{x}_B = B^{-1}\bar{b} - B^{-1}N\,\tilde{x}_N$$

and f(x) can be written in terms of $\tilde{x}_N$, as $f(B^{-1}\bar{b} - B^{-1}N\tilde{x}_N, \tilde{x}_N)$ the reduced gradient for changes in $\tilde{x}_N$ is therefore:[1]

$$\text{(A9)} \qquad r_{x_N} = \nabla f_{\tilde{x}_N}(\bullet) - \nabla f_{\tilde{x}_B}(\bullet)\,B^{-1}N$$

Since $f(\bullet)$ is monotonically increasing in x_N and x_B, the resource constraints will continue to be binding since the optimization criterion will maximize those activities with a nonnegative reduced gradient until the reduced gradient is zero or the upper bound calibration constraint $\tilde{x}_N + \varepsilon$ is encountered. Since m<n, the model overspecializes in the more profitable crops when subject only to constraint set I. Under the specification in problem P2 the most profitable activities will not have a zero reduced gradient before being constrained by the calibration set II at values of $\tilde{x}_N + \varepsilon$. Thus, the binding constraint set I remains binding under the ε perturbation.

The resource vector for the resource constrained crop activities (x_B) now is:

$$\text{(A10)} \qquad \bar{b} - N(\tilde{x}_N + \varepsilon) \qquad \text{and from (A8)}$$

$$x_B = B^{-1}[\bar{b} - N(\tilde{x}_N + \varepsilon)].$$

Since B is of full rank m, exactly m values of x_B are determined by the binding resource constraints, which depend on the input requirements for the subset of calibrated crop acre values $\tilde{x}_N + \varepsilon$.

The slackness in the m calibration constraints associated with the m resource constrained output levels x_B, follows from the monoticity of the production function in the rational stage of production. Since the production function is monotonic, the input requirement functions are also monotonic, and expansion of the output level of the subset of crop acreage to $\tilde{x}_N + \varepsilon$ will have a nonpositive effect on the resource vector remaining for the vector of crop acreages constrained by the right hand side, x_B. That is:

(A11) $$\bar{b} - N(\tilde{x}_N + \varepsilon_1) \leq \bar{b} - N\tilde{x}_N \quad \text{for } \varepsilon_1 > 0$$

But since the input requirement functions for the x_B subset are also monotonic (A11) and (A8) imply that

(A12) $$x_B \leq \tilde{x}_B \qquad \text{or} \qquad x_B < \tilde{x}_B + \varepsilon_2 \qquad \text{for } \varepsilon_2 > 0.$$

From (A12) it follows that the m perturbed upper bound calibration constraints associated with x_B will be slack at the optimum solution. Given (A5) and (A12), the constraints at the optimal solution to the perturbed problem P2 are:

(A13) $$\begin{bmatrix} B & N \\ \hat{A}_1 & \hat{A}_2 \\ I_1 & \\ & I_2 \end{bmatrix} \begin{bmatrix} x_B \\ \tilde{x}_N + \varepsilon \end{bmatrix} \begin{matrix} = \bar{b} \\ < \hat{b} \\ < \tilde{x}_B + \varepsilon \\ = \tilde{x}_N + \varepsilon \end{matrix}$$

Thus, there are k binding constraints, $\bar{b}$ (mxl) and $x_n + \varepsilon$ ((k–m)xl).

The dual constraints to this solution are

(A14) $$\begin{bmatrix} B' & 0 \\ N' & I_2 \end{bmatrix} \begin{bmatrix} \lambda_1^* \\ \text{-----} \\ \lambda_2^* \end{bmatrix} = \begin{bmatrix} \nabla_{x_B} f(x^*) \\ \nabla_{x_N} f(x^*) \end{bmatrix}$$ using the partitioned inverse,

$$(A15)\ \text{OR}\quad \begin{matrix}\lambda_1^* \\ \lambda_2^*\end{matrix} = \begin{bmatrix} P & 0 \\ Q & I \end{bmatrix} \begin{bmatrix} \nabla_{x_B} f(x^*) \\ \nabla_{x_N} f(x^*) \end{bmatrix}$$

where $P = B'^{-1}$ and $Q = -N'B'^{-1}$.

Thus, the ε perturbation on the upper bound constraint set II decouples the dual values of constraint set I from constraint set II, and ensures that k constraints are binding.

Footnotes to Appendix I

[1]A short intuitive explanation of the reduced gradient is that the net effect of a change in $\tilde{x}_N$ is the gradient of the direct effect of $\tilde{x}_N$ on $f(\bullet)$ less the effects of reductions forced on $\tilde{x}_B$. The cost of reduction of $\tilde{x}_B$ is clearly influenced by $\nabla f_{\tilde{x}_B}(\bullet)$ and the relative marginal physical products from the scarce resources $B^{-1}N$.

Figure 1. L.P. Problem with Calibration Constraints
Two Activity/One Resource Constraint

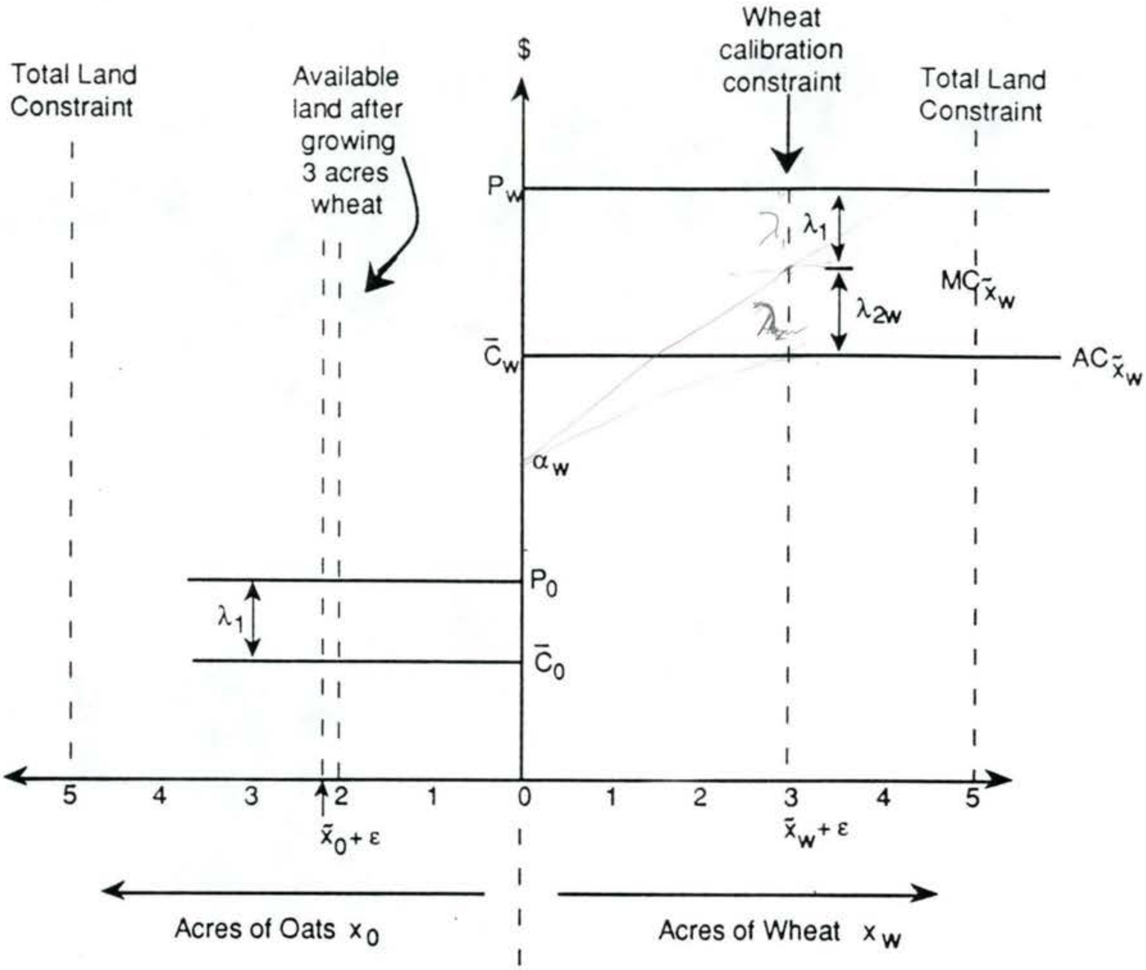

Figure 2. PMP Cost Function on Wheat

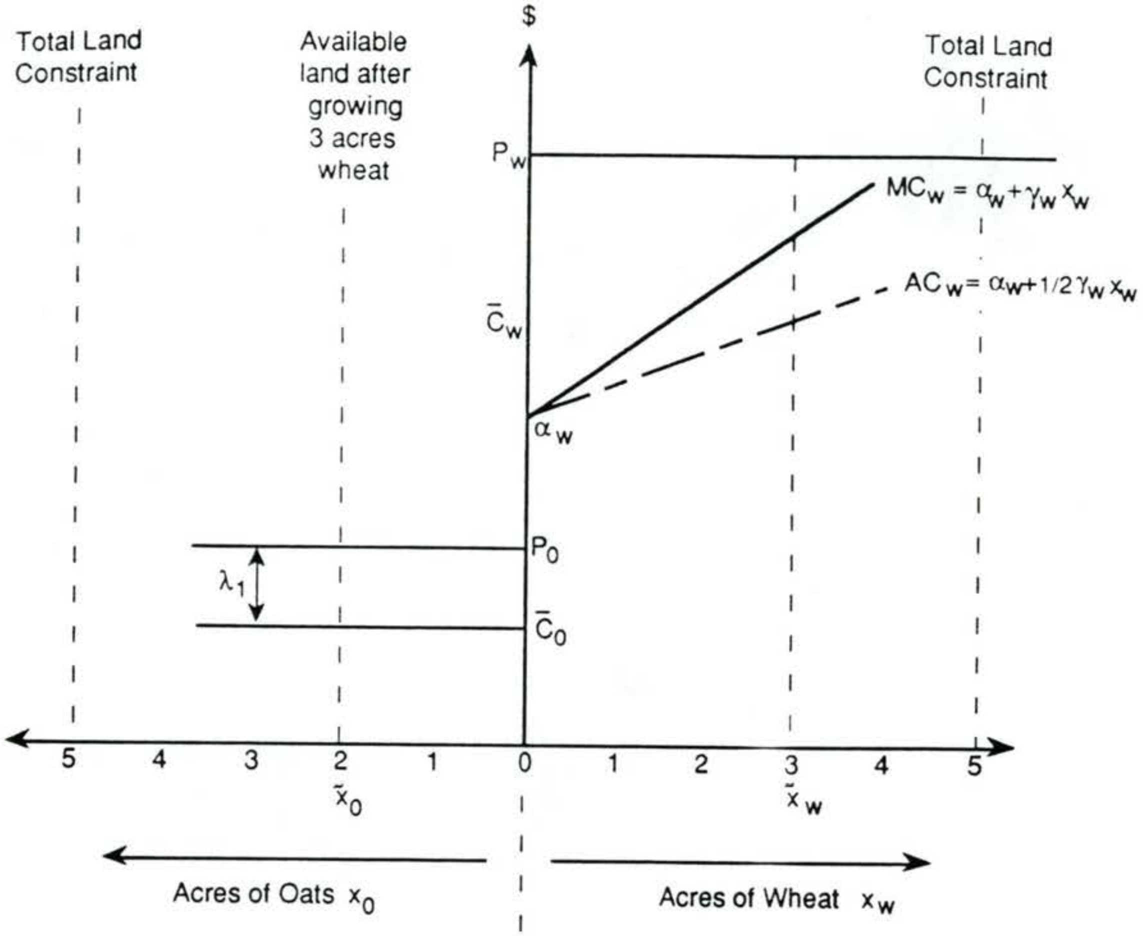

Figure 3. PMP Model – Quadratic Costs on all Crops

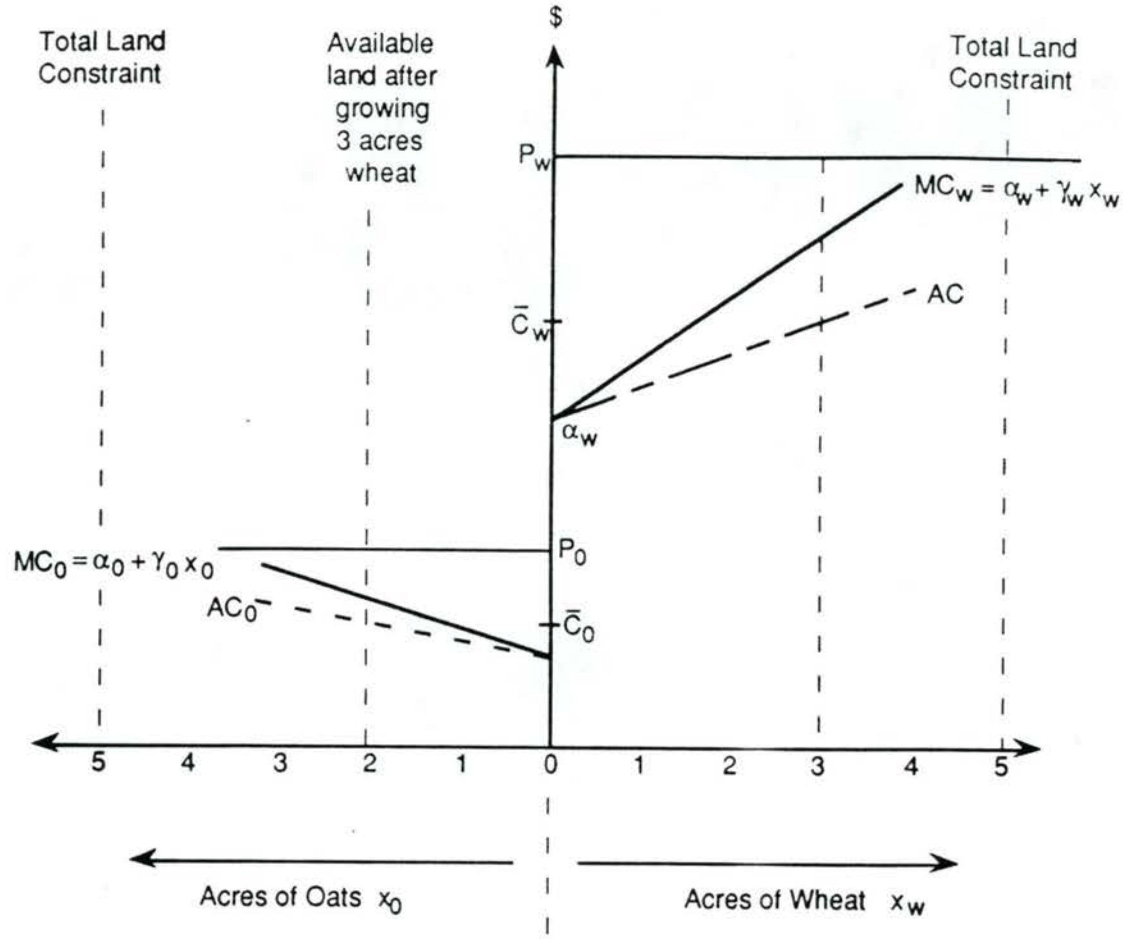

Figure 4. PMP Model – Calibrating "Rotational" Crops