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ECONOMIC OPTIMISATION OF SPRINKLER IRRIGATION CONSIDERING UNCERTAINTY OF SPATIAL WATER DISTRIBUTION*

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The study is focused on the development and the application of a stochastic economic optimisation model by which optimal levels of applied water and sprinkler spacing are determined. Data on crop-water production function and uniformity of water application are taken from a sprinkler irrigation plot of sweet corn. It was found that a saving of irrigation water can be achieved not only by raising water prices but also by increasing application uniformity.

The spatial variability of applied irrigation water has been well recognised by both agricultural researchers and growers for many years. The level of application uniformity is highly dependent on the type and performance of the irrigation method. The yield of a given crop, grown during a specific season in a certain field and under certain management and cultivation conditions, is also spatially variable and is assumed to be dependent directly on the spatially variable water application. It was generally noticed that the variation of the applied water as well as of the yield is not completely disordered in space but can be analysed within the frame of stochastic modelling, that is, regarding the applied water and the resulting yield of a given field as random functions of space coordinates characterised by their probability density function (pdf) and correlation structure, rather than by their deterministic values.

Most economic studies of efficient water use under non-uniform irrigation focus on optimisation with respect to the quantity of irrigation water and tend to ignore the optimisation with respect to the uniformity level (for example, Seginer 1978; Feinerman, Letey and Vaux 1983; Feinerman, Bresler and Dagan 1985). In a few previous economic studies (for example, Hill and Keller 1980; Chen and Wallender 1984; Gohring and Wallender 1987), the joint effects of uniformity and quantity of applied water on irrigation system selection or on economical sprinkler spacing were investigated. The dependence of the irrigation system cost on the uniformity was derived by varying the sprinkler spacing and calculating the resulting changes in the uniformity and in the cost. However, the spatial variation of the irrigation water was regarded as deterministic, uncertainty was not accounted for in the economic application, and the attitude of the decision maker toward risk was ignored. Seginer (1987) presented a comprehensive review describing a general approach to economic optimisation of the irrigation system considering the quantity of

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applied water and application uniformity under spatially variable conditions. However, his theoretical framework was not supported by, or validated by, real field data.

In the present study, we developed and applied a stochastic–economic optimisation model aimed at determining the optimal levels of applied water and the optimal application uniformity (via sprinkler spacing), taking into account the farmer's attitude to risk.

The analysis is applicable to many sprinkler-irrigated crops enabling an economic evaluation which will optimise the use of the irrigation system. The results may be generalised where large scale natural resources, such as mines, aquifers and the upper soil layer, display a wide variation in their properties. The yield of the appropriate resource is a quantity of primary interest in any economic analysis, and it is assumed to depend on the level of economic inputs (control variables) and on some relevant spatially variable properties. Consequently, the yield is also spatially variable and its prediction under field conditions is subjected to uncertainty. In other words, even if an accurate yield model under controlled conditions is available, prediction of yield on a large scale is still error-prone, since it is affected by spatially variable and uncertain parameters.

Examples of typical yield in applications related to hydrology, mining and agriculture are volume of water pumped from an aquifer, ore quantity and crop production as given in this paper, respectively. The respective typical control variables are number and size of wells, depth of the mine and quantity of agricultural inputs such as irrigation water. The corresponding spatially variable properties in the same applications are aquifer transmissivity and storativity, ore concentration, and variability of applied irrigation water.

Theoretical Model and Methodology

Consider a field of area A hectares (ha). Let $x \in A$ be the coordinate vector of a point in the field. To avoid unnecessary complications, it is assumed that there is no run-off or run-on and that precipitation is negligible, so that crop yield responds to the amount of water which reaches the point x. The depth of water, $Q(x)$, at any single point x in the field, is related to the spatial field average depth of applied water by

$$(1)\quad Q(x)=\bar{Q}\beta(x)$$

where $\bar{Q}$ is the spatial field average of Q defined by

$$(2)\quad \bar{Q}=(1/A)\int_A Q(x)\,\mathrm{d}x$$

and $\beta(x)$ is a space-dependent spatial random function, representing the degree of water application uniformity. Note that $\bar{Q}A$ is the total quantity of water applied to the field and that

$$(3)\quad (1/A)\int_A \beta(x)\,\mathrm{d}x=\bar{\beta}=1$$

Assuming the ergodic hypothesis (Lumley and Panofsky 1964) the spatial variations of a given property in one realisation represent all possible variations in all realisations of the ensemble [that is, the distribution of $\beta(x)$ is the same for every $x \in A$] so that $E[\beta(x)] = \bar{\beta}$, where E is the expectation operator.

Variability (or uniformity) of water application can be depicted by the variance σ_β^2 of the random function β,

$$(4) \quad \sigma_\beta^2 = (1/A)\int_A \beta^2(x)\,\mathrm{d}x - 1$$

While $E(\beta) = \bar{\beta}$ is always unity, the variance σ_β^2 depends on the irrigation method and, for a given method, on its performance (for example, spacing between sprinklers, emitters, etc.). Hence, σ_β^2 is, in addition to $\bar{Q}$, a decision or man-control variable of the farmer and is not exogenous. An increase in the value of σ_β^2 describes a situation in which water application uniformity decreases and the probability moves from the centre towards the tails of the pdf of β, while the mean ($\bar{\beta}$) remains unchanged. This holds, regardless of the form of the pdf of β. A change of this type is known as a 'mean preserving spread' (MPS) of the distribution under consideration (for example, Sandmo 1971).

The physical model describing the spatial non-uniformity of water application is tied to the economic optimisation model (see below) by crop-water production function. Such a function relates the commercial yield to the depth of applied water. For K commercial yield components (for example, kernels, total dry matter in corn, lint and seeds in cotton, etc.),

$$(5) \quad Y_j(x) = f_j[Q(x)] = f_j[\bar{Q}\beta(x)] \qquad j = 1, \ldots, K$$

where $Y_j(x)$ is the yield per unit area of component j at the point x in the field, and f_j is its associated production function.

The quantities of interest in the economic optimisation are the spatial field averages over the field of all the relevant commercial yield components, which are given by

$$(6) \quad \bar{Y}_j = (1/A)\int_A f_j[\bar{Q}\beta(x)]\,\mathrm{d}x \qquad j = 1, \ldots, K$$

Note that $\bar{Y}_j (j = 1, \ldots, K)$ is a random variable which depends on the random function $\beta(x)$ and on the decision (or man-control) variable $\bar{Q}$, and its expectation is given by

$$(7) \quad E(\bar{Y}_j) = (1/A)\int_A E\{f_j[\bar{Q}\beta(x)]\}\,\mathrm{d}x$$

$$= (1/A)\int_A\int_{-\infty}^{\infty} f_j[\bar{Q}\beta(x)]\, g[\beta(x)/\sigma_\beta^2]\,\mathrm{d}x\,\mathrm{d}\beta$$

Here, $g[\beta(x)/\sigma_\beta^2]$ is the probability density function of β which is conditioned on the variable σ_β^2 which, in turn, can be partly controlled.

The approximate relationships between the average yield expectation $E(\bar{Y}_j)$ and the expectation $(\bar{\beta})$, and the variance (σ_β^2) of β can be obtained by employing a second-order Taylor expansion of $Y_j(x)$ about $f_j(\bar{\beta}\bar{Q})$ which yields

$$(8) \quad E(\bar{Y}_j) \approx f_j(\bar{\beta}\bar{Q}) + (\bar{Q}^2\sigma_\beta^2)\frac{\partial^2 f_j(\cdot)}{\partial(\beta\bar{Q})^2}\bigg|_{\beta=\bar{\beta}} \qquad j=1, \ldots, K$$

Hence, as long as the yield is a concave function of applied water, that is, $\frac{\partial^2 f_j(\cdot)}{\partial(\beta\bar{Q})^2} < 0$, expected yield will increase as the variance of β decreases.

The highest value of $E(\bar{Y}_j)$ for a given level of $\bar{Q}$ will be achieved under completely uniform water application conditions, that is, when $\sigma_\beta^2 = 0$.

The variance $\sigma_{\bar{Y}_j}^2$, of $\bar{Y}_j$, is given by

$$(9) \quad \sigma_{\bar{Y}_j}^2 = E[\bar{Y}_j - E(\bar{Y}_j)]^2 = \int_{-\infty}^{\infty} [\bar{Y}_j - E(\bar{Y}_j)]^2 g[\beta(x)/\sigma_\beta^2]\, d\beta$$

For the economic evaluation of the optimal level of the control variable $\bar{Q}$ and of σ_β^2, we define a profit function π (in \$ per unit area)

$$(10) \quad \pi(\bar{Q}, \sigma_\beta^2) = \sum_{j=1}^{K} P_j \bar{Y}_j(\bar{Q}, \sigma_\beta^2) - P_Q\bar{Q} - C(\sigma_\beta^2)$$

where P_j is the price per unit of the j-th crop yield's component, P_Q is the price per unit of water, and $C(\sigma_\beta^2)$ is a cost function, the level of which depends on the variance of β (application uniformity) such that $\partial C/\partial\sigma_\beta^2 \equiv C' < 0$. Being dependent on the random $\bar{Y}_j$'s, the profit itself is a random variable.

To select optimal values for the two decision variables $\bar{Q}$ and σ_β^2, two possible objective functions will be discussed here. These two functions express the farmer's attitude toward risk. For a risk-neutral farmer one has to maximise

$$(11) \quad \underset{\bar{Q},\, \sigma_\beta^2}{\text{maximum}} \left\{ \sum_{j=1}^{K} P_j E[\bar{Y}_j(\bar{Q}, \sigma_\beta^2)] - P_Q\bar{Q} - C(\sigma_\beta^2) \right\}$$

with first-order conditions for optimum

$$(11a) \quad \sum_{j=1}^{K} P_j E(\partial\bar{Y}_j/\partial\bar{Q}) - P_Q = 0$$

$$(11b) \quad \sum_{j=1}^{K} P_j E(\partial\bar{Y}_j/\partial\sigma_\beta^2) - C' = 0$$

The second objective function to be tested here is relevant for a risk-averse farmer with a subjective concave utility function $U(\pi)$ which assigns an appropriate utility to each possible π. The optimal levels of $\bar{Q}$ and σ_β^2 are then those which maximise the expected utility.

Formally

(12) $\underset{\bar{Q},\,\sigma_\beta^2}{\text{maximum}}\, E\{U[\pi(\bar{Q}, \sigma_\beta^2)]\}$

with first-order conditions for optimum

(12a) $E\left\{U'(\pi)\left[\sum_{j=1}^{K} P_j(\partial \bar{Y}_j/\partial \bar{Q}) - P_Q\right]\right\} = 0$

(12b) $E\left\{U'(\pi)\left[\sum_{j=1}^{K} P_j(\partial \bar{Y}_j/\partial \sigma_\beta^2) - C'\right]\right\} = 0$

Before proceeding with the analysis it is interesting to compare the optimal amounts of irrigation water for the risk-neutral farmer with the risk-averse farmer. With the ergodic and stationarity assumptions, the average yield function in equation (6) can be rewritten (after omitting the index j for convenience) as

(13) $\bar{Y} = h(\bar{Q}\beta)$

For a given level of σ_β^2 the optimisation problem for the risk-neutral farmer is now

(14) $\max_{\bar{Q}}\{PE[h(\bar{Q}\beta)] - P_Q\bar{Q} - C\} \rightarrow \bar{Q} = \bar{Q}_n$

where Q_n is the optimal water application of the risk-neutral farmer. The first-order condition is

(14a) $PE[\partial h/\partial \bar{Q}] = P_Q$

The optimisation problem for the risk-averse farmer is

(15) $\max_{\bar{Q}} E\{U[Ph(\bar{Q}\beta) - P_Q\bar{Q} - C]\} \rightarrow \bar{Q} = \bar{Q}_a$

where $\bar{Q}_a$ is optimal water application of the risk-averse farmer. The first-order conditions for equation (15) may be derived by a method first used by Horowitz (1970):

(15a) $PE[\partial U/\partial \pi]E[\partial h(\bar{Q}\beta)/\partial \bar{Q}] + P\,\mathrm{cov}[\partial U/\partial \pi, \partial h(\bar{Q}\beta)/\partial \bar{Q}] = P_Q E[\partial U/\partial \pi]$

Dividing both sides of equation (15a) by $E[\partial U/\partial \pi]$ yields

(15b) $PE[\partial h/\partial \bar{Q}] - P_Q = -P/E[\partial U/\partial \pi]\,\mathrm{cov}[\partial U/\partial \pi, \partial h/\partial \bar{Q}]$

Now, $\partial(\partial U/\partial \pi)/\partial \beta = (\partial^2 U/\partial \pi^2)(\partial \pi/\partial \beta)$; consequently, $\partial^2 U/\partial \pi^2 < 0$ (risk aversion) and $\partial \pi/\partial \beta = P\bar{Q}\;\; \partial h/\partial(\bar{Q}\beta) > 0$ imply that $\partial(\partial U/\partial \pi)/\partial \beta < 0$. Additionally, assuming concavity of the relevant part of the production function, $\partial(\partial h/\partial \bar{Q})/\partial \beta = \bar{Q}\partial^2 h/\partial(\bar{Q}\beta)^2 < 0$. The work of Lehman (1966) allows us to conclude that $\mathrm{cov}(\partial U/\partial \pi, \partial h/\partial \bar{Q}) > 0$. Hence, the right-hand side of equation (15b) is negative, permitting us to write

(16) $PE[\partial h/\partial \bar{Q}] < P_Q$

Thus [compare with equation (14a)] the risk-averse farmer will demand more water than the risk-neutral one (that is $\bar{Q}_a > \bar{Q}_n$) and water, then, can be characterised as a marginally risk-reducing input (see Figure 1). This is so because the risk-averse farmer utilises more

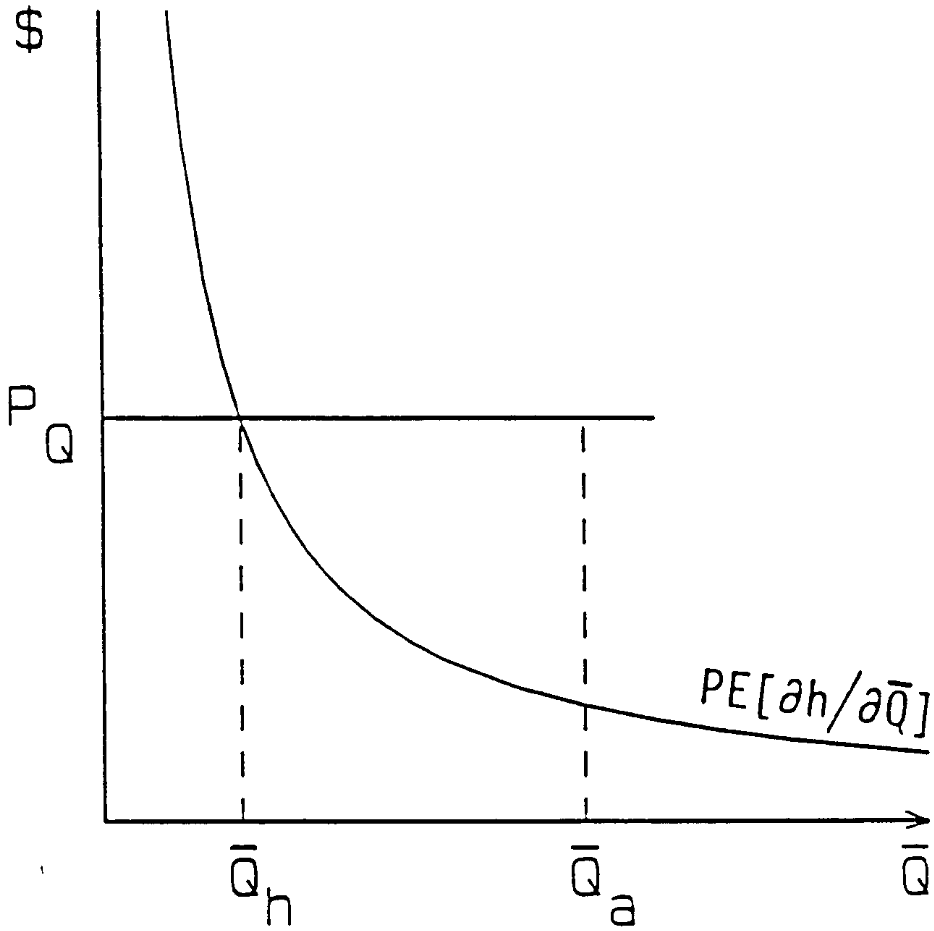

FIGURE 1—Optimal Quantities of Applied Irrigation Water for the Risk-Neutral ($\bar{Q}_n$) and the Risk-Averse ($\bar{Q}_a$) Farmer.

water than the risk-neutral farmer when other input conditions are fixed (Pope 1979).

The aforementioned optimisation model has been applied to sprinkle-irrigated sweet corn (variety 'Jubilee'), a crop with two commercial yield components ($K=2$): marketable kernels and total dry matter yield. Water quantities and their variability were achieved by a single line source (Hanks, Keller, Rasmussen and Willson 1976) and imaginary spacing between adjusting lines.

Experimental data from field trials conducted from a single line source were used to estimate the appropriate response relationships. Detailed information on the experiments is available from the authors.

Cost Estimates of Water Distribution Patterns

To quantify water distribution pattern and the degree of application non-uniformity, a number of characterising indexes have been suggested. The most common one was proposed by Christiansen (1942) as the Christiansen Uniformity Coefficient (*CUC*)

$$(17) \quad CUC=\left(1-\sum_{i=1}^{n}|Q_i-\bar{Q}|/n\bar{Q}\right)$$

where Q_i is the quantity of water measured in the i-th collection can, n is the number of collection cans in the arrangement, and $\bar{Q}$ is the average defined by $\bar{Q}=\Sigma Q_i/n$.

An additional common uniformity coefficient is the Statistical Uniformity Coefficient (SUC)

(18) $SUC=1-CV_Q$

where $CV_Q=\sigma_Q/\bar{Q}$ is the coefficient of variation of Q and σ_Q is the standard deviation estimate defined by

$$\sigma_Q=\left[\sum_{i=1}^{n}(Q_i-\bar{Q})^2/(n-1)\right]^{1/2}$$

By inspecting equation (1) it can be easily verified that $CV_Q=\sigma_\beta$. From this identity and for a known pdf (compare, for example, Warrick 1983) the relationship between CUC and σ_β for normal, lognormal and uniform pdf of β can be obtained respectively from

(19a) $CUC=1-0{\cdot}798\sigma_\beta$

(19b) $CUC=3-4P\{0{\cdot}5[\ln(1+\sigma_\beta^2)]^{0{\cdot}5}\}$

(19c) $CUC=0{\cdot}866\sigma_\beta$

when $P\{\cdot\}$ represents the probability of standard normal variables.

From the water distribution pattern of the observation sample taken from the single line source in the irrigation experiment (Figure 2) it is

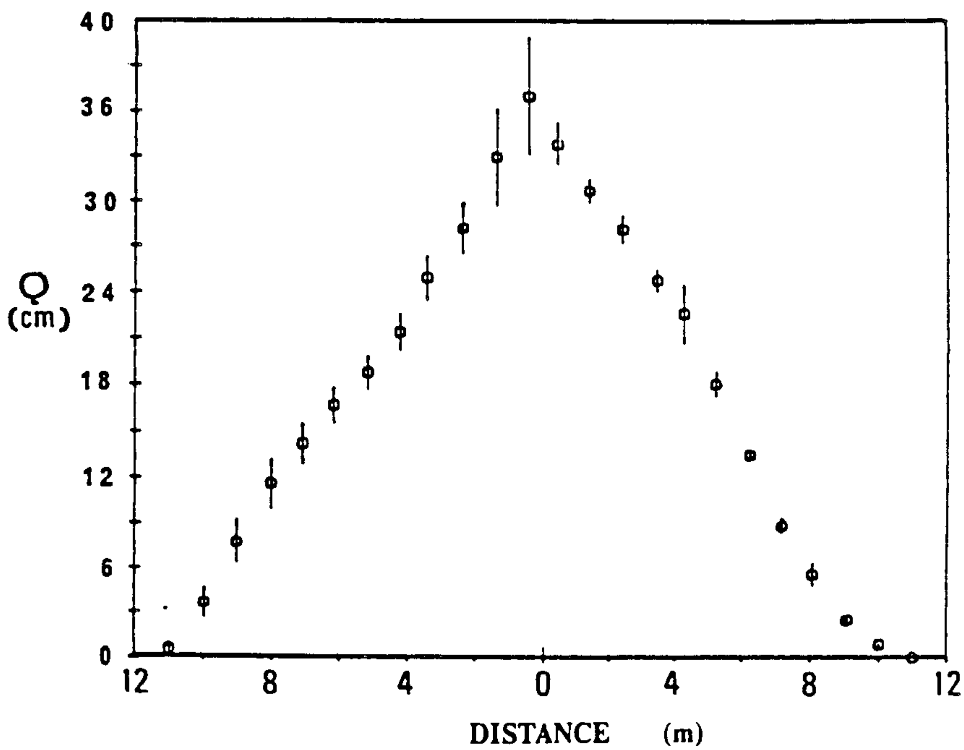

FIGURE 2—Average Depth of Applied Irrigation Water (o) and its Associated Standard Deviation (|) as Functions of the Distance From the Sprinkler Line.

possible to calculate the *CUC*, σ_β or σ_Q for a number of line sources depending on spacing between the lines. Calculations are based on imaginary overlapping and superposition of water quantities from adjacent lines. The circles in Figure 2 are average water quantities along the lines of sprinklers and other arbitrary lines parallel to the single sprinkler source. As such, the circle at zero distance identifies the average water quantity along the line of the sprinklers, and the length of the vertical bar shows the standard deviation of the water quantities that were measured along the line itself. The other points are equivalent measures at other distances parallel to the line source. The spacing between lines and hence *CUC* are subject to the farmer's decision as is illustrated in Figure 3. The variance of β considers overlapping among an infinite number of lines (Figure 2) when the distance between two adjacent lines is as in Figure 3. This figure demonstrates that there is an inverse relationship between application uniformity as characterised by *CUC* and spacing between the lines. Generally, uniformity increases as spacing decreases (especially when the distance between lines exceeds 12 m), and therefore (because of the concave production function) a higher yield per unit of $\bar{Q}$ is achieved. Unfortunately, however, some cost is involved in increasing application uniformity because it requires more labour and equipment per unit crop area.

The cost as a function of *CUC* has been estimated for spacing between 12 and 30 m (the contribution to improve uniformity of spacing less than 12 m is relatively small, as seen in Figure 3). The quadratic regression estimate, using prices typical to Israel, is

(20) $C = 40{\cdot}3 - 40{\cdot}5(CUC) + 72{\cdot}8(CUC)^2 \qquad (R^2 = 0{\cdot}93)$

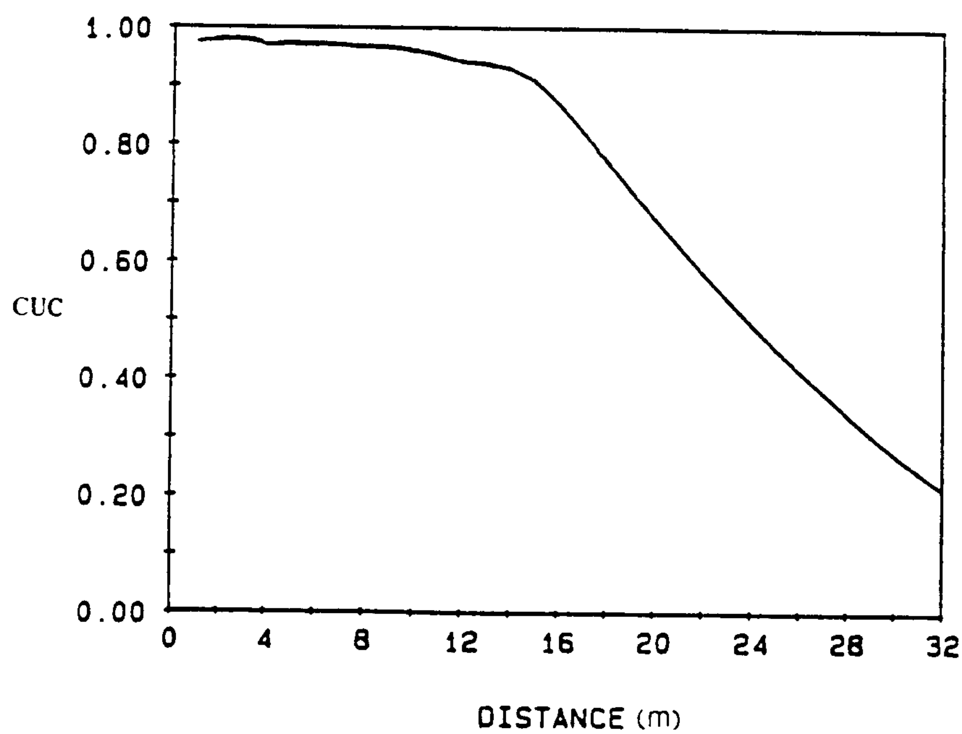

FIGURE 3—*CUC* as a Function of the Sprinkler-Line Spacing.

where C is the price, in \$/0·1 ha. Note that if the pdf of $\beta(x)$ is known, it is possible, in the optimisation problems (11) and (12), to replace the decision variable CUC in equation (20) by its equivalent σ_β^2. [See equations (19a) to (19c).]

Estimations of Alternative Yield Production Functions

The functional relationships between the two yield components of sweet corn (marketable kernels and total dry matter) and seasonal irrigation water quantity were estimated from water quantity measurements in each corn row and average yield of the associated row. This averaging procedure was performed because water application along any row parallel to the sprinkler line was quite uniform, with negligible standard deviation, as can be seen in Figure 2.

The results of each yield component were best fitted to form alternative production functions of the following types:

(i) Sigmoid type

(21) $F(Q) = \alpha_0 \exp[-\alpha_1 \exp(-\alpha_2 Q)]$

(ii) Parabolic type

(22) $F(Q) = \beta_0 + \beta_1 Q + \beta_2 Q^2$

(iii) Exponential (Mitcherlich) type

(23) $$F(Q) = \begin{cases} 0 & \text{for } Q < Q_0 \\ \delta_0\{1 - \exp[-\delta_1(Q - Q_0)]\} & \text{for } Q \geq Q_0 \end{cases}$$

(iv) Piecewise linear type

(24) $$F(Q) = \begin{cases} 0 & \text{for } Q < Q_{min} \\ \theta_0 + \theta_1 Q & \text{for } Q_{min} \leq Q \leq Q_{max} \\ Y_{max} & \text{for } Q > Q_{max} \end{cases}$$

Here, Q is the seasonal irrigation quantity (m^3/0·1 ha), which does not include the 5 cm pre-irrigation, Q_0, Q_{min} are the threshold Q values [for production functions (23) and (24), respectively] that must be exceeded to attain yield, and Q_{max} is the minimum value of Q for attaining maximum yield Y_{max}. The parameters of the four production functions were estimated by the maximum likelihood (ML) procedure (for example, Theil 1971), assuming that each yield component was sampled from a multivariate normal population. The justification of this assumption is discussed under 'Economic Optimisation and Sensitivity Analysis'.

The estimated parameters of the four yield production functions for the two yield components are given in Table 1. Sampling points and the best estimated regression lines are illustrated in Figure 4 (a to d) for marketable kernels and in Figure 5 (a to d) for total dry matter. From inspection of Figures 4 and 5, it is clear that 5 cm of pre-irrigation ($Q = 50$) was sufficient for a small production of dry matter (0·035 to 0·068 tons/0·1 ha) but insufficient for marketable kernels. The threshold seasonal water quantity for the latter was between 11 and 12·7 cm, depending on the type of production function. These are significant findings (Table 1) since, except for the Q_0 estimate for the

TABLE 1

Estimated Parameters of the Four Yield Production Functions

Production function	Estimated parameter											
	α_0 (T/0·1 ha)	α_1 $(cm)^{-1}$	α_2 $(cm)^{-1}$	β_0 (T/0·1 ha)	β_1 (T/0·1 ha/cm)	β_2 (T/0·1 ha/cm)	γ_0 (T/0·1 ha)	Y_{max} (T/0·1 ha)	Q_{max} (cm)	δ_0	δ_1	Q_0
Dry matter												
Sigmoid	1·083 (53·32)[a]	2·769 (23·26)	0·0924 (33·93)									
Parabolic				0·035 (2·53)	0·040 (58·68)	$-3{\cdot}73\times10^{-4}$ (−15·59)						
Piecewise							0·049 (2·44)	0·917 (57·77)	26·67 (37·73)			
Exponential										1·748 (44·58)	0·0234 (32·21)	−1·018 (−2·04)
Kernels												
Sigmoid	1·881 (36·35)	30·041 (16·0)	0·159 (59·03)									
Parabolic				−1·523 (−41·95)	0·135 (91·75)	$-1{\cdot}2\times10^{-3}$ (−15·59)						
Piecewise							−1·176 (−15·37)	1·626 (49·70)	30·67 (71·02)			
Exponential										3·906 (29·19)	0·0261 (23·12)	12·67 (27·23)

[a]Figures in parentheses denote the calculated t values.

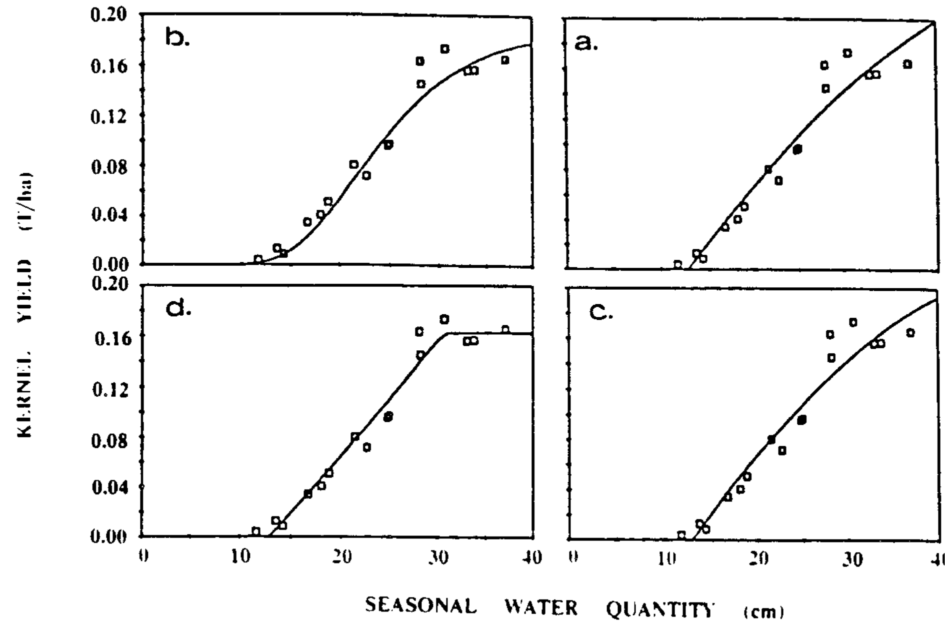

FIGURE 4—Actual Data and Estimated Regression Lines of the Four Kernel Yield Production Functions: (a) Exponential, (b) Sigmoid, (c) Parabolic and (d) Piecewise Linear.

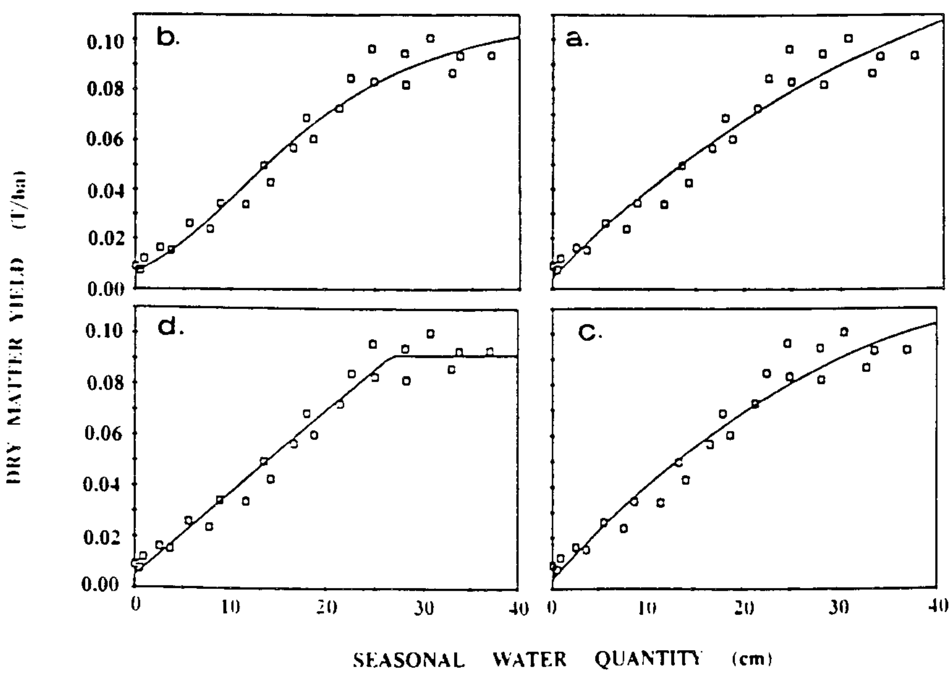

FIGURE 5—Actual Data and Estimated Regression Lines of the Four Dry Matter Yield Production Functions: (a) Exponential, (b) Sigmoid, (c) Parabolic and (d) Piecewise Linear.

dry matter using the exponential type function, all other estimated parameters differ significantly from zero at the 5 per cent significance level.

To compare the four production functions in relation to their fit to the empirical data, we used Akaike Identification Criteria (*AIC*) (for example, Carrera and Neuman 1986):

(25) $AIC = 2F + 2M$

where $F = \ln L$, with L being the maximum value of the likelihood function and M the number of estimated parameters by ML (three parameters + the disturbance term). The smaller the value of the *AIC* for a given production function, the better the fit of the model to the empirical data. Using these criteria, the decreasing order of fit is: piecewise linear model, sigmoid, parabolic and exponential model (*AIC* for kernels are, respectively: −28·1, −26·0, −22·4 and −21·7; for dry matter: −5·2, −2·6, −0·17 and −0·12). Although the piecewise linear model gave the best fit, the sigmoid model was selected for our analysis because: (i) its *AIC* value is close to the piecewise linear model, and (ii) the sigmoid production function does not contain an undefined second-order derivative of yield versus applied water as does the piecewise linear model at $Q = Q_{max}$.

Calculations of Statistical Moments of Space Average Yields

The quantities of interests for economic optimisation are averages over the field space of the commercial yield components. The values of the two first statistical moments of $\bar{Y}_j$, that is, $E(\bar{Y}_j)$ and $\sigma^2_{\bar{Y}_j}$, $j = 1, \ldots, K$ [equations (7) and (9)], depend upon the production function f_j, on $g(\beta)$ the pdf of β, and on the decision variables $\bar{Q}$ and σ^2_β (or alternatively on *CUC*). For example, assuming normal water application distribution, that is, $\beta(x) \simeq N(\bar{\beta}, \sigma^2_\beta)$, and the sigmoid production function [equation (21)], the average yield expectation and variance are estimated from

$$(26) \quad \hat{E}(\bar{Y}_j) = 1/(2\pi\hat{\sigma}^2_\beta \bar{Q}^2 A^2)^{1/2} \int_0^\infty \hat{\alpha}_{0j}[\exp\{-\hat{\alpha}_{1j}\exp[\hat{\alpha}_{2j}\beta(x)\bar{Q}]\}] \exp\{-0{\cdot}318\bar{Q}[\beta(x)-1]/(0{\cdot}798\sigma_\beta\bar{Q})^2\} \mathrm{d}\beta$$

$$(27) \quad \hat{\sigma}^2_{\bar{Y}_j} = \int_0^\infty [\bar{Y}_j - \hat{E}(\bar{Y}_j)]^2 g[\beta(x)/\sigma^2_\beta] \mathrm{d}\beta$$

Equations (26) and (27) were evaluated numerically and the relative yield expectations for kernel yield and dry matter as a function of *CUC* and $\bar{Q}$ are presented in Figures 6 and 7. These figures demonstrate the substitution between the two decision variables *CUC* (or σ_β) and $\bar{Q}$. For example, an expected relative dry matter yield of 0·75 can be obtained by the following combinations of ($\bar{Q}$, *CUC*): (40, 19), (35, 30), (30, 42) and (25, 65). These are points on an isoquant in the ($\bar{Q}$, *CUC*) coordinate system with decreasing marginal rate of technical substitution between $\bar{Q}$ and *CUC*.

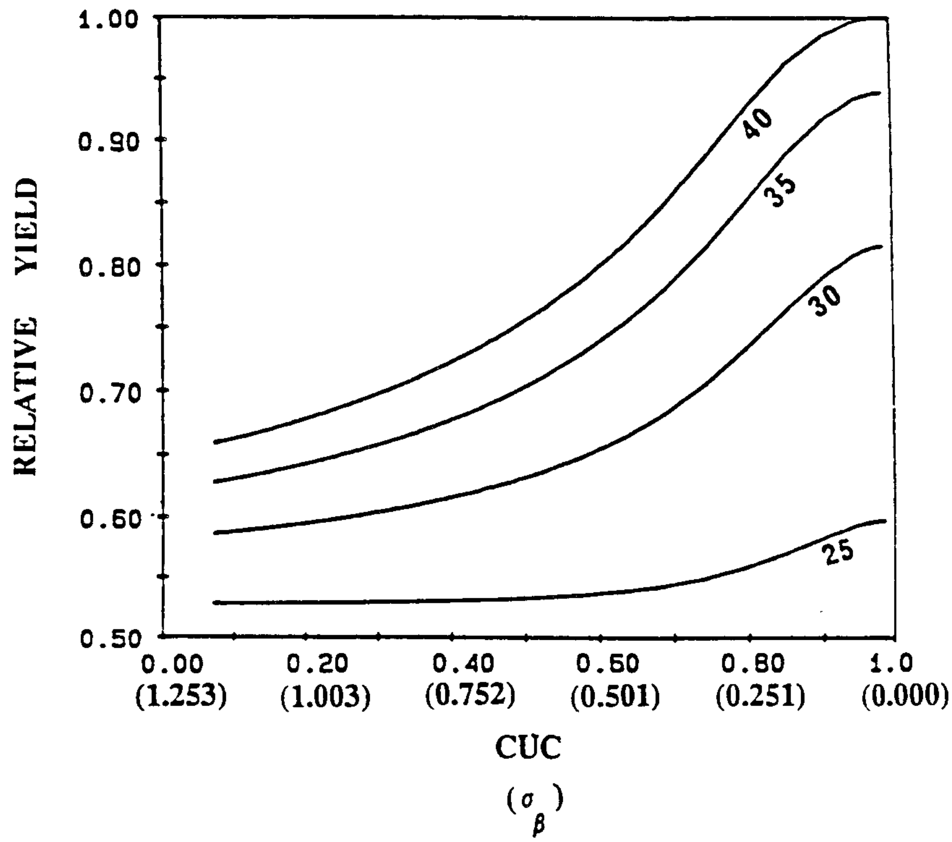

FIGURE 6—Expected Kernel Yield Relative to a Yield of $\bar{Q}=40$ cm and $CUC=1$ as a Function of CUC (Numbers Labelling the Lines Indicate the Average Depth of Applied Water, $\bar{Q}$).

Examination of Figures 6 and 7 shows that for any given water quantity $\bar{Q}$ the shape of the expectation function follows the sigmoid model. Hence, the expected marginal production of *CUC* increases at small values of *CUC*, decreases at high *CUC* and approaches zero at *CUC* greater than 0·95. Expected yield depression resulting from decreasing *CUC* at small water quantity (for example, $\bar{Q}=20$ cm) is more moderate than the depression at high $\bar{Q}$ (for example, $\bar{Q}=40$ cm). The reason is that at low $\bar{Q}$ water stress has a dominant effect on yield while at high $\bar{Q}$ the effect of water stress is small and uniformity of application is dominant.

The effects of alternative pdf of β have also been examined, by calculating $\hat{E}(\bar{Y}_j)$ with the lognormal and uniform distribution, as well as the normal distribution. Similar to Warrick (1983), it was found that the differences among the three pdfs in their effect on average yield expectations are small and can be neglected for all practical purposes in the relevant range of $0{\cdot}5 \leq CUC \leq 1$.

Calculations of $\hat{\sigma}^2_{\bar{Y}_j}$ from equation (27) with sigmoid production function and the three alternative pdf of β show the following. (i) For a given *CUC* the variance of average yield (kernels and dry matter) decreases as $\bar{Q}$ increases. Obviously, for a given $\bar{Q}$, $\hat{\sigma}^2_{\bar{Y}_j}$ is inversely

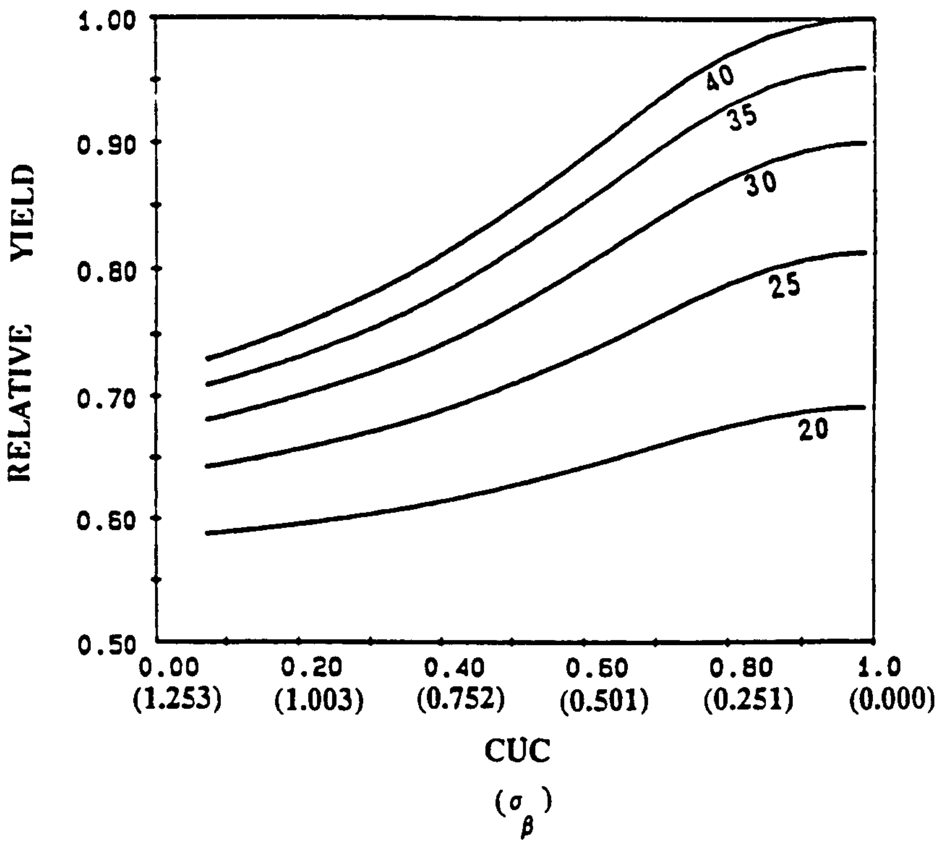

FIGURE 7—Expected Dry Matter Yield Relative to a Yield of $\bar{Q}$ = 40 cm and CUC = 1 as a Function of CUC (Numbers Labelling the Lines Indicate the Average Depth of Applied Water, $\bar{Q}$).

related to CUC. (ii) For a given $\bar{Q}$ and CUC the smallest $\hat{\sigma}^2_{\bar{Y}_i}$ is generally achieved when β distribution is uniform. Conversely, the largest $\hat{\sigma}^2_{\bar{Y}_i}$ was calculated for normal pdf. (iii) The smaller the average seasonal water quantity $\bar{Q}$ the larger the decrease in $\hat{\sigma}^2_{\bar{Y}_i}$ in response to a given increase in CUC values. In other words, the positive contribution of a high CUC in decreasing $\hat{\sigma}^2_{\bar{Y}_i}$ is more effective as $\bar{Q}$ is smaller.

Economic Optimisation and Sensitivity Analysis

The optimisation purpose of this study is to determine the level of the two decision variables $\bar{Q}$ and σ^2_β (or CUC) so as to maximise the relevant objective function. Two objective functions are considered here: (i) maximising profit expectation [equation (11)], which is related to a risk-neutral farmer, and (ii) maximising utility expectation [equation (12)], which is suitable for a risk-averse farmer. For the latter we assumed a negative exponential utility function which is quite common for economic empirical applications (for example, Deaton and Muellbauer 1980, p. 401; Buccola 1982; Feinerman *et al.* 1985).

This function is given formally by

(28) $U(\pi) = -\exp(-\theta\pi)$

where θ is the Arrow–Pratt measure of absolute risk aversion. For a sigmoid production function and normal water distribution, the optimisation problem for the risk-neutral farmer is

$$(29)\quad \underset{\bar{Q},\, CUC}{\text{maximum}} \left\{ \sum_{j=1}^{2} P_j \hat{E}[\bar{Y}_j(\bar{Q},\, CUC)] - P_Q \bar{Q} - C(CUC) \right\}$$

Similarly, for the risk-averse farmer:

$$\underset{\bar{Q},\, CUC}{\text{maximum}} \hat{E}\{U[\theta\pi(\bar{Q},\, CUC)]\} =$$

$$(30)\quad \underset{\bar{Q},\, CUC}{\text{maximum}} \left\{ -\exp(-\theta \sum_{j=1}^{2} P_j \hat{E}[\bar{Y}_j(\bar{Q},\, CUC)]) - P_Q \bar{Q} - C(CUC) + 0{\cdot}5\theta^2 \left[\sum_{j=1}^{2} P_j^2 \hat{\sigma}^2_{\bar{Y}_j}(\bar{Q},\, CUC) + 2P_1P_2 \operatorname{cov}(\bar{Y}_1,\, \bar{Y}_2) \right] \right\}$$

where $\hat{E}(\bar{Y}_j)$ and $\hat{\sigma}^2_{\bar{Y}_j}$ are given by equations (26) and (27), respectively, $C(CUC)$ is given by equation (20), and $\operatorname{cov}(\bar{Y}_1,\, \bar{Y}_2) = \hat{\sigma}_{\bar{Y}_1 \bar{Y}_2} r(\bar{Y}_1,\, \bar{Y}_2)$ is the covariance between the two yield components (marketable kernels and dry matter) and $r(\bar{Y}_1,\, \bar{Y}_2)$ is their correlation coefficient.

Model (30) correctly reflects an exponential utilist's expected utility only if profit is normally distributed (Freund 1956). Note, however, that the spatial average of $\bar{Y}_j$ [equation (6)] is composed of the sum of random variables $Y_j(x)$ [equation (5)] with common and finite mean and variance. Since these variables are asymptotically not correlated (that is, the covariance $\operatorname{cov}[Y_i(x_i),\, Y_j(x_j)]$ goes to zero as the distance $|x_i - x_j|$ exceed some upper limit), it is reasonable to assume that they are asymptotically independent as well. Therefore, the work of Dvoretzky (1977) allows us to assume that $\bar{Y}_j$ is asymptotically normally distributed and hence, π [equation (10)] which is a linear function of $\bar{Y}_j$ is also asymptotically normally distributed.

The optimisation process is performed in two stages as follows:

In the first stage, optimal $\bar{Q}$ was calculated for series of $CUC = 75, 76, 77, \ldots, 99, 100$, with $Z_1[\bar{Q}_n(CUC)]$ and $Z_2[\bar{Q}_a(CUC)]$ being optimal values of the objective functions (29) and (30), respectively, for risk-neutral ($\bar{Q}_n$) and risk-averse ($\bar{Q}_a$) cases.

In the second stage, CUC values were selected such that Z_1 and Z_2 were maximised.

The results are based on variable water prices of $P_Q = 0{\cdot}05, 0{\cdot}10, \ldots, 0{\cdot}40$, kernel price of $P_1 = \$157$/ton, dry matter price of $P_2 = \$35$/ton, and a correlation coefficient kernel dry matter of $r(\bar{Y}_1,\, \bar{Y}_2) = 0{\cdot}9$. There is no uniformity in views on the appropriate value of the risk aversion coefficient θ. King and Robinson (1981), for example, suggested a range of values for the risk aversion coefficient between $-0{\cdot}0001$ (a risk-loving farmer) and $0{\cdot}001$. Yassour, Zilberman and Rausser (1981) assumed a range for θ between $0{\cdot}001$ and $0{\cdot}01$ whereas Musser and Stamoulis (1981) parameterised θ in the range of $0{\cdot}00017$ to $0{\cdot}0175$. Sensitivity of the results to the assumed value of θ in this study was investigated by assuming four levels of θ: $0{\cdot}001$, $0{\cdot}01$, $0{\cdot}05$ and $0{\cdot}1$.

TABLE 2

Optimal Water Quantities ($m^3/0{\cdot}1$ ha) for a Risk-Neutral Farmer ($\bar{Q}_n$) and a Risk-Averse Farmer ($\bar{Q}_a$) for Five Values of the Christiansen Uniformity Coefficient (CUC) and Three Different Water Prices (P_Q)

	P_Q=\$0.05/m³					P_Q=\$0.20/m³					P_Q=\$0.40/m³				
	θ					θ					θ				
	0	·001	·01	·05	·1	0	·001	·01	·05	·1	0	·001	·01	·05	·1
CUC	$\bar{Q}_n$	$\bar{Q}_a$				$\bar{Q}_n$	$\bar{Q}_a$				$\bar{Q}_n$	$\bar{Q}_a$			
0·75	670	680	745	791	827	474	480	530	671	740	389	393	433	558	627
0·80	620	625	661	700	730	463	467	499	585	646	389	392	422	506	564
0·85	573	574	585	618	644	447	447	465	507	538	385	387	404	452	485
0·90	540	540	543	553	562	432	433	438	456	471	378	379	386	408	426
0·95	522	522	523	525	527	423	423	424	429	433	373	373	375	381	388

Table 2 presents values of $\bar{Q}_n$ and $\bar{Q}_a$ for different values of *CUC*, θ and P_Q as obtained from the first stage of the optimisation. It is seen that optimal $\bar{Q}_n$ and $\bar{Q}_a$ decreased as application uniformity increased (that is, $d\bar{Q}_n/dCUC$ and $d\bar{Q}_a/dCUC<0$). This is true no matter what P_Q is, but the rate $d\bar{Q}/dCUC$ decreased as water price increased (that is, $|d[(d\bar{Q}_n/dCUC)]/dP_Q|$, $|d[d\bar{Q}_a/dCUC]/dP_Q|<0$), as illustrated by Figure 8 (a, b) for $\theta=0{\cdot}01$. As expected, Table 2 shows that at any value of *CUC* and P_Q, a risk-averse farmer will use more water for irrigation than the risk-neutral farmer, that is, $\bar{Q}_a(CUC)>\bar{Q}_n(CUC)$, ($\bar{Q}_a=\bar{Q}_n$ for high *CUC* and $\theta=0{\cdot}001$ as a result of rounding), and that an increase in the degree of risk aversion (θ) implies an increase in the optimal water application. Also, the drop in $\bar{Q}$ application which is caused by the reduced variability is larger for the risk-averse farmer than for the risk-neutral one. This is expected because only the risk-averse farmer does care about the effects of the variance of β.

It is obvious that for given *CUC* values optimal water quantities decrease as water prices increase. Hence, the results given in Table 2 point out that a saving of irrigation water in areas where water is scarce can be achieved not only by raising water prices but also by subsidising irrigation systems with higher field application uniformity. This subsidy will be more effective at lower water prices and for the risk-averse farmer than for the risk-neutral one. The higher the degree of risk aversion the more effective the impact of the subsidisation.

So far we have presented the results of the first stage, while the second stage determines optimal *CUC* associated with the first stage optimal $\bar{Q}$. In Table 3 $(CUC)_n$, $E(\pi_n)$, $(CUC)_a$ and $E(\pi_a)$ the optimal expected profit and *CUC* values that maximised the objective functions (29) and (30), respectively, are presented after completion of the second stage. Also given in Table 3 are optimal water quantities $\bar{Q}$, together with a comparison with fully uniform and certain conditions [that is, $CUC=1$ or alternatively $\beta(x)$ being a deterministic function of x at level $\bar{\beta}$ everywhere in the field]. The results summarised in Table 3 show, first, that optimal water quantities under uncertain conditions ($\bar{Q}_n$ and $\bar{Q}_a$) are larger than under certain $\bar{Q}$, for every level of water cost P_Q. However, the differences decrease with the increase in water cost.

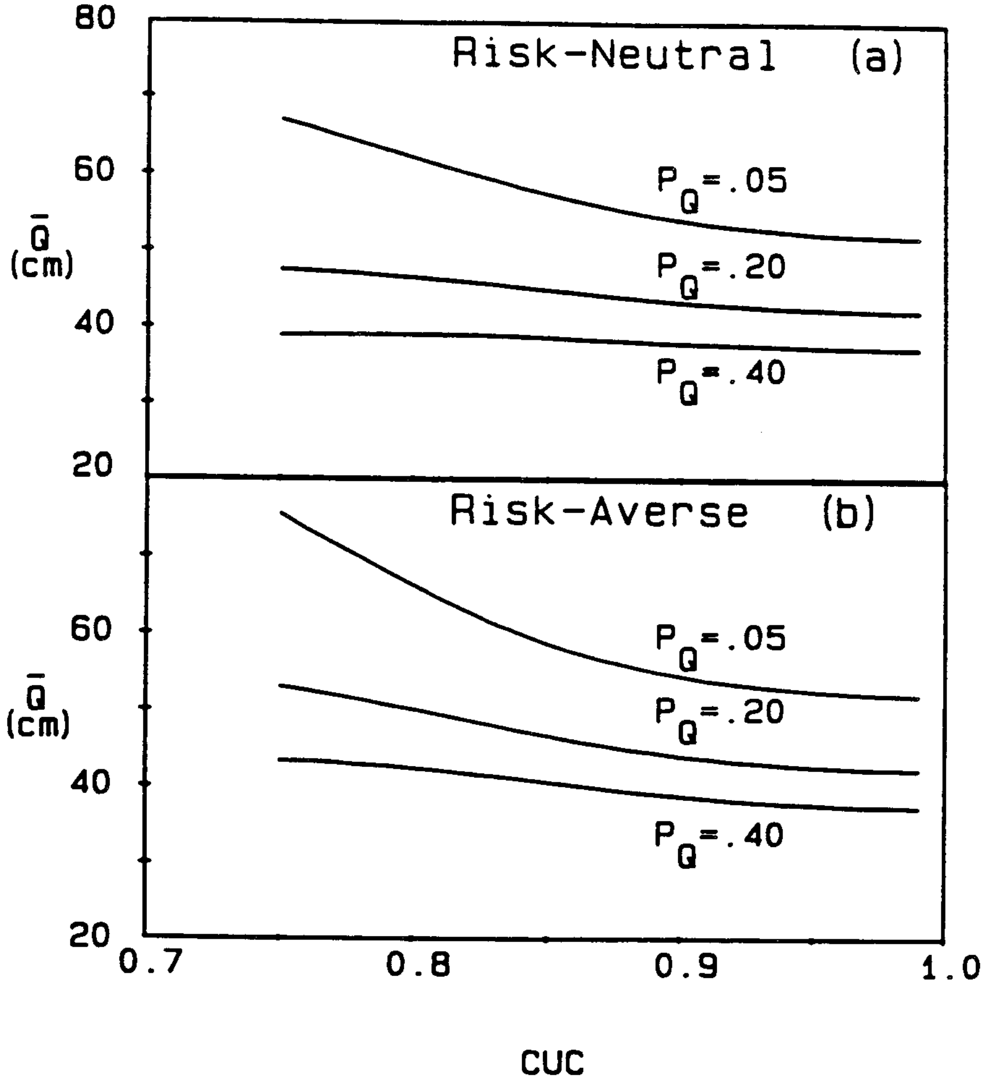

FIGURE 8—Optimal Depth of Applied Irrigation Water ($\bar{Q}$) as a Function of *CUC* for Three Water Prices (P_Q), and for (a) a Risk-Neutral and (b) a Risk-Averse Farmer with $\theta = 0{\cdot}01$.

Also, optimal water application increases with the degree of risk aversion (θ). These results demonstrate the quantitative contribution of water application uniformity to water saving. Because completely uniform conditions are quite expensive, the optimal profit under certain conditions is always lower than $E(\pi_n)$ and $E(\pi_a)$ received under the uncertain conditions. Here again, the differences diminish as the water cost rises. It should be remembered that certain profits are being compared with expectation of profits under uncertainty, while the 'true' profit will always be different from its expectation.

Note also that $E(\pi_n)$ is higher than $E(\pi_a)$ (the equalities for $\theta = 0{\cdot}001$ are due to rounding). A risk-averse farmer pays for his risk aversion by using more water $\bar{Q}$, which reduces the profit.

TABLE 3

Optimum Values of Water Quantities, CUC, and Expected Profit Under Different Uncertainty Conditions, Certainty Conditions and Prices Per Unit of Water

	Risk-averse											
	$\bar{Q}_a$ (m³/0·1 ha)				$(CUC)_a$				$E(\pi_a)$ ($/0·1 ha)			
	θ				θ				θ			
P_Q	·001	·01	·05	·10	·001	·01	·05	·10	·001	·01	·05	·10
0·05	593	598	606	622	·83	·84	·87	·86	259·9	259·8	259·0	258·8
0·10	506	508	523	527	·87	·87	·88	·89	233·0	232·9	232·4	231·9
0·15	462	464	476	480	·88	·90	·90	·91	209·0	208·8	208·5	207·9
0·20	433	435	449	453	·90	·91	·91	·92	186·7	186·6	186·2	185·6
0·25	414	415	427	431	·91	·92	·92	·93	165·6	165·4	165·1	164·5
0·30	400	399	410	413	·92	·93	·93	·94	145·3	145·0	144·7	144·2
0·35	386	388	399	396	·92	·93	·93	·95	125·6	125·5	125·1	124·6
0·40	376	376	385	388	·93	·94	·94	·95	106·6	106·4	106·1	105·5

Certainty ($CUC=1$)			Risk-neutral		
P_Q ($/m³)	$\bar{Q}$ (m³/0·1 ha)	π ($/0·1 ha)	$\bar{Q}_n$ (m³/0·1 ha)	$(CUC)_n$	$E(\pi_n)$ ($/0·1 ha)
0·05	516·9	252·6	591	0·83	259·9
0·10	468·3	222·6	504	0·86	233·0
0·15	440·1	203·5	461	0·88	209·0
0·20	420·1	182·0	432	0·90	186·7
0·25	404·5	161·4	413	0·91	165·6
0·30	391·6	141·5	399	0·92	145·3
0·35	380·7	122·2	386	0·92	125·6
0·40	371·2	103·5	376	0·92	106·6

Table 3 also shows that the values of $\bar{Q}_a$ and $(CUC)_a$ are equal to or only a little higher than $\bar{Q}_n$ and $(CUC)_n$. The differences are larger as the value of θ increases and are probably dependent on the type of the selected production function. At any rate, the fact that $\bar{Q}_a-\bar{Q}_n\geq 0$ and that $CUC_a-CUC_n\geq 0$ is logical because when a risk-neutral farmer becomes risk-averse, he will first increase $\bar{Q}_n$ to $\bar{Q}_a^*$ and then will improve uniformity by changing from CUC_n to CUC_a and thereby will reduce the risk (uncertainty) from which he is now averted. The increase in CUC will then be associated with a decrease back from $\bar{Q}_a^*$ to $\bar{Q}_a$ that will still be higher than $\bar{Q}_n$. It should be emphasised (Table 3) that always $E(\pi_n)\geq E(\pi_a)$, because the objective of a risk-neutral farmer is to maximise the expectation of the profit, while that of the risk-averse individual is to maximise the expectation of the utility of the profit. The differences, however, are negligibly small. Note also that the expectation of the profit decreases slightly with the degree of risk aversion.

Finally, Table 3 shows that an increase in water cost is associated with a decrease in optimal water quantities and an increase in CUC of the risk-averse as well as of the risk-neutral farmer. The rates of water quantity decrease and of uniformity increase are higher with higher water cost. Obviously, the increase in water prices results in a decrease in the profit (under certain conditions) and in the expectation of the profit (under uncertain conditions).

Summary and Conclusions

In this study we developed and applied a stochastic economic optimisation model by which optimal levels of applied water and sprinkler spacing are determined. The model considers the farmer's attitude toward risk. Data on crop-water production function were taken from a sprinkler irrigation plot of sweet corn. Two yield components were considered: total dry matter and marketable kernels.

Complete irrigation uniformity is unattainable and in general economically undesirable, but poor application uniformity may entail significant losses in yield and revenue. Under certain conditions, controlling irrigation uniformity may be as important as controlling irrigation water quantities. In other words, uniformity of water application should be treated as an endogenous control variable rather than as a predetermined exogenous variable. It was found (Figures 6 and 7) that increased water application ($\bar{Q}$) can be substituted for uniformity (*CUC*) with decreasing marginal rate of technical substitution between $\bar{Q}$ and *CUC*. It was also found that: (i) expected yield depression, which resulted from decreasing water uniformity at low water quantity, is more moderate than the depression at high $\bar{Q}$; (ii) for a given *CUC*, the variance of average yield decreases as $\bar{Q}$ increases; and (iii) the positive contribution of high *CUC* in reducing the variance of the average yield is more effective as $\bar{Q}$ is smaller.

The principal findings of the applied stochastic economic optimisation and the sensitivity analysis can be summarised as follows: the optimal levels of applied water for both risk-neutral and risk-averse farmers decrease as application uniformity increases and the rate of the decrease is inversely related to the level of for higher water prices.

Increased uniformity of water application (as well as increased water prices) will lead to a decline in optimal $\bar{Q}$ and to a saving of scarce irrigation water, especially for lower water prices and for the more risk-averse farmers. Optimal water quantities under uncertain conditions are greater than the optimal quantities under certain conditions, regardless of the cost of the water. The optimal values of applied water and uniformity of application for the risk-averse farmer are only a little higher than those for the risk-neutral farmer. A risk-averse farmer pays for his aversion by using more water. Finally, an increase in water prices entails a decrease in optimal water quantities and an increase in optimal uniformity for the risk-averse as well as for the risk-neutral farmer. The rates of water quantity decrease and of uniformity increase are higher for higher water prices.

The results obtained in this study indicate that using a predetermined arbitrary value of the Christiansen Uniformity Coefficient as a criterion for planning and operating sprinkler irrigation systems is not appropriate. In sensitive crops, for example sweet corn, such a practice may decrease yields, waste water and lower profits. Knowing the relationships between irrigation uniformity, water quantities and crop may enable the farmer to optimise the use of a sprinkler irrigation system.

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