Australian Journal of Agricultural Economics, Vol. 33, No. 2 (August 1989), pp. 108–122

A FAMILY OF AGRONOMIC PRODUCTION FUNCTIONS WITH ECONOMIES OF SCOPE

P. J. LLOYD
Department of Economics, University of Melbourne, Parkville, Vic. 3052

This paper derives a family of multiple-output multiple-input production functions from the underlying technologies. These technologies are represented by average yield functions for each of the commodities (crops) produced from a common pool of resources. The production functions are implicitly separable. Examples include the CRETH, translog, generalised power and generalised McFadden functions. Moreover, given a function which is a member of this family, the individual commodity yield and production functions can be recovered. Such implicitly separable multiple-output production functions may exhibit economies or diseconomies of scope which reflect the interactions between outputs sharing a common pool of resources.

Intrinsic jointness has always been recognised as a feature of many agricultural production processes (see, for example, Bishop and Toussaint 1958, ch. 11). It is customary to derive multiple-output production functions from desirable analytical properties. For example, the CET (constant elasticity of transformation) multiple-output production function has the property that the partial elasticities of transformation between pairs of outputs are equal to a constant. The CRETH (constant ratio of elasticities of transformation, homothetic) function has the somewhat less restrictive property that the partial elasticities may vary along isoquants and differ between pairs of inputs but, for any two pairs, the ratios are equal everywhere. The translog has the property that it is linear in logarithms. All three of these functional forms have been used by agricultural economists [see Powell and Gruen (1968), Vincent, Dixon and Powell (1980) and Akridge and Hertel (1986) respectively].

In the last 15 years various flexible functional forms that provide a second order differential approximation to any arbitrary twice continuously differential function have been developed (Diewert 1974). Two of the most commonly used flexible functional forms are the generalised Leontief production function introduced by Diewert (1971) and the translog production function which, for a multiple-output production process, was introduced by Christensen, Jorgensen and Lau (1973).[1] More recently, a second generation of flexible functions has been developed. These enable correct curvature and other appropriate conditions such as constant returns to scale to be imposed. These functions include the generalised McFadden multiple-output cost function introduced by Diewert and Wales (1987). However, the choice of flexible form is itself arbitrary and imposes restrictions on the technology. It is desirable, if possible, to

[1] An alternative approach is to specify a dual cost or profit function. For example, McKay, Lawrence and Vlastuin (1983) have used the dual profit functions to describe multiple-output agricultural production processes. With appropriate regularity conditions, duality theory ensures that descriptions of a technology given by, say, a profit function has an equivalent description in terms of a production or cost function.

derive a production function using *a priori* knowledge of the properties of the technology which the functional form represents.

In some cases, it may be possible to build up a production function from a lower-level description of the physical production process or technology. In a recent review of Kenneth Arrow's contribution to production theory, Starrett (1987, p. 93) comments in this vein. 'There are very few instances where empirical or theoretical economic considerations have been used to determine functional form or shape restrictions beyond the general requirements of monotonicity and concavity (or convexity). Rather, attention has focused on building flexibility into functional forms in the hope that the 'true' form will be close to some element of the class estimated (an example would be the trans-log functional form). Perhaps it is the best that can be done, but surely it would be more satisfying if functional form restrictions could be justified on the basis of experimental or empirical evidence (as they are in physics).'

In the manufacturing sector there are a few examples of 'engineering production functions' which have been derived in this way. This paper derives a number of production functions for multiple-output agricultural production from a specification of the technologies in terms of average yield functions. Hence, the resulting multiple-output production functions are called 'agronomic production functions'. This paper also discusses the implications of this family of functions for the analysis of economies of scope in agriculture. Economies of scope are of major interest in multiple cropping systems in particular. The source of economies has been the subject of debate among economists (see, for example, Baumol, Panzar and Willig 1982, ch. 4C). In agronomic production functions, economies of scope are due to output interactions arising from the biological growth processes involving crops that use common resources which is a different source of economies of scope than the presence of quasi-public inputs identified by Panzar and Willig (1981).

In the first section, the production function is derived from the average yield functions when there are only two outputs. It is also shown that it is possible to recover the average yield functions from the multiple-output production function. In the second section, these functions are generalised to allow more than two outputs and more complex interactions among them and specific inputs. This generates a family of implicitly separable multiple-output production functions. In later sections, the economies of scope properties of such functions and the main advantages of the approach used in this paper are discussed.

Derivation of the Production Function from Yield Functions

Agricultural scientists have for a long time been investigating the interrelationships between the outputs of multiple crops in intercropping situations (Willey 1979*b*). Mostly, this has concerned the interrelationships between outputs of two crops. A common device for representing the agronomic relationships that have been found empirically is to calculate the yields obtained from a replacement series. This is a series of treatments which contains the pure stands of each species and some mixtures formed by replacing given proportions of one species with the equivalent proportions of the other.

More generally, we can consider a tract of land of a certain size, which can be used to produce two agricultural outputs, y_1 and y_2. Let s_1 and s_2 be the proportions of the land devoted to outputs one and two respectively when they are produced together. It is reasonable to suppose that output yields are a function of s_1 and s_2 but independent of the size of the plot because any agricultural production process on one area can be replicated on other identical areas. There is then for each output a yield function which shows the yield per unit area as a function of its share of resources only (since $s_1 + s_2 = 1$):

$$(1) \quad y_i/L = \alpha_i f_i(s_i) \qquad \alpha_i > 0;\ i = 1, 2$$

with $f_i(0) = 0$, $f_i(1) = 1$. $df_i/ds_i \geq 0$ and $df_i/ds_i > 0$ for $s_i \in (0, 1)$. These $f_i(s_i)$ are interaction functions which determine the sign and magnitude of the interaction between the two outputs. When $s_i = 1$, $y/L = \alpha_i$. Hence, α_i are parameters that determine the efficiency of production, given units of measurement of the outputs.

A representative sample of three such functions for two cropped products is given in Figure 1. One vertical axis shows the yield from one species and the other shows the yield from the second. The dashed lines show the yield of one crop if it is intercropped with the second crop and there are no interactions between the yields of the intercropped

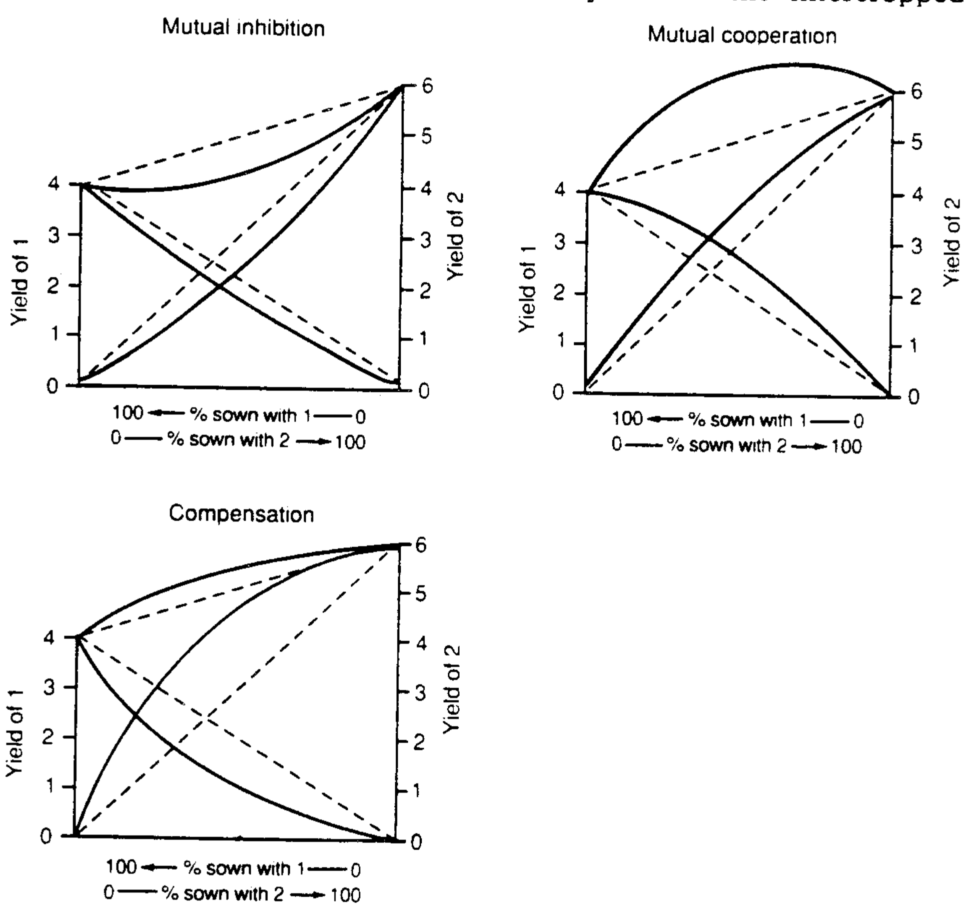

FIGURE 1—Yield Interactions in Intercropping.

outputs. This function is commonly called the expected yield. If, however, there is a positive interaction on the yield of the one crop when it is intercropped with the other crop, the actual yield function lies above the expected yield function. This is known as 'cooperation'. If, instead, there is a negative interaction on the yield of the first crop, the actual yield function lies below the expected yield function. This is known as 'inhibition'. For different crops, these interaction effects have been attributed to weeds or pests or disease or sunshine or shade or physical effects associated with temporal rotation of crops. These effects cause the intercropping yields to deviate from the expected yields.

One or both crops may exhibit some non-zero interaction. If both crops exhibit a positive interaction this is known as 'mutual cooperation' and if both exhibit a negative interaction this is known as 'mutual inhibition'. If one exhibits a positive interaction while the second exhibits a negative interaction, this is known as 'compensation'. The three panels in Figure 1 provide an example of these three types of interaction. Such interactions have been found for a wide variety of multiple crops. Willey (1979*a*, *b*) provides a number of examples of mutual cooperation in which both crops have a positive interaction or one crop has a positive interaction and the other a zero interaction: beans and sorghum or beans and corn, other legumes and non-legumes, millet and sorghum, sunflower and fodder crops, and others. Mutual inhibition will not normally be observed since intercropping in this case is less efficient than sole cropping.

While these average yield functions contain valuable information concerning the interrelationships between the crops or other outputs, they are in effect only contours of the multiple-output production function which describes the technology, if this function exists. Agricultural scientists have added the yields of the two crops. Such aggregation provides the solid upper lines in the figures. However, this aggregation is meaningless unless the two crops produce a homogeneous output. Plainly, intercropping will rarely, if ever, do so.

This problem of aggregation is overcome if both outputs are mapped on to the separate axes of a two-dimensional figure. In fact, this yields the transformation function which is a standard device in multiple-output production models. In particular, if there is mutual cooperation the transformation function will be a non-linear function which lies outside the line that maps the two expected yields. Figure 2 gives an example of such a transformation function when there is mutual cooperation. The dashed line in Figure 2 connecting the end points maps the two expected yields. This line shows the output vectors that would result if there had been zero interactions (or pure stands in varying proportions). This line is obtained by assuming zero interaction in which case $f_i(s_i)=s_i$ for $i=1, 2$.

A transformation function is defined when the resources used in the production of the two outputs are fixed. We need to know what happens when the resources and the outputs are freely variable. The natural device to represent these relationships is the multiple-output production function. This production function can be derived from output yield functions such as those plotted in Figure 1 in the following way.

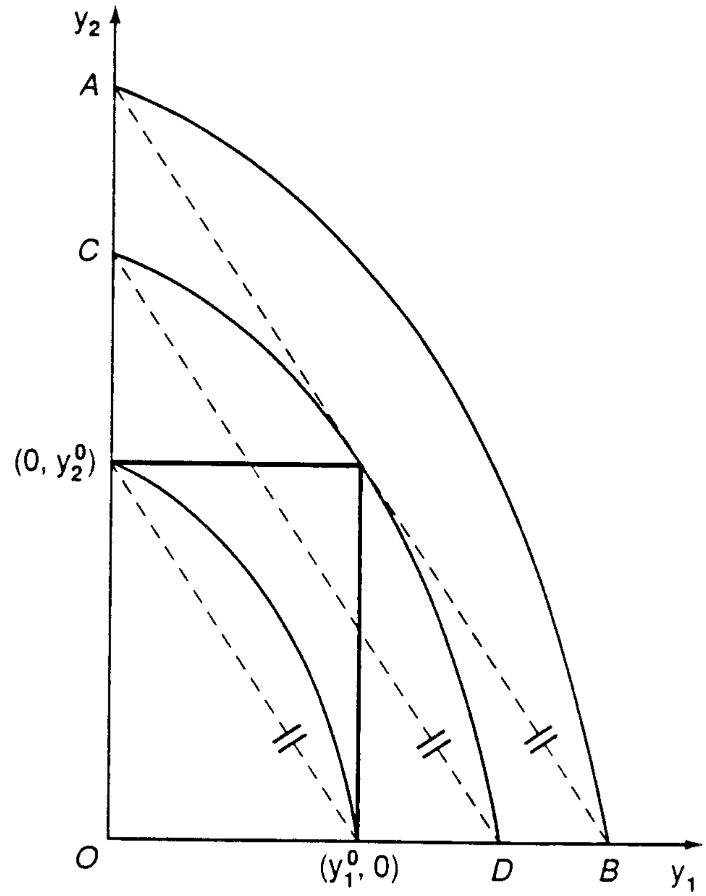

FIGURE 2—Intercropping Transformation Function.

More generally, what is called land is a composite index of productive capacity or resources.[2] If there is some substitution among inputs used in the production of these outputs, this index of productive capacity is given by

(2) $L = g(x) \qquad x = (x_1, \ldots, x_m)$

where x is a vector of non-specific inputs and $g(x)$ is homogeneous of degree one. Together these assumptions specify the whole of a multiple-input multiple-output technology. Substituting equation (2) in equations (1) gives the general form of the two yield functions

(3) $y_i/g(x) = \alpha_i f_i(s_i) \qquad i = 1, 2$

Cross-multiplying gives

(4) $y_i = \alpha_i f_i(s_i) g(x) \qquad i = 1, 2$

These equations in (4) are production functions for the two individual outputs but the dependence of each output on its share of the inputs makes the technology intrinsically joint. Such functions apply to any

[2] Kislev and Peterson (1982) constructed a realistic measure of 'augmented' land in which units of mechanical services are produced by labour and machines in variable proportions and then combined with land in fixed proportions.

multiple cropping or multiple production process where the two outputs are produced by distributing between them a common pool of resources.

These commodity production functions can be used to derive a multiple-output production function of the usual implicit form $F(y, x)=0$ where $y=(y_1, y_2)$ is the vector of outputs. Inverting equations (3) and utilising $s_1+s_2=1$ gives this function:

$$(5)\quad \sum_{i=1,2} f_i^{-1}[\alpha_i^{-1} y_i/g(x)]-1=0$$

This is the general form of the implicit production function when the technology can be described by average yield functions for each output as in equation (3).

Note that the implicit production function in equation (5) has the property of almost-homogeneity of degree one (constant returns to scale) (Hasenkamp 1976). An implicit production function is almost-homogeneous of degree $+1$ if, when $F(y, x)=0$, then $F(\lambda y, \lambda x)=0$ for $\lambda>0$. Constant returns to scale holds because the technology is replicable.

An example will now be given.

Example 1. Two-output CRETH production function

Suppose the interaction functions $f_i(s_i)$ are approximated by the power functions

$$(6)\quad f_i(s_i)=s_i^{\beta_i} \qquad i=1, 2;\ 0<\beta_i \gtreqless 1$$

If $\beta_i<(>)$ 1 there is a gain (loss) of output in that the output is greater (less) if the crops are grown together than if they are grown on separate plots as sole crops. Substituting equations (6) in equations (4) gives the two individual production functions

$$(7)\quad y_i=\alpha_i s_i^{\beta_i} g(x) \qquad i=1, 2$$

Inverting equations (6) and utilising $s_1+s_2=1$ gives the multiple-output production function

$$(8)\quad \delta_1[y_1/g(x)]^{\gamma_1}+\delta_2[y_2/g(x)]^{\gamma_2}-1=0 \qquad \delta_i=\alpha_i^{-1/\beta_i},\ \gamma_i=1/\beta_i$$

This function can be recognised as a two-output version of the CRETH multiple-output production function with a change in the normalisation such that the constant equals unity. If, further, $g(x)$ is CRESH (constant ratio of elasticities of substitution, homothetic), $F(y, x)=0$ is a CRETH–CRESH multiple-output production function.

For any given level of inputs, the slope of the transformation curve for the CRETH function is

$$(9)\quad \partial y_2/\partial y_1=-(\beta_2/\beta_1)(y_1/L)^{(1-\beta_1)/\beta_1}(y_2/L)^{-(1-\beta_2)/\beta_2}<0$$
$$=\psi(y_1/y_2)$$

If $\beta_1\leq 1$ and $\beta_2\leq 1$ and at least one inequality holds strictly, then $\partial^2 y_2/\partial y_1^2>0$, that is, the transformation function exhibits an increasing marginal rate of transformation and hence it is globally strictly quasi-convex. If $\beta_1=\beta_2=\beta$, the function reduces to

$$(10)\quad [\sum_i \gamma_i y_i^{1/\beta}]^{\beta}=g(x) \qquad \gamma_i>0$$

This is recognisable as the CET output transformation function with an input index, $g(x)$. If, further, $g(x)$ is a constant elasticity of substitution (CES) function, $F(y, x)=0$ is a CET–CES multiple-output multiple-input production function.

In this example, a specific form of the average yield functions, namely a power function, was seen to imply a specific form of the multiple-output production function, namely the CRETH function. It is an interesting question whether the converse applies. That is, does a CRETH multiple-output production function necessarily imply a power functional form of the average yield function? First, we can observe that variables s_i, the shares of resources devoted to the production of each output, do not appear in the CRETH function. However, the average yields, $y_i/g(x)$ do appear. For convenience, let $z_i = y_i/g(x)$. Moreover, the terms $\delta_i(z_i)^{1/\beta_i}$ are positive and sum to unity and hence they are positive fractions. Thus, these terms can only be interpreted as shares of the common pool of resources. Therefore, the assumption that the functional form of the multiple-output function is CRETH implies that these shares are

$$(11)\quad s_i = \delta_i z_i^{1/\beta_i}$$

and, by inversion,

$$(12)\quad z_i = \alpha_i s_i^{\beta_i} \qquad \alpha_i = \delta_i^{-\beta_i}$$

These yield functions are the equations given by (7).

For any multiple-output production function of the form in equation (5), the yield functions can be recovered by interpreting $f_i^{-1}(\alpha_i^{-1} z_i)$ as share functions and inverting them.

The General Form with n-outputs and m-inputs

The model of production outlined in the previous section can be generalised in three directions: by extending the number of output commodities, by introducing more general interaction functions and by introducing specific inputs. These will now be examined.

Suppose there is a production process with any (fixed) number of outputs, say n. Any two or more commodities may interact. However, for the time being, we shall continue to assume there are no specific inputs. We should allow a set of interaction functions which are more general than $f_i(s_i)$. The most general form is $\phi_i(s_1, \ldots, s_n)$. In place of equations (1) we now have

$$(13)\quad z_i = \alpha_i \phi_i(s_1, \ldots, s_n) \qquad i = 1, \ldots, n$$

with $\phi_i(s_i=0)=0$, $\phi_i(s_i=1)=1$ and $\partial\phi_i/\partial s_i \geq 0$. This allows the average yield for output commodity i to vary not only with i's share of resources but also with the share(s) going to one or more of the other individual commodities with which it may interact positively or negatively. If ϕ_i exist, the Implicit Function Theorem asserts that, under fairly general conditions, there are a set of inverse functions

$$(14)\quad s_i = \psi_i[\alpha_1^{-1} y_1/g(x), \ldots, \alpha_n^{-1} y_n/g(x)] \qquad i = 1, \ldots, n$$

Consequently, we have the multiple-output production function

$$(15)\quad \sum_i \psi_i[\alpha_1^{-1} y_1/g(x), \ldots, \alpha^{-1} y_n/g(x)] - 1 = 0$$

This is the general form of the n-output m-input production function with interactions of the general form in equation (13).

Consider the following example of such a function.

Example 2. Generalised power function

Suppose the yield functions have the form

$$(16) \quad z_i = \alpha_i \prod_{j=1}^{n} s_j^{\beta_{ij}} \qquad i = 1, \ldots, n; \ \beta_{ij} \geq 0$$

This system of equation is linear in the logs of the variables. Let $z' = (\log z_1, \ldots, \log z_n)$, $\alpha' = (\log \alpha_1, \ldots, \log \alpha_n)$ and $s' = (\log s_1, \ldots, \log s_n)$. Then taking logarithms, equation (13) can be rewritten as

$$(17) \quad z = \alpha + \beta s$$

Hence,

$$(18) \quad s = \gamma(z - \alpha) \qquad \gamma = \beta^{-1}$$

Using $\sum_i s_i = 1$, this gives the particular multiple-output production function

$$(19) \quad \sum_i \prod_j [\alpha_j^{-1} y_j / g(x)]^{-\gamma_{ij}} - 1 = 0$$

where γ_{ij} is the i, j-th element of γ. Equation (18) allows many patterns of interaction and complementarity/substitutability between pairs of outputs. From equation (16) if $\beta_{ij} > 0$ for all i and j, $i \neq j$, this means all outputs interact with each other. However, β_{ij} may be zero for some i and j.

One special case is that in which all of the off-diagonal elements of β are zero. This gives average yield functions of the form

$$(20) \quad y_i / g(x) = \alpha_i s_i^{\beta_i} \qquad i = 1, \ldots, n; \ \beta_i \geq 0$$

and the derived implicit multiple-output production function is

$$(21) \quad \sum_i \delta_i [y_i / g(x)]^{\gamma_i} - 1 = 0 \qquad \delta_i = \alpha_i^{-1/\beta_i}, \ \gamma_i = 1/\beta_i$$

This is recognisable as the general n-output CRETH functional form. It permits different rates of transformation between different pairs of commodities. However, the derivation from the underlying yield functions in equation (20) allows the yield per unit of resources for each commodity to vary only with the share of resources devoted to this commodity. That is, if there is any interaction it is between this commodity and the other commodities as a group. It is independent of the composition of the group. In particular, this rules out the possibility that the positive effect of producing one output may be linked to the growing of only one or a proper subset of the other outputs. Thus, the CRETH form is quite restrictive when $n > 2$.

Not all multiple-input multiple-output production functions can be written in the form of equation (15). As an example consider the translog function.

Example 3. Almost-homogeneous input–output-separable translog function

Christensen *et al.* (1973) specified a multiple-output multiple-input production function which is quadratic in the logarithms of outputs and inputs.

$$(22)\quad \alpha+\sum_i \beta_i \log y_i+\sum_h \gamma_h \log x_h+\sum_i\sum_j \delta_{ij} \log y_i \log y_j + \sum_h\sum_k \varepsilon_{hk} \log x_h \log x_k+\sum_i\sum_h \psi_{ih} \log y_i \log x_h=0$$

This functional form is flexible and has become popular. Under certain additional restrictions it is input–output separable in the usual sense that $F(y, x)=h(y)-g(x)=0$, where $h(y)$ is a transformation function for outputs and $g(x)$ is an input index function (Lau 1972), that is

$$(23)\quad \sum_i\sum_j \delta_{ij} \log y_i \log y_j=\sum_h\sum_k \varepsilon_{hk} \log x_h \log x_k$$

Input–output separability implies there is an index for a group of inputs which may be used to produce multiple outputs in different proportions and which satisfies $h(y)=g(x)$. This input–output-separable form of the translog function cannot be written in the form of equation (15) in general. However, suppose the translog function is also almost-homogeneous of degree one. Then $g(x)$ must be homogeneous of degree one and, setting $\gamma=1/g(x)$, we have

$$(24)\quad \sum_i\sum_j \delta_{ij} \log [y_i/g(x)] \log [y_j/g(x)]-1=0$$

The terms summed over i sum to unity. If they are all positive, the almost-homogeneous input–output-separable translog function is another case of equation (15).

Not all functions of the form of equation (15) are input–output separable; for example, the CRETH function with $\beta_1 \neq \beta_2$. Thus, the form of separability in equation (15) is less restrictive than the usual form of separability in multiple-output multiple-input technologies. This holds essentially because it is a property of the implicit form of the production function. It can therefore be called implicit separability.

It is now possible to see the relationship between implicitly separable multiple-output production functions and flexible functional forms. Any multiple-output multiple-input production function which is input–output separable and almost-homogeneous of degree one can be written in the form

$$(25)\quad h[y/g(x)]-1=0$$

Flexible forms such as the translog or generalised McFadden or generalised Barnett functions (Diewert and Wales 1987) are linear in the parameters. Consequently, if these flexible forms for the primal function are also assumed to be input–output separable and almost-homogeneous, they may be written in the implicitly separable form of equation (14). Thus, these flexible functions with additional restrictions imposed are further examples of agronomic production functions but, without these additional restrictions, they do not conform to these technologies.

It might be thought that the restrictions of implicit input–output separability and almost-homogeneity (constant returns to scale) are severe. However, these are precisely the restrictions which must be satisfied if and when one has a prior belief that the agricultural production process conforms to that described in equations (3) or (4).

The analysis can be extended to accommodate specific inputs. If these inputs are proportional to outputs one merely adds more equations, which do not affect equation (15). If there is substitutability between specific and non-specific inputs, the natural method is to generalise the individual commodity production functions in equation (13) to

$$(26)\quad z_i = \alpha_i \phi_i(s_1, \ldots, s_n, v_i) \qquad i = 1, \ldots, n$$

where v_i is a vector of inputs which is specific to output i and measured per unit of land, a composite of non-specific inputs. These equations define an implicit multiple-output production function which can be obtained by maximising y_1, given the values of $y_2, \ldots, y_n$. Thus, it is always possible to derive a multiple-output production function from the underlying technology in the presence of specific inputs. If these functions in equation (26) can be inverted, we have

$$(27)\quad s_i = \Psi_i(\alpha_i^{-1} z_i, v_i) \qquad i = 1, \ldots, n$$

and the multiple-output production function is

$$(28)\quad \sum_i \Psi_i(\alpha_i^{-1} z_i, v_i) - 1 = 0$$

These examples show that this family of functions is distinct from input–output-separable multiple-input multiple-output production functions of the form $F(y, x) = h(y) - g(x) = 0$. To my knowledge, the family defined by equation (15) [or most generally by equation (28)] has not been discussed generically before.[3] However, it is a family which derives from the traditional agronomic concept of yield functions.

Economies of Scope in Agronomic Production Functions

Economies of scope were defined by Panzar and Willig (1981) as a property of the dual multiple-output cost function, $C(y, w)$ where y is the vector of outputs and w the vector of input prices. For two distinct commodity sets, one and two, there are economies of scope at $y = (y_1, y_2)$ if and only if, for any w,

$$(29)\quad C(y_1, y_2, w) < C(y_1, 0, w) + C(0, y_2, w)$$

As defined this is a local and strict property. Thus, economies of scope is the property that the cost function be locally and strictly orthogonally

[3] The family of multiple-output production functions which are both input–output separable in the usual sense and almost-homogeneous has been considered by Hasenkamp (1976). Hanoch (1975) developed the concept of implicit additivity for single-output production functions. The family of functions above is an extension with multiple-outputs and the additional property of almost-homogeneity, and it has been derived from the description of the technology for individual commodities.

sub-additive. There are diseconomies of scope at (y_1, y_2) if and only if, for any w,

$$(30) \quad C(y_1, y_2, w) > C(y_1, 0, w) + C(0, y_2, w)$$

Economies (diseconomies) of scope hold globally if and only if the cost sub- (super-) additivity holds for all (y, w).

Economies of scope are related to the complementarity of outputs in the cost function. Two outputs h and j are said to be locally complementary if the multiple-output cost function has the property $\partial^2 C/\partial y_h \, \partial y_j < 0$. If there are economies of scope, from equation (29), we have

$$(31) \quad [C(y_1, y_2, w) - C(0, y_2, w)] - [C(y_1, 0, w) - C(0, 0, w)] < 0$$

since in the long run $C(0, 0, w) = 0$. Dividing both sides by $\Delta y_2 = (y_2 - 0)$ we have

$$(32) \quad \Delta(\Delta C/\Delta y_1)/\Delta y_2 < 0$$

This relation describes complementarity over the arc from 0 to (y_1, y_2). Consequently, it is not surprising that Baumol *et al.* (1982, p. 75) showed that local complementarity is a sufficient condition for economies of scope in the production of two outputs.

When the production technology is represented by a multiple-output production function, economies of scope can also be characterised in terms of the property that the multiple-output production function is strictly orthogonally super-additive. This property is defined as follows. Let y' and y'' be disjoint vectors. The function is strictly orthogonally super-additive if, when $F(y', x') = 0$ and $F(y'', x'') = 0$, there exists an output vector such that $F(y, x) = 0$ with $x = x' + x''$ and $y = y' + y''$ (that is, $y_i = y_i' + y_i''$ for all i and $y_i > y_i' + y_i''$ for some i). In other words, when two outputs are produced together a greater output of one or both can be produced than when they are produced singly, using a given quantity of inputs. This property may hold locally or globally.

In the literature on intercropping of two crops, economies of scope have long been recognised. Mutual cooperation can now be identified as a strong form of economies of scope in terms of a super-additive production function. Economies of scope may also arise if there is a positive interaction with one crop and a zero interaction with another, or even if there is a situation of compensation. Therefore, it is not surprising that agricultural scientists have also stated these economies in terms of 'complementarities', in much the same manner as Baumol *et al.* (1982, ch. 7).[4] Agricultural scientists have even devised measures of the local degree of economies of scope in terms of input usage such as the 'land equivalent ratio'. This is the ratio of the land required to grow a given quantity of two crops when they are sole cropped to the land required when they are intercropped (see Willey 1979*a*).

[4] For example, '. . . usually a yield advantage occurs because component crops differ in their use of growth resources in such a way that when they are grown in combination they are able to 'complement' each other and so make better overall use of resources than when grown separately. In terms of competition, this means that in some way the component crops are not competing for exactly the same overall resources and thus intercrop competition is less than intracrop competition. Maximizing intercropping advantages is therefore a matter of maximizing the degree of 'complementarity' between components and minimizing intercrop competition' (Willey 1979*a*, p. 7).

When one has the description of the technology given in full by the multiple-output production function this ratio can be calculated. for any vector of outputs. $y=(y_1, y_2)$, directly from the function. With sole cropping the land required to obtain y_1 and y_2 is obtained by solving $F[y_1, 0, g(x)]=0$ and $F[0, y_2, g(x)]=0$ respectively for $g(x)$. Call these land requirements $g_1(x)$ and $g_2(x)$. With intercropping, the land required to obtain (y_1, y_2) is obtained by solving $F[y_1, y_2, g(x)]=0$. Call this land requirement $g_{1+2}(x)$. Then the land equivalent ratio is $[g_1(x)+g_2(x)]/g_{1+2}(x)$. This is greater than unity if there are economies of scope. Alternatively, the economies of scope can be measured in terms of the ratio of costs $C(y_1, y, w)/[C(y_1, 0, w)+C(0, y_2, w)]$. This will be less than unity. Moreover, with given w, this ratio of costs is equal to the inverse of the land equivalent ratio. Both measures are local measures only. The extent of economies of scope as measured by these ratios will vary over the output space.

The two definitions of economies of scope in terms of the properties of the multiple-output production function and those of the multiple-output cost function can be related. Duality theory shows that the cost function and the production function are two equivalent ways of representing the production technology. It follows that any property possessed by one function. say the production function. must be repeated in an equivalent way in the dual function. It is reasonable to suppose that if two (or more) outputs can be produced at lesser cost in combination than singly, then the resources used to produce two output levels singly will produce a higher output when the two outputs are grown in combination, and conversely; that is, the multiple-output production function is (globally) orthogonally super-additive if and only if the dual multiple-output cost function is (globally) orthogonally sub-additive. Rosse (1970) proved this formally.

A geometric example of economies of scope expressed in terms of the transformation function is given in Figure 2. Suppose there is a given quantity of land which can be used to grow two crops. $(y_1^0, 0)$ and $(0, y_2^0)$ are the output vectors when the land is used to produce each crop singly. If there is zero interaction between the two crops, the dashed line joining the two end points would show the possible combinations of y_1 and y_2 which could be grown. If there are positive interactions between one or both crops, the transformation curve showing the combinations of y_1 and y_2 which can be grown by switching resources between them will lie outside the line. In Figure 2 the curve is convex. With constant returns to scale there will be a map of such curves in which each curve is a radial blow up of the first curve. The lines connecting the end points will be parallel. Now consider the combination (y_1^0, y_2^0). This must lie on a line which is parallel to the line joining $(0, y_2^0)$ and $(y_1^0, 0)$. This output point is achievable if the two crops are sole cropped on twice the area which will produce $(y_1^0, 0)$ or $(0, y_2^0)$. However, since there are positive interactions intercropping increases yields as shown by all points on the curve AB. Alternatively, the output vector (y_1^0, y_2^0) lies on the curve CD which shows that it is achievable using less resources than those required to produce $(y_1^0, 0)$ and $(0, y_2^0)$. This figure represents graphically economies of scope in the dual primal production function and the cost function.

As an example of a function exhibiting economies of scope or diseconomies of scope consider the generalised power function in equations (16) or (19). In this function the β_{ij} are the elasticities of average yield with respect to the shares. If resources are switched from the production of commodity k to production of commodity i there are positive (negative) interactions and associated economies (diseconomies) of scope as $(\beta_{ii}-\beta_{ik})<(>)\ 1$.

These economies of scope must also appear in the dual cost function, as noted above. It is not possible to derive an exact expression for the dual of equation (19). However, we can obtain the cost function for the special case of the power function in equation (10) above. This production function is super-additive. It has the dual cost function

$$(33)\quad C(y,w)=h(y)c(w)$$
$$=[\sum_i \gamma_i y_i^{1/\beta}]^{\beta}c(w)$$

where $c(w)$ is a linearly homogeneous index of the cost of producing a unit of land, $g(x)$. If $\beta_i \leq 1$ for $i=1$ and 2, and $\beta_i<1$ for $i=1$ or 2, $h(y)$ is strictly quasi-convex. Hence, the cost function is strictly orthogonally sub-additive, namely

$$(34)\quad C(y,w)=[\sum_i \gamma_i y_i^{1/\beta}]^{\beta}c(w)$$
$$<\sum_i[\gamma_i y_i^{1/\beta}]^{\beta}c(w)=C(y_1,0,w)+C(0,y_2,w)$$

That is, there are economies of scope in the multiple-output production function and its dual cost function.

The source of economies of scope in farms that utilise such technologies is very different than the use of quasi-public inputs to which Panzar and Willig (1981) attributed economies of scope in firms. Panzar and Willig defined quasi-public inputs as 'inputs which, once procured for the production of one output, would be also available (either wholly or in part) to aid in the production of other outputs'. In these agronomic technologies, the derivation of the multiple-output production function showed that the individual commodities use common inputs but the sum of the shares is always unity. That is, these common inputs are not quasi-public inputs. Instead, the economies of scope derive from output interactions arising from the biological growth processes, as specified in equations (3) or (4); for example, the benefits of inter-cropping or of crop rotations.[5]

Quasi-public inputs may also be important in some agricultural technologies. For example, at the farm level some items of farm machinery may be available at zero cost for the production of other commodities. (This assumes that such surplus capacity cannot be sold to other farms.)

Economies of scope due to common inputs and economies due to quasi-public inputs can be combined in one technology. The yield

[5] These interactions in turn can be modelled by a sub-model of the biological growth process for each crop (see Willey 1979*a*, pp. 7–10). However, these sub-models are not relevant to economic decisions as the physical inputs such as light, water, soil nutrients and nitrogen are not purchased from markets.

functions for individual commodities in equation (13) can be generalised further to

$$(35) \quad z_i = \alpha_i \Theta_i(s_1, \ldots, s_n, K)$$

where K is a vector of quasi-public inputs. If these functions can be inverted, we have

$$(36) \quad s_i = \mu_i(\alpha_i^{-1} z_i, K)$$

and the multiple-output multiple-input production function with both common and quasi-public inputs is

$$(37) \quad \sum_i \mu_i(\alpha_i^{-1} z_i, K) - 1 = 0$$

This form is distinct from equation (13) above.

Conclusions

The derivation of production functions from the underlying technologies has two main advantages. First, by choosing a functional form from the family of functions defined by equations (15) or (28) or (37) one has a function with properties that are appropriate to the technology. It is possible to obtain any desired pattern of interactions and associated economies or diseconomies of scope. Second, it gives a much deeper understanding of the multiple-output production and cost functions by making it possible to relate them back to the yield functions of individual agricultural commodities. This is an example of the general method of constructing a production function from the underlying technology which has the features appropriate to the technology.

References

Akridge, J. and Hertel, T. (1986), 'Retail fertilizer plants and multiproduct theory', *American Journal of Agricultural Economics* 68(4), 928–38.

Baumol, W., Panzar, J. C. and Willig, R. D. (1982), *Contestable Markets and the Theory of Industry Structure*, Harcourt, Brace and Jovanovich, New York.

Bishop, C. E. and Toussaint, W. D. (1958), *Introduction to Agricultural Economic Analysis*, John Wiley, New York.

Christensen, L. R., Jorgensen, D. W. and Lau, L. J. (1973), 'Transcendental production frontiers', *Review of Economics and Statistics* 55(1), 28–45.

Diewert, W. E. (1971), 'An application of Shephard duality theorem: A generalized Leontief production function', *Journal of Political Economy* 79(3), 481–507.

—— (1974), 'Applications of duality theory' in M. D. Intriligator and D. A. Kendrick, *Frontiers of Quantitative Economics*, II, North-Holland, Amsterdam.

—— and Wales, T. J. (1987), 'Flexible functional forms and global curvature conditions', *Econometrica* 55(1), 43–68.

Hanoch, G. (1975), 'Production and demand models with direct or indirect implicit additivity', *Econometrica* 43(3), 395–419.

Hasenkamp, G. (1976), *Specification and Estimation of Multiple-Output Production Functions*, Springer-Verlag, Wien.

Kislev, Y. and Peterson, W. (1982), 'Prices, technology and farm size', *Journal of Political Economy* 90(3), 578–95.

Lau, L. J. (1972), 'Profit functions of technologies with multiple inputs and outputs', *Review of Economics and Statistics* 54(3), 281–9.

McKay, L., Lawrence, D. and Vlastuin, C. (1983), 'Profit, output supply and input demand functions for multiproduct firms: The case of Australian agriculture', *International Economic Review* 24(2), 323–39.

Panzar, J. C. and Willig, R. D. (1981), 'Economies of scope', *American Economic Review* 71(2), 268–72.

Powell, A. A. and Gruen, F. H. (1968), 'The constant elasticity of transformation frontier and linear supply system', *International Economic Review* 9(3), 315–28.

Rosse, J. N. (1970), 'Equivalent structures of production, cost and input demands with multiple inputs and outputs', Department of Economics, Stanford University, *Studies in Industry Economics*, No. 13.

Starrett, D. A. (1987), '*Production and Capital*: Kenneth Arrow's contributions in perspective – a review article', *Journal of Economic Literature* 25(1), 92–102.

Vincent, D. P., Dixon, P. B. and Powell, A. A. (1980), 'The estimation of supply response in Australian agriculture: The CRESH/CRETH production system', *International Economic Review* 21(1), 221–42.

Willey, R. W. (1979*a*), 'Intercropping – its importance and research needs. Part 1. Competition and yields advantages', *Field Crop Abstracts* 32(1), 1–10.

——(1979*b*), 'Intercropping – its importance and research needs. Part 2. Agronomy and research approaches', *Field Crop Abstracts* 32(2), 73–85.