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WORKING PAPER SERIES

University of California, Davis
Department of Agricultural Economics

COMPUTATIONAL EXPERIENCE WITH KARMARKAR'S ALGORITHM FOR LINEAR PROGRAMMING. PART II. ALGORITHMIC VARIANTS

by

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COMPUTATIONAL EXPERIENCE WITH KARMARKAR'S ALGORITHM FOR LINEAR PROGRAMMING

PART II. ALGORITHMIC VARIANTS

1. Introduction

This paper reports the results of several variants of Karmarkar's algorithm for linear programming. While all these variants work, their discussion is presented here with the purpose of better understanding the algorithm.

A summary of the algorithm as originally presented by Karmarkar will be given in section 2. A more detailed discussion of it can be found in Paris. The proposed variants will be briefly introduced in section 3. Each of the variants will be analyzed geometrically and numerically in section 4. Tentative conclusions will be offered in section 5.

2. The Sphere Algorithm

The characterizing geometrical figure of Karmarkar's algorithm for linear programming is a sphere. For this reason, it is called the sphere algorithm, as opposed to Khachiyan algorithm, known also as the ellipsoid algorithm.

Consider the following LP problem:

$$\text{(1)} \qquad \min \quad c^T x = \text{CMIN}$$
$$\text{subject to:} \quad Ax = b, \quad x \geq 0$$

where A is a (mxn) matrix of rank m. Consider a strictly interior point $x_o > 0$ for problem (1) an define a projective transformation using the diagonal matrix D_o such that $D_o = \text{diag}(x_o)$:

$$\text{(2)} \qquad x = \frac{D_o x'}{e^T D_o x' + 1} = \frac{D_o x'}{x'_{n+1}}$$

where $x'_{n+1} = 1 - e^T x'$ and e is a vector of ones. This transformation allows the re-specification of (1) as follows:

(3)
$$\min \left[c^T D_o x' - CMIN x'_{n+1}\right]$$
$$\text{subject to: } AD_o x' - bx'_{n+1} = 0$$
$$e^T x' + x'_{n+1} = 1$$
$$x'_{n+1} \geq 0,\ x'_{n+1} > 0$$

or, in more compact form:

(4)
$$\min c'^T x'$$
$$\text{subject to: } A'x' = 0$$
$$e^T x' = 1$$
$$x' \geq 0$$

where $x'^T = \left[x'^T,\ x'_{n+1}\right]$, $A' = [AD_o,\ -\ b]$, $c'^T = [c^T D_o,\ -\ CMIN]$

Consider now a second projective transformation

(5)
$$x' = \frac{Dx''}{e^T Dx''}$$

where D is a (n+1)x(n+1) diagonal matrix, D = diag(x'). Problem (4) can now be transformed into the reference working framework of Karmarkar by substituting (5) for x' and replacing the nonegativity conditions by a sphere as follows:

(6)
$$\min\ c'^T D x''$$
$$\text{subject to: } A'Dx'' = 0$$
$$e^T x'' = 1$$
$$x'' = a_o - \alpha r \hat{c}$$

where $a_o = e/(n+1)$ is the center of the simplex $S'' = \{x'' | e^T x'' = 1, x'' \geq 0\}$ and of the sphere $B''(a_o, \alpha r)$, r is the radius of the largest inscribed sphere and α is a number between 0 and 1.

Phase II of the algorithm, as defined by Karmarkar requires the knowledge of a strictly interior point, $x' > 0$. The various steps of the algorithm can now be listed as follows:

Phase II - Iterative Loop

B.1 Define the diagonal matrix $D = \mathrm{diag}(x')$.

B.2 Define the matrix $B = \begin{bmatrix} A'D \\ \hline e^T \end{bmatrix}$.

B.3 Compute the orthogonal projection of the objective function's gradient

$$c_p = [I - B^T(BB^T)^{-1}B]Dc'.$$

B.4 Normalize c_p, that is $\hat{c} = c_p/(c_p'c_p)^{1/2}$.

B.5 Compute the next interior feasible point using the recursive relation

$$x'' = a_o - \alpha r\hat{c}.$$

B.6 Compute the inverse projective transformation

$$x' = Dx''/e^T Dx''.$$

B.7 Verify the convergence criterion

$$\frac{c'^T x'}{c'^T a_o} < 2^{-q}$$

where q is the precision of the available computing machine. If the convergence relation is satisfied, stop. Otherwise, return to B.1.

Phase I - Initial Interior Point

To begin Phase II, an initial, strictly interior, feasible point $x' > 0$ for the system

(7) $$A'x' = 0$$
$$e^T x' = 1$$
$$x' \geq 0$$

is required. To achieve this objective consider the original system

(8) $$Ax = b$$
$$x \geq 0$$

A.1 Choose $x_o > 0$ to be an arbitrary point, strictly interior to the positive orthant.

A.2 Define the discrepancy vector

$$d = Ax_o - b.$$

A.3 Define the artificial problem

(9) $$\min \ \lambda$$
$$\text{subject to: } Ax - \lambda d = b$$
$$x \geq 0, \ \lambda \geq 0$$

Notice that $x = x_o$ and $\lambda = 1$ constitute a strictly interior point for (9).

A.4 Apply Karmarkar's algorithm as defined in Phase II to problem (9). The optimum value of λ is $\lambda = 0$ which, when achieved, will be associated with an explicit, strictly interior, feasible point x' for problem (7).

3. Variants of Karmarkar's Algorithm

Two principal variants of Karmarkar's algorithm will illustrate its essential features:

1. elimination of the capacity constraint $e^T x'' = 1$;
2. elimination of Phase II.

In spite of the seemingly essential requirement, it can be shown that the constraint $e^T x'' = 1$ is not crucial for the working of the algorithm. The reason is based upon the following fact. First of all, from the projective transformation

$$x' = \frac{Dx''}{e^T Dx''}$$

it is seen that the coordinates of the x' vector of problem (4) add up to one regardless of whether the constraint $e^T x'' = 1$ is imposed explicitly on problem (6). In fact,

$$e^T x' = \frac{e^T Dx''}{e^T Dx''} = 1.$$

The elimination of the constraint $e^T x'' = 1$ from the specification of problem (6) has, in general, positive effects. For example, the condition number of the B matrix (which now has one less row) is improved by an order of magnitude which depends on n, the number of variables in the original problem.

Intuitively, the possibility of successfully solving the given LP problem without imposing the constraint $e^T x'' = 1$ is due to the essential role played by the sphere $B''(a_o, \alpha r)$, represented in the algorithm by the recursive relation in B.5. Since x'' is a vector whose coordinates differ only slightly from those of a_o, we can be sure that they will always be positive and will constitute a feasible solution of the problem. Then, whether or not they add up to one is not an essential aspect of Karmarkar's algorithm.

The other major variant deals with the possibility of eliminating Phase II and concentrating in Phase I the problem of finding an interior feasible and optimal point. The required modification of the artificial problem as defined in A.3 is easily justified. Define

$$d_1 = c^T x_o - \text{CMIN}$$

as the discrepancy in the objective function when it is evaluated at the arbitrary point $x_o > 0$. Then, the new artificial problem is defined as

(10) $$\min \ \{c^T x - \lambda d_1\} = \text{CMIN}$$

$$\text{subject to: } Ax - \lambda d = b$$

$$x \geq 0, \ \lambda \geq 0.$$

A value of CMIN (easy to guess) substantially above the optimal value is required. At this point, Karmarkar's algorithm is applied to problem (10). If the value of λ can be driven to zero, the resulting interior point is also optimal. Otherwise, the original problem does not possess a solution.

4. Detailed Discussion of the Variants

In the following section a small numerical example will be analyzed with the purpose of providing empirical evidence for the deductions discussed above. Six scenarios are studied:

E.1 A Phase I problem with the $e^T x'' = 1$ constraint;

E.2 The same Phase I problem without the $e^T x'' = 1$ constraint;

E.3 A Phase II problem with the $e^T x'' = 1$ constraint;

E.4 The same Phase II problem without the $e^T x'' = 1$ constraint;

E.5 An optimizing Phase I problem with the $e^T x'' = 1$ constraint;

E.6 The same optimizing problem without the $e^T x'' = 1$ constraint.

E.1 Phase I with the $e^T x'' = 1$ constraint

On pages 8 through 10 the numerical analysis of this case is presented. It is preceded by the solution of the corresponding LP problem performed by means of the simplex method. From the printout it can be seen that the matrix

A is of dimensions (2x3). The results are those expected, that is $e^T x'' = 1$, $e^T x' = 1$, $e^T \hat{c} = 0$. The condition number at the beginning and at the end of the process is 213,94 and 194,61, respectively.

E.2 Phase I <u>without</u> the $e^T x'' = 1$ constraint

Pages 11 through 13 report the solution of the same numerical problem without the capacity constraint $e^T x'' = 1$. Notice that, now:

$$e^T x'' \neq 1,\ e^T x' = 1,\ e^T c = 1,\ \text{and}\ e^T x'' + \alpha/\sqrt{n(n+1)} = 1.$$

This means that x'' does not lie on the simplex, but this fact does not prevent the achievement of the goal of finding a feasible, strictly interior point. In fact, by comparing the two interior points (with and without $e^T x'' = 1$) we notice that they are the same (except for rounding-off errors). It took the same number of iterations to solve the two problems but the CPU time in this latter case is 20 percent less than that with $e^T x'' = 1$. The condition number is also considerably better than in the case with $e^T x'' = 1$.

E.3 Phase II <u>with</u> $e^T x'' = 1$

Pages 14 and 15 give the results of solving a LP problem following strictly Karmarkar's layout. The same optimal solution achieved by the simplex method is reached in 16 iterations and 19.67 seconds of CPU time. Condition numbers are better than in Karmarkar's Phase I and Phase II. As required, $e^T x'' = 1$, $e^T x' = 1$, and $e^T c = 0$.

E.4 Phase II <u>without</u> $eTx'' = 1$

The same optimal solution is achieved with the same number of iterations but less CPU time. The conditions numbers are better than in the case with

```
:_MINSIMPLEX
EXECUTION STARTED
LPMIN(C,A,B,0:PRIMAL=P,DDUAL=D) =  6.7273

   A
 *****
 2 1 3
 5 2 2


  B   C    PRIMAL    DDUAL   CPULP
 **   *   *******   ******   *****
  6   3   1.6364    .36364   .4
 10   4   0         .45455
      2    .90909

MANUAL MODE
:_$
:_$
:_$
:_$  EXAMPLE OF PHASE ONE WITH ETX = 1.
:_$
:_$
:_PHASE1V
EXECUTION STARTED

  P H A S E     O N E      B E G I N S

 INPUT ALPHA Q
:>ALPHA=.5
:>Q=20
:><<< NULL LINE ENTERED >>>
INPUT TABUL INCREMENT
:>TABUL=5
:>INCREMENT=4
:><<< NULL LINE ENTERED >>>
INPUT ARE THE MATRIX A AND VECTOR B DEFINED? (Y/N) Y=1 N=Y=0,Y
:>Y=1
:><<< NULL LINE ENTERED >>>

   A      B
 *****   **
 2 1 3    6
 5 2 2   10

INPUT ARE DATA OK? (Y/N) Y=1 N=Y=0,  Y
:><<< NULL LINE ENTERED >>>

 AUGMENTA
 ********
 2 1 3 4
 5 2 2 7


 DISCREP   RHS   ARTIFC
 *******   ***   ******
   -4       6     0
   -7      10     0
                  0
                  1


 ITER   PBTROOTS
 ****   ********
```

SIMPLEX

4 .023371
2.3015
5

CONDITION NUMBER = 213.94

ITER	BP1	X	CP	CHAT
5	.22999	.36478	-.0035593	-.26822
	.22674	.29074	-.0031732	-.23913
	.22681	.23718	-.0031819	-.23979
	.10045	.006656	.011816	.89042
	.21602	.10065	-.0019014	-.14328

DD					S
.35672	0	0	0	0	-.09568
0	.2884	0	0	0	-.068963
0	0	.23519	0	0	-.056395
0	0	0	.014903	0	.01327
0	0	0	0	.10479	-.015015

ITER	BP1	X	CP	CHAT
9	.22521	.37116	-1.1793E-4	-.22547
	.22507	.29236	-1.1728E-4	-.22423
	.22509	.23877	-1.1736E-4	-.22438
	.1	2.5989E-4	4.6781E-4	.89442
	.22464	.09745	-1.1525E-4	-.22035

DD					S
.37083	0	0	0	0	-.083609
0	.29228	0	0	0	-.065538
0	0	.23869	0	0	-.053557
0	0	0	5.8477E-4	0	5.2303E-4
0	0	0	0	.097613	-.021508

ITER	BP1	X	CP	CHAT
13	.22501	.37141	-4.5645E-6	-.22368
	.225	.29242	-4.5636E-6	-.22363
	.225	.23883	-4.5637E-6	-.22364
	.1	1.014E-5	1.8252E-5	.89443
	.22499	.097326	-4.5605E-6	-.22348

DD					S
.3714	0	0	0	0	-.083074
0	.29242	0	0	0	-.065394
0	0	.23883	0	0	-.053411
0	0	0	2.2815E-5	0	2.0407E-5
0	0	0	0	.097332	-.021752

ITER	BP1	X	CP	CHAT
17	.225	.37142	-1.7805E-7	-.22361
	.225	.29242	-1.7804E-7	-.22361

```
      .225    .23884    -1.7804E-7  -.22361
      .1     3.9565E-7   7.1217E-7   .89443
      .225    .097321   -1.7804E-7  -.2236


                  DD                                    S
***************************************   *********
 .37142 0      0         0         0          -.083053
0        .29242 0        0         0          -.065388
0       0        .23884 0          0          -.053406
0       0       0        8.9022E-7 0          7.9623E-7
0       0       0        0            .097321 -.021761


RESULTS AFTER  ITER =  19    ITERATIONS

 CONVERG   CPUF1     CMIN        CMAX
*********  *****  *********  **********
8.7923E-7  27.06  6.0229E-7  8.0334E-12


  XPRIME       OLDX        DUAL       BBTROOTS  INTERIOR
*********  *********  ***********  ********  ********
 .37142    1.2721     4.134E-13    .025692  .37142
 .29242    1.0016     5.5529E-13  1.8368    .29242
 .23884     .81904    7.913E-8    5         .23884
1.7585E-7  6.0229E-7                         .097321
 .097321   0
```

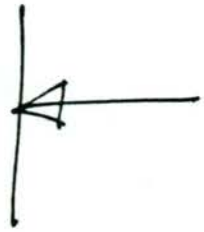
→ CONDITION NUMBER 194.61

```
    P H A S E     O N E     E N D S

MANUAL MODE
:_#
:_#
:_TABULATE BB BP1 XPRIME CP CHAT
IN LINE 'TABULATE BB BP1 XPRIME CP CHAT'  TABULATION FAILED+

  BB (A 3 BY 5 MATRIX)
  .24761     .097475     .23884      1.5826E-6 -.58392
  .61903     .19495      .15922      2.7696E-6 -.97321
  1          1           1           1          1
```

← $e^T x'' = 1$

```
  BP1 (A VECTOR WITH 5 COMPONENTS)
  .225  .225  .225  .1    .225

  XPRIME (A VECTOR WITH 5 COMPONENTS)
  .37142     .29242      .23884      1.7585E-7  .097321

  CP (A VECTOR WITH 5 COMPONENTS)
 -7.9131E-8 -7.913E-8  -7.913E-8   3.1652E-7 -7.9129E-8

  CHAT (A VECTOR WITH 5 COMPONENTS)
 -.22361 -.22361 -.22361  .89443 -.2236

:_TABULATE BB


                  BB
***************************************
 .24761   .097475   .23884 1.5826E-6 -.58392
 .61903   .19495    .15922 2.7696E-6 -.97321
1        1         1        1         1
```

```
:_TABULATE DD

                        DD
******************************************
  .37142 0          0          0            0
 0         .29242 0            0            0
 0          0          .23884 0            0
 0          0          0          3.9565E-7 0
 0          0          0          0            .097321

:_SUM(BP1)
SUM(BP1) =   1
:_SUM(XPRIME)
SUM(XPRIME) =   1
:_SUM(CHAT)
SUM(CHAT) = -3.4694E-18
:_$
:_$
:_$
:_$ EXAMPLE OF PHASE ONE WITHOUT ETX = 1.
:_$
:_$
:_PHASEW01
EXECUTION STARTED

  P H A S E     O N E     B E G I N S

 INPUT ALPHA Q
:>ALPHA
ALPHA =   .5
:>Q
Q =   20
:><<< NULL LINE ENTERED >>>
INPUT ARE THE MATRIX A AND VECTOR B DEFINED? (Y/N) Y=1 N=Y=0,Y
:>JOURNAL/OFF
:>PHASEW21
EXECUTION STARTED

  P H A S E     O N E     B E G I N S

 INPUT ALPHA Q
:>ALPHA
ALPHA =   .5
:>Q
Q =   20
:><<< NULL LINE ENTERED >>>
INPUT TABUL INCREMENT
:>TABUL=5
:>INCREMENT
INCREMEN =   4
:><<< NULL LINE ENTERED >>>
INPUT ARE THE MATRIX A AND VECTOR B DEFINED? (Y/N) Y=1 N=Y=0,Y
:>Y
Y =   1
:><<< NULL LINE ENTERED >>>

   A      B
 *****   **
 2 1 3    6
 5 2 2   10

INPUT ARE DATA OK? (Y/N) Y=1 N=Y=0,   Y
:>Y
```



Y = 1
•<<< NULL LINE ENTERED >>>

AUGMENTA			
********			
2	1	3	4
5	2	2	7

DISCREP	RHS	ARTIFC
*******	***	******
-4	6	0
-7	10	0
		0
		1

ITER	BETROOTS
****	********
4	.023296
	2.327

CONDITION NUMBER = 99.93

ITER	BP1	X	CP	CHAT
****	*******	********	**********	********
5	.20451	.36467	-6.2277E-4	-.040362
	.2015	.29052	-2.0706E-4	-.01342
	.20156	.23717	-2.1494E-4	-.01393
	.088629	.0068709	.01537	.99615
	.19157	.10077	.0011639	.075436

DD					DDCHAT
************************************					*********
.35626	0	0	0	0	-.014379
0	.28907	0	0	0	-.0038657
0	0	.23509	0	0	-.0032749
0	0	0	.015489	0	.015429
0	0	0	0	.1051	.007928

ITER	BP1	X	CP	CHAT
****	*******	*********	**********	**********
9	.20019	.37127	-9.8797E-7	-.0016751
	.20006	.2922	-3.2875E-7	-5.574E-4
	.20008	.23881	-4.1184E-7	-6.9828E-4
	.088197	2.6008E-4	5.8978E-4	.99999
	.19967	.097461	1.7358E-6	.0029431

DD					DDCHAT
**					**********
.37093	0	0	0	0	-6.2137E-4
0	.29212	0	0	0	-1.6283E-4
0	0	.23873	0	0	-1.667E-4
0	0	0	5.8979E-4	0	5.8979E-4
0	0	0	0	.097625	2.8732E-4

ITER	BP1	X	CP	CHAT
****	*******	*********	***********	**********
13	.20001	.37152	-1.4173E-9	-6.3547E-5
	.2	.29226	-4.7156E-10	-2.1144E-5

```
        .2          .23888     -5.9456E-10  -2.6659E-5
        .088197   9.8351E-6     2.2303E-5     1
        .19999     .097336      2.4838E-9     1.1137E-4


                          DD                                  DDCHAT
****************************************************     **********
 .37151 0          0          0           0               -2.3608E-5
0         .29226 0          0           0               -6.1795E-6
0        0          .23887 0           0               -6.368E-6
0        0         0          2.2303E-5 0                 2.2303E-5
0        0         0          0            .097342        1.0841E-5


ITER    BP1           X              CP             CHAT
****  *******   *********    ***********    **********
 17   .2          .37153     -2.0271E-12    -2.4034E-6
      .2          .29226     -6.7447E-13    -7.9969E-7
      .2          .23888     -8.5059E-13    -1.0085E-6
      .088197   3.7193E-7     8.4342E-7       1
      .2          .097331     3.5522E-12     4.2117E-6


                          DD                                  DDCHAT
****************************************************     **********
 .37153 0          0          0           0               -8.9294E-7
0         .29226 0          0           0               -2.3372E-7
0        0          .23888 0           0               -2.4091E-7
0        0         0          8.4342E-7 0                 8.4342E-7
0        0         0          0            .097331        4.0993E-7

*MEMORY EXHAUSTED - UNMAPPING LINKULES*
*UNLOAD LOADED LINKULES TO INCREASE MEMORY*

 RESULTS AFTER   ITER =  18     ITERATIONS

  CONVERG    CPUF1      CMIN        CMAX
*********  *****  *********  **********
8.2008E-7  23.05  5.6171E-7  7.0972E-12


  XPRIME       OLDX         DUAL        BBTROOTS   INTERIOR
*********  *********  **********   ********   ********
 .37153     1.2724      3.6551E-13    .025703   .37153
 .29226     1.0009      4.9042E-13   1.8373     .29226
 .23888      .8181                               .23888
1.6402E-7   5.6171E-7                            .097331
 .097331    0
```

CONDITION NUMBER = 73.278

```
     P H A S E     O N E     E N D S

MANUAL MODE
:_$$
:_$
:_$
:_SUM(BP1)
SUM(BP1) =   .8882
:_SUM(BP1)+ALPHA/SQRT(N*NP1)
SUM(BP1)+ALPHA/SQRT(N*NP1) =   1
:_SUM(CHAT)
SUM(CHAT) =   1
:_TABULATE BB
```

```
                                PE
 *************************************
 .24768 .097421 .23888 1.4877E-6 -.58399
 .61921 .19484  .15925 2.6035E-6 -.97331
```

← WITHOUT $e^T x'' = 1$

```
:_TABULATE DD

                                DD
 ***************************************
  .37153 0         0         0         0
 0         .29226 0          0         0
 0         0         .23888 0          0
 0         0         0         3.7193E-7 0
 0         0         0         0          .097331

:_$
:_$
:_$
:_JOURNAL OFF
:_$  EXAMPLE OF PHASE TWO WITH EXT = 1.
:_$
:_$
:_PHASETWO
EXECUTION STARTED

      P H A S E        T W O        B E G I N S

INPUT  IS THE PROBLEM A MIN=1 OR A MAX(MIN=-1) ?, MIN
:>MIN=1
:><<< NULL LINE ENTERED >>>

   A      B   C
 *****   **   *
 2 1 3    6   3
 5 2 2   10   4
              2


 ITER  BBTROOTS
 ****  ********
  2      .036059
        2.328
        4
```

CONDITION NUMBER = 110.93

```
 RESULTS AFTER  ITER =  16    ITERATIONS
*MEMORY EXHAUSTED - UNMAPPING LINKULES*
*UNLOAD LOADED LINKULES TO INCREASE MEMORY*

  CONVERG    CPUF2   CMIN     CMAX
 *********  *****  ******   ******
 6.6554E-7  19?67  6.7273   6.7273


  XPRIME      OLDX        DUAL       BBTROOTS
 *********  *********  *********   ********
  .56842    1.6364      .36364      .048787
 5.2354E-7  1.5071E-6   .45455     2.9597
  .31579     .90909    2.5381E-8   4
  .11579    0
```

CONDITION NUMBER = 81.99

```
        P H A S E          T W O          E N D S

MANUAL MODE
:_$
:_$
:_TABULATE BB

                  BB
 ********************************
  .37895 4.0719E-7  .31579  -.69474
  .94737 8.1439E-7  .21053 -1.1579
 1       1          1        1

:_SUM(BP1)
SUM(BP1) =  1
:_SUM(X)
SUM(X) =  1
:_SUM(CHAT)
SUM(CHAT) =  6.9222E-11
:_$
:_$
:_$ EXAMPLE OF PHASE TWO WITHOUT EXT = 1.
:_$
:_$
:_PASE2W01
EXECUTION STARTED

      P H A S E        T W O        B E G I N S

INPUT  IS THE PROBLEM A MIN=1 OR A MAX(MIN=-1) ?, MIN
:>MIN
MIN =   1
:><<< NULL LINE ENTERED >>>
INPUT TABUL INCREMENT
:>TABUL=5
:>INCREMENT
INCREMEN =   4
:><<< NULL LINE ENTERED >>>

   A      B   C
 *****   **   *
 2 1 3    6   3
 5 2 2   10   4
              2


 ITER   BBTROOTS
 ****   ********
  2      .029299
        2.005


 ITER    BP1      X          CP          CHAT
 ****   ******  *******  *********   *******
  5     .293    .56147   -.0073895   -.29795
        .12501  .010324   .021477     .86596
        .29157  .31307   -.0071428   -.288
        .29041  .11514   -.0069441   -.27999


                  DD                         S
 ******************************          ********
  .55238 0         0          0          -.16458
```

Handwritten annotations: "$e^T x'' = 1$" (arrow pointing to the row of 1s in BB); arrow pointing to SUM(X) = 1; "EXT = 1." underlined; "CONDITION NUMBER = 68.43" beside BBTROOTS.

0	.023807	0	0	.020616
0	0	.30952	0	-.089143
0	0	0	.11429	-.031998

ITER	BP1	X	CP	CHAT
****	******	*********	**********	*******
9	.29169	.56819	-2.4833E-4	-.28896
	.12498	3.5125E-4	7.4465E-4	.86616
	.29164	.3157	-2.4805E-4	-.28852
	.29161	.11577	-2.4781E-4	-.28825

DD				S
*******************************				*********
.56787	0	0	0	-.16403
0	8.1932E-4	0	0	7.0966E-4
0	0	.31557	0	-.09105
0	0	0	.11574	-.033361

ITER	BP1	X	CP	CHAT
****	******	*********	**********	*******
13	.29136	.56841	-8.289E-6	-.28656
	.12469	1.1822E-5	2.5112E-5	.86814
	.29136	.31579	-8.2887E-6	-.28655
	.29136	.11579	-8.2884E-6	-.28654

DD				S
*******************************				*********
.5684	0	0	0	-.16298
0	2.7623E-5	0	0	2.3981E-5
0	0	.31578	0	-.090487
0	0	0	.11579	-.033178

ITER	BP1	X	CP	CHAT
****	******	*********	**********	*******
17	.28628	.56842	-2.3167E-7	-.25135
	.12006	3.8279E-7	8.2978E-7	.90026
	.28628	.31579	-2.3167E-7	-.25135
	.28628	.11579	-2.3167E-7	-.25135

DD				S
*******************************				*********
.56842	0	0	0	-.14287
0	9.1275E-7	0	0	8.2172E-7
0	0	.31579	0	-.079373
0	0	0	.11579	-.029104

RESULTS AFTER ITER = 17 ITERATIONS

CONVERG	CPUF2	CMIN	CMAX
*********	*****	******	******
4.9865E-7	16	6.7273	6.7273

XPRIME	OLDX	DUAL	BBTROOTS
*********	********	******	********
.56842	1.6364	.36364	.048787

```
3.8279E-7  1.102E-6  .45455  2.9597
 .31579     .90909
 .11579     0
```

CONDITION NUMBER = 60.66

```
          P H A S E          T W O          E N D S

MANUAL MODE
:_$
:_$
:_TABULATE BB

                BB
 ******************************
 .37895 3.0425E-7 .31579  -.69474
 .94737 6.085E-7  .21053 -1.1579
```

← WITHOUT $e^T x'' = 1$

```
:_SUM(BP1)
SUM(BP1) =  .9789
:_SUM(BP1)+ALPHA/SQRT(N*NP1)
SUM(BP1)+ALPHA/SQRT(N*NP1) =  1.1232
:_SUM(CHAT)
SUM(CHAT) =  .14621
:_SUM(BP1)+ ALPHA/SQRT(N*NP1)*SUM(CHAT)
SUM(BP1)+ ALPHA/SQRT(N*NP1)*SUM(CHAT) =  1
:_JOURNAL OFF
```

$e^Tx'' = 1$. Notice that $e^Tx'' \neq 1$, $e^Tx' = 1$, and $e^Tc \neq 0$. Hence, the x'' vector does not lie on the simplex, but this fact does not interfere with the attainment of the LP optimal solution.

E.5 Optimal Phase I with $e^Tx'' = 1$

A numerical example is solved using the regular Phase I - Phase II Karmarkar's algorithm. The corresponding results are given on pages 19 and 20. Notice that it took a combined 57 seconds of CPU time and 43 iterations to achieve convergence. The initial and final condition numbers of the (BB^T) matrix are 213,94 and 81,99, respectively.

Next, the same numerical example is solved using only the Phase I variant with $e^Tx'' = 1$. A value of CMIN = 1000 was entered at the beginning. The algorithm will update this estimate at each iteration as in Phase II of the regular Karmarkar's procedure.

It took 32.42 seconds of CPU time and 27 iterations to achieve convergence. These figures compare favorably with those taken by the Phase I and Phase II procedure. The condition numbers at the beginning and the end are 203.3 and 102.5.

E.6 Optimal Phase I without $e^Tx'' = 1$

Finally, the results of optimizing the problem using Phase I without the capacity constraint are presented on pages 23 and 24.

It took 21.36 seconds of CPU time and 26 iterations, a considerable savings over either procedure reported in E.5. The condition numbers are also improved as they are 96.69 and 60.67 for the beginning and ending iterations. At the optimum $e^Tx'' = .92642$, $e^Tx' = 1$, and $e^Tc = .65814$.

REGULAR ALGORITHM

```
EXECUTION STARTED

  P H A S E     O N E     B E G I N S
```
19
```
 INPUT ALPHA Q
:>ALPHA=.5
:>Q=25
:>>>>> NULL LINE ENTERED >>>
INPUT ARE THE MATRIX A AND VECTOR B DEFINED? (Y/N) Y=1 N=Y=0,Y
:>Y=1
:>>>>> NULL LINE ENTERED >>>

   A       B
 *****   **
 2 1 3    6
 5 2 2   10

INPUT ARE DATA OK? (Y/N) Y=1 N=Y=0,   Y
:>>>>> NULL LINE ENTERED >>>

 AUGMENTA
 ********
 2 1 3 4
 5 2 2 7


 DISCREP   RHS   ARTIFC
 *******   ***   ******
   -4       6      0
   -7      10      0
                   0
                   1


 ITER   BBTROOTS
 ****   ********
  4       .023371
         2.3015
         5
```
CONDITION NUMBER = 213.94

```
RESULTS AFTER  ITER =  23   ITERATIONS

 CONVERG     CPUF1      CMIN        CMAX
*********   *****   *********   **********
1.5247E-8   30.46   1.0445E-8   2.4159E-15


  XPRIME       OLDX        DUAL       BBTROOTS   INTERIOR
*********   *********   **********   ********   ********
 .37142     1.2722      1.2432E-16    .025692   .37142
 .29242     1.0016      1.6699E-16   1.8368     .29242
 .23884      .81804     1.3722E-9    5          .23884
3.0494E-9   1.0445E-8                           .097321
 .097321    0
```
→ COND. # = 194.61

```
    P H A S E     O N E     E N D S


    P H A S E     T W O     B E G I N S

INPUT  IS THE PROBLEM A MIN=1 OR A MAX(MIN=-1) ?, MIN
```

```
:>MIN=1
:>>>> NULL LINE ENTERED >>>

   A       B  C
 *****   **   *
 2 1 3    6   3
 5 2 2   10   4
              2

 ITER   BBTROOTS
 ****   ********
  2       .036051
         2.3276
         4
```

COND. # = 110.95

```
*MEMORY EXHAUSTED - UNMAPPING LINKULES*
*UNLOAD LOADED LINKULES TO INCREASE MEMORY*

 RESULTS AFTER  ITER =  20   ITERATIONS

  CONVERG    CPUF2   CMIN   CMAX
 *********   *****  ******  ******
 2.4359E-8   26.76  6.7273  6.7273
```

23 43 30.46 57.22

```
   XPRIME      OLDX        DUAL       BBTROOTS
 *********  *********  **********   ********
  .56842    1.6364       .36364       .048787
 1.7676E-8  5.0885E-8    .45455      2.9597
  .31579     .90909    8.3701E-11    4
  .11579    0

          P H A S E         T W O           E N D S

MANUAL MODE
:_$
:_$
:_$
:_$
:_$ EXAMPLE WITH OPTIMAL PHASE ONE WITH EXT=1
:_$
:_$
```

COND. # = 81.99

```
MANUAL MODE
:_JOURNAL OFF
:_OPTIF1V
EXECUTION STARTED

  P H A S E     O N E     B E G I N S

 INPUT ALPHA Q
:>ALPHA
ALPHA =   .5
:>Q
Q =  25
:><<< NULL LINE ENTERED >>>
INPUT ARE THE MATRIX A AND VECTORS B AND C DEFINED? (Y/N) Y=1 N=Y=0,Y
:><<< NULL LINE ENTERED >>>

   A      B  C
 *****   **  *
 2 1 3    6  3
 5 2 2   10  4
             2

INPUT ARE DATA OK? (Y/N) Y=1 N=Y=0,   Y
:><<< NULL LINE ENTERED >>>
INPUT IS THE PROBLEM A MIN=1 OR A MAX=-MIN=-1 ?, MIN
```

```
:>MIN
MIN =  1
:>>>> NULL LINE ENTERED >>>
INPUT CMIN
:CMIN=1000
:>>>> NULL LINE ENTERED >>>
```

AUGMENTA
****

2	1	3	4
5	2	2	7

DISCREP	RHS	ARTIFC
-4	6	3
-7	10	4
		2
		998

ITER	BBTROOTS
4	.024594
	2.3676
	5

COND. # = 203.30

RESULTS AFTER ITER = 27 ITERATIONS

RESULTS AFTER ITER = 27 ITERATIONS

CONVERG	CPUF1	CMIN	CMAX
1.8305E-8	32.42	6.7274	6.7274

XPRIME	OLDX	DUAL	BBTROOTS	INTERIOR
.56841	1.6363	.36366	.048785	.56841
1.4691E-5	4.2291E-5	.45454	2.9596	1.4691E-5
.31579	.90909	4.657E-6	5	.31579
1.3444E-8	3.8703E-8			.11579
.11579	0			

→ COND. # = 102.49

BP1	X	CP	CHAT
.24083	.56841	-1.2347E-5	-.36516
.13876	1.4691E-5	1.8521E-5	.54773
.24082	.31579	-1.2347E-5	-.36515
.13876	1.3444E-8	1.852E-5	.54771
.24082	.11579	-1.2347E-5	-.36514

XPRIME	OLDX	DUAL	BBTROOTS
.56841	1.6363	.36366	.048785
1.4691E-5	4.2291E-5	.45454	2.9596
.31579	.90909	4.657E-6	5
1.3444E-8	3.8703E-8		
.11579	0		

```
 P H A S E     O N E     E N D S

MANUAL MODE
:_JOURNAL OFF
:_$
:_$
:_$
:_$ EXAMPLE OF OPTIMAL PHASE ONE WITHOUT EXT = 1.
:_$
:_$
:_OPTP1W01
EXECUTION STARTED

 P H A S E     O N E     B E G I N S

 INPUT ALPHA Q
:>ALPHA
ALPHA =   .5
:>Q
Q =  25
:><<< NULL LINE ENTERED >>>
INPUT ARE THE MATRIX A AND VECTORS B AND C DEFINED? (Y/N) Y=1 N=Y=0,Y
:>Y
Y =  1
:><<< NULL LINE ENTERED >>>

   A     B   C
 *****  **   *
 2 1 3   6   3
 5 2 2  10   4
             2

INPUT ARE DATA OK? (Y/N) Y=1 N=Y=0,  Y
:><<< NULL LINE ENTERED >>>
INPUT IS THE PROBLEM A MIN=1 OR A MAX=-MIN=-1 ?, MIN
:>MIN
MIN =  1
:><<< NULL LINE ENTERED >>>
INPUT CMIN
:>CMIN=1000
:><<< NULL LINE ENTERED >>>

 AUGMENTA
 ********
 2 1 3 4
 5 2 2 7


 DISCREP  RHS  ARTIFC
 *******  ***  ******
   -4      6      3
   -7     10      4
                  2
                998

*MEMORY EXHAUSTED - UNMAPPING LINKULES*
*UNLOAD LOADED LINKULES TO INCREASE MEMORY*

 ITER  BBTROOTS
 ****  ********
  4     .024565
       2.3752
```

COND. N. = 96.69

```
RESULTS AFTER  ITER =  26    ITERATIONS
```



```
RESULTS AFTER  ITER =  26    ITERATIONS
```



```
 CONVERG    CPUF1   CMIN     CMAX
*********   *****  ******  ******
1.9694E-8   21.36  6.7274  6.7274

  XPRIME       OLDX       DUAL     BBTROOTS    INTERIOR
*********   *********   ******   ********   *********
 .56841      1.6363     .36367    .048784    .56841
1.8045E-5   5.1947E-5   .45454   2.9596     1.8045E-5
 .31578      .90909                          .31578
1.6513E-8   4.7539E-8                        .11579
 .11579      0
```

Cond. # = 60.67

```
  BP1        X           CP          CHAT
******   *********   **********   *******
.2243     .56841     -9.6309E-6   -.21736
.12676   1.8045E-5    2.9027E-5    .6551
.2243     .31578     -9.6304E-6   -.21735
.12676   1.6513E-8    2.9026E-5    .65508
.2243     .11579     -9.6301E-6   -.21734

  XPRIME       OLDX       DUAL     BBTROOTS
*********   *********   ******   ********
 .56841      1.6363     .36367    .048784
1.8045E-5   5.1947E-5   .45454   2.9596
 .31578      .90909
1.6513E-8   4.7539E-8
 .11579      0

    P H A S E     O N E     E N D S

MANUAL MODE
:_$
:_$
:_SUM(BP1)
SUM(BP1) =  .92642
:_SUM(X)
SUM(X) =  1
:_SUM(CHAT)
SUM(CHAT) =  .65814
:_JOURNAL OFF
```

References

Khachiyan, L. G., "A Polynomial Algorithm in Linear Programming," Doklady Akademiia Nauk SSSR 244:S (1979):1093-1096. Translated in Soviet Mathematics Doklady 20:1(1979):191-194.

Karmarkar, N., "A New Polynomial-Time Algorithm for Linear Programming," Mimeographed, Undated. AT&T Bell Laboratories, Murray Hill, NJ.

Paris, Q., "A Primer on Karmarkar's Algorithm for Linear Programming," Department of Agricultural Economics, University of California, Davis, January 1985.