FORMULATING AND ESTIMATING DYNAMIC STOCHASTIC PRODUCTION MODELS

by

John M. Antle
Arthur Havenner

University of California, Davis
Department of Agricultural Economics

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ABSTRACT

Both the dynamic and stochastic aspects of economic life are increasingly being recognized and incorporated in current analytics. In this paper we examine these aspects in production models, with the aim of achieving the following goals: (i) Unifying the various existing efforts into one more general analytic framework; (ii) Evaluating the tractability of the two dominant functional forms in the presence of dynamics and uncertainty; (iii) Exhaustively analyzing the estimation issues of the general form in stochastic multiperiod and multistage problems with observable and unobservable intermediate outputs; (iv) Determining the conditions under which production function and factor demand estimation is separable, i.e., when knowledge of the production process is not required for efficient factor demand estimation, and vice-versa; (v) Examining the existence of analytic factor demands in the general dynamic stochastic model; and finally, (vi) Providing alternative conditions that lead to an assortment of tractable, estimable models for empirical use.

We are encouraged by the conclusion that, subject to avoiding certain pitfalls, the general production model can be extended to include both multiple periods and uncertainty, with increased empirical relevance.

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The aim of this paper is to develop the general econometric structures and to identify appropriate estimators for dynamic production models. This research is motivated by the growing recognition that production is typically a dynamic, time-dependent phenomenon and that dynamic technological relations have important theoretical and empirical implications (Hansen and Sargent 1980, Kydland and Prescott 1982, Long and Plosser 1983, Antle 1983a, 1983b). Surprisingly, the production function estimation literature has largely ignored dynamic production relations, despite innovations in other areas such as risk and duality. In contrast, dynamic factor demand models have a long tradition, originating with the Nerlovian partial adjustment model and culminating in recent cost of adjustment models explicity incorporating the firm's optimization problem (see Berndt, Morrison, and Watkins 1981).

Our approach to formulating dynamic production models is based on the observation that the fundamental dynamic structures of production processes are due to basic technological "facts of life," such as the time required to make physical capital investment, the biological processes in agricultural production which give rise to crop rotations, and so forth. We begin with general representations of the production processes which encompass these fundamental dynamic technological relations. Following Antle (1983b), we model the firm's input choice problem as a discrete time stochastic control problem. Two general sources of production dynamics are identified, input dynamics and output dynamics. The former results in models in moving average form, while the latter leads to autoregressive models. These two fundamental dynamic structures imply that, generally, both the production functions and the input demand functions are dynamic, recursive equation systems. Thus our analysis shows that the structural equations of dynamic production models necessarily consist of both production functions and input demand functions, in contrast to the literature which treats production function estimation and dynamic factor demand equation estimation as distinct problems. Analysis of the statistical properties of these models shows that, depending on the dynamic structure and observability of the model being considered, either single equation estimation, seemingly unrelated equations estimation, or simultaneous equations estimation is appropriate.

Our approach to formulating dynamic production models recognizes production as both a dynamic and stochastic phenomenon, and thus generalizes static econometric production function models and deterministic cost of adjustment models. The approach also permits a broader class of dynamic phenomena to generate production dynamics than the cost of adjustment model. Explicitly modeling the production problem as stochastic also gives insights into the questions of error specification and functional form, two issues of major concern in the static production literature not yet investigated in a dynamic framework. We show that the quadratic model which has been used in the cost of adjustment literature is not tractable for a broad class of dynamic phenomena. We also show that, remarkably, the dynamic Cobb-Douglas production model, even though nonlinear, is tractable in the stochastic control framework. Moreover, the Cobb-Douglas model can be used to represent a much broader class of dynamic phenomena than the quadratic model.

Modeling production as a stochastic phenomenon brings a number of issues to empirical research which do not arise in deterministic models. One major difficulty is that the conventional duality relations between production, cost, and profit functions generally do not hold. This is a major justification for our "primal" approach to formulating and estimating dynamic production models. The stochastic properties of dynamic production processes simply cannot be represented by dual functions because the maximization problem under uncertainty is defined over an expected value taken with respect to the random variables in the model. Therefore, the dual functions of this expected value are nonstochastic functions of the parameters of the distributions of the random variables in the firm's objective function. Hence, the duals cannot be used to represent the stochastic structure of the production process.

The paper begins with examples from the literature illustrating some properties of the DPM. The second section characterizes the general structure of the DPM. The third section discusses functional forms including the properties of the quadratic and Cobb-Douglas DPM. The next section discusses general estimation considerations and the appropriate methods for important special cases of the DPM, followed by the conclusion.

1. Examples of Dynamic Production Models

Before providing a general structure for dynamic production models (DPM) it is useful to review five examples from the literature which illustrate how dynamic relations enter production processes. These seemingly disparate cases will prove to be useful referents during our development of the general DPM.

1.1 Cost of Adjustment Model

Hansen and Sargent (1980) developed a discrete-time cost of adjustment model. Here we provide a modified version of their model. The firm's net returns in period t are

(1) $\pi_t = p_t O_t - w_t n_t - r_t \Delta k_t$

where p_t, w_t and r_t are prices of output O_t, labor input n_t and capital k_t, and Δ is the first difference operator. The production technology is quadratic,

$$(2) \quad O_t = \alpha n_t + \beta k_t + \left(\frac{1}{2}\gamma_1 \Delta k_t^2 + \frac{1}{2}\gamma_2 \Delta n_t^2 + \gamma_3 \Delta k_t \Delta n_t\right) + \varepsilon_t$$

where ε_t is a shock to technology, α, $\beta > 0$; γ_1, $\gamma_2 < 0$; and $(\gamma_1\gamma_2 - \gamma_3^2) < 0$. These conditions ensure that the technology is concave. The expression in parentheses represents costs of adjustment in terms of the effects of input changes on output.

The firm's objective function at time 0 is

$$(3) \quad \max_{\{n_t, k_t\}} E_0 \sum_{t=0}^{\infty} \delta_t \pi_t.$$

Since π_t is quadratic in n_t and k_t, the resulting factor demand equations are linear. For our purposes, several features of this formulation are notable. First, the only nonlinearities in the model are introduced through the quadratic adjustment cost technology. Second, dynamics are generated through lagged <u>inputs</u> in the production function. Lagged outputs play no role in the model.

1.2 Time to Build

Kydland and Prescott (1982) propose an aggregate production technology without adjustment costs but with a "time to build" requirement for investment projects. The production function is

$$(4) \quad O_t = n_t^{\theta} \left[(1 - \sigma)k_t^{-\nu} + \sigma y_t^{-\nu}\right]^{-(1 - \theta)/\nu} \varepsilon_t$$

where O_t, n_t, k_t, and ε_t are as defined above and y_t is inventories at time t. Dynamics are introduced by assuming J time periods are required to build new capital. Define s_{jt} as the number of projects j periods from completion,

$j=1, \ldots, J-1$. Then s_{1t} is the actual addition to the capital stock in t, and capital evolves according to

$$(5) \quad k_{t+1} = (1-\delta)k_t + s_{1t}.$$

Due to the time to build assumption, s_{1t} represents the completed investment projects initiated at time t-J. Combining (5) and (4) shows that the time to build assumption induces a lag structure on the investment inputs s_{jt} in the production technology.

1.3 Multiple-Use Commodity Model

Long and Plosser (1983) use a simple dynamic Cobb-Douglas technology to model real business cycles. In the model outputs are either consumed in period t or are combined with labor in period t to produce period t+1 output. The model is

$$(6) \quad Q_{i,t+1} = n_{it}^{b_i} \prod_{j=1}^{N} x_{ijt}^{a_{ij}} \varepsilon_{it}$$

where the i^{th} output in t+1, $Q_{i,t+1}$, is a function of labor n_{it} and commodity inputs x_{ijt}. If we let ρ_{ijt} be the proportion of commodity j used to produce commodity i in period t, then

$$Q_{i,t+1} = n_{it}^{b_i} \prod_{j=1}^{N} (\rho_{ijt} Q_{it})^{a_{ij}} \varepsilon_{it}$$

$$= n_{it}^{b_i} \prod_{j=1}^{N} (\rho_{ijt} n_{jt-1}^{b_j} \prod_{k=1}^{N} x_{jkt-1}^{a_{jk}} \varepsilon_{jt-1})^{a_{ij}}$$

and so forth. This model is different from those above in one very important respect: dynamics enter production through lagged <u>outputs</u>. $Q_{i,t+1}$ is therefore a function of past inputs <u>and</u> past production shocks ε_{it}. The implicit presence of past production shocks in (6) marks a fundamental difference from examples 1.1 and 1.2.

1.4 Production Stages and Sequential Decision Making

Antle (1983b) uses a multistage Cobb-Douglas production model to investigate the implications of sequential decision making for production function estimation. The simple two-stage crop production model is composed of first and second stage production functions

$$
(7) \qquad \begin{aligned} Q_{i1} &= \beta_0 \, n_{i1}^{\beta_1} \, A_i^{\beta_2} \, e^{\varepsilon_{i1}} \\ Q_{i2} &= \gamma_0 \, Q_{i1}^{\gamma_1} \, n_{i2}^{\gamma_2} \, e^{\varepsilon_{i2}} \end{aligned}
$$

where i indexes the observation, A_i is acreage planted in the crop, n_{i1} is planting and cultivation labor input in the first stage, ε_{i1} is a random shock, and the unharvested crop is Q_{i1}; n_{i2} is harvest labor and ε_{i2} a random shock, giving harvested output Q_{i2}. As in example 1.3, this model generates dynamics because outputs are interrelated over time. Substituting Q_{i1} into Q_{i2} shows that Q_{i2} depends on both lagged inputs and production shocks.

1.5 Technological Change and Learning by Doing

Arrow (1961) postulated that a firm's rate of productivity growth using technological innovations is a function of its learning about efficient use of the new technology. He hypothesized that learning could be measured in terms of cumulative output produced with the new technology, so define learning as

$$
L_t = \sum_{j=t_0}^{t-1} Q_j
$$

where the new technology is adopted at t_0. Then the production function is assumed to depend on L_t

$$Q_t = f[x_t, L_t, \varepsilon_t],$$

and hence, Q_t implicity depends on past outputs. Feder and O'Mara (1982) have shown that a Bayesian learning model implies that technological adoption is a function of cumulative past adoption, which may be equivalent to cumulative output as hypothesized by Arrow.

2. The General Structure of Dynamic Production Models

In all of the above examples, dynamic relations enter production processes in two ways. A process with _input dynamics_ is of the moving average (MA) form

$$(8) \quad Q_t = f_t[x_t, x_{t-1}, \ldots, \varepsilon_t]$$

where Q_t is (observable) output in period t, x_{t-j} is the input chosen in t-j, and ε_t is the random shock to production in period t realized after x_t is chosen; f_t is a production function satisfying standard regularity conditions. Examples 1.1 and 1.2 are special cases of input dynamics. In contrast, a process with _output dynamics_ is of the autoregressive (AR) form

$$(9) \quad Q_t = f_t[x_t, Q_{t-1}, Q_{t-2}, \ldots, \varepsilon_t]$$

where f_t is regular with respect to x_t and the Q_{t-j}. Examples 1.3, 1.4, and 1.5 are special cases of (9). Repeated substitutions show that (9) can be expressed as

$$(10)\quad \begin{aligned} Q_t &= f_t[x_t, f_{t-1}[x_{t-1}, \ldots, \varepsilon_{t-1}], \ldots, \varepsilon_t] \\ &= h_t[x_t, x_{t-1}, \ldots, \varepsilon_t, \varepsilon_{t-1}, \ldots]. \end{aligned}$$

Both (8) and (10) show Q_t as a function of past x_t's, but (10) also depends on past ε_t's. This difference has important consequences for formulating and estimating the DPM.

2.1 Multistage Versus Multiperiod

Dynamic production may involve two types of decision problems classified by the timing of output sales. In the case of a multistage production process, a sequence of inputs is applied over time to a sequence of intermediate production stages which produce only intermediate products, not saleable final output. (To be distinguished from single stage production, multistage processes must exhibit output dynamics, although they may have both input and output dynamics.) The multistage criterion is

$$(11)\quad \max_{\{x_t\}} E[\delta_T p_T Q_T - \sum_{t=1}^{T} \delta_t w_t x_t]$$

where δ_t is a discount factor, w_t is the unit input cost, and p_T is the price of final output Q_T, and there are T production stages.

The other type of decision problem corresponds to the case in which inputs are chosen to produce output that is sold in each period. With this multiperiod decision problem, the objective is to

$$(12)\quad \max_{\{x_t\}} E[\sum_{t=1}^{T} \delta_t(p_t Q_t - w_t x_t)].$$

Note that multiperiod problems may involve either input dynamics or output dynamics or both.

2.2 General Solution

The general solution to DPM's is given by the dynamic programming algorithm. Let $x^t = (x_1, \ldots, x_t)$ and $Q^t = (Q_1, \ldots, Q_t)$, and define μ_t as the parameter vector of the decision maker's subjective joint probability distribution function of $p_t, \ldots, p_T, w_{t+1}, \ldots, w_T$, and $Q_t, \ldots, Q_T$, conditioned on information available at time t. Then the (optimal) factor demands implied by the multiperiod decision problem (12) are

$$(13) \quad x_t^* = x_t^*\left[x^{t-1}, Q^{t-1}, \mu_t, w_t\right] \qquad t = 1, \ldots, T.$$

showing that the input decision in t depends on the price of x_t which is known at time t, past inputs and outputs, and the subjective expectations of future outputs and prices. Note that at the beginning of the t^{th} production period, the output Q_t and output price p_t are unknown. The solution of the multistage problem (11) is analogously obtained by redefining μ_t to include only period T output price and future input prices. Thus, the DPM generally is composed of a recursive system of production functions and input demand equations.

The assumptions made about the information set used by the decision maker to solve dynamic optimization problems have important effects on the properties of the resulting structural model. Antle (1983b) shows that there are important differences between the open loop solution, which implies the decision maker does not sequentially update the information set, and solutions such as open loop feedback and closed loop which do imply the information set is sequentially updated. We can expect decision makers to use feedback solutions when feasible. In terms of equation (13), the open loop solution would have $E_1\left[Q_{t-j}\right]$ in place of Q_{t-j}, where E_1 denotes the expectation operator conditioned on the information available at the beginning of period 1. Below, we show that the dependence of input decisions on past output has important consequences for econometric properties of the DPM. Note, however, that this difference appears only in models with output dynamics.

The T input demands (13) are the result of the individual optimization of expected profits [either (11) or (12)], and thus are not stochastic at the level of the decision maker; with appropriate (individual) data these functions would fit exactly. Frequently, however, the observer/econometrician does not have access to data this specific, so that there are individual factors observed by the decision maker but unknown to the observer that can be meaningfully represented as random disturbances--in many cases the Central Limit Theorem would suggest normality. [For example, Hansen and Sargent (1980) use this information-based argument.] In the following discussion we sometimes assume the existence of this observer/decision maker dichotomy, writing the input demand equations with random error terms.

3. Functional Forms for the DPM

Control theorists have long known that closed form solutions to the general nonlinear optimal stochastic control problem do not exist (Aoki 1967). This problem arises for two reasons. First, the fully optimal closed loop solution must account for the fact that decision makers know they will sequentially update their information sets and decisions. This "closing" of the information loop requires specification of the updating rules for the joint probability distribution functions in each time period, a problem that is tractable only for certain distribution functions. One simplification is to approximate the control solution by the open loop feedback solution, thus preserving the sequential updating of the information set. This solution is obtained by applying the open loop solution in each period conditional on the information available at that time; the possibility of future updating is ignored.[1] Thus, the open loop feedback solution implies decision makers use currently available information but ignore the fact that they will be able to revise their information set in the future. We use the open loop feedback solution in our following discussions of the DPM.

The second difficulty in solving control problems is that they generally are highly nonlinear systems of equations that do not admit analytical solutions. This has led researchers to resort to the quadratic control model, which is tractable. The quadratic production function also has the desirable property of self-duality, but is restrictive in other respects, e.g., it gives linear input demand functions which imply unusual demand elasticity behavior. Another major limitation of the quadratic is that it is parameter-intensive. This is not a serious limitation in highly aggregated models with few inputs, but is a very serious problem when micro data with many inputs is used. In this section, we first consider other restrictive properties of the quadratic DPM and then discuss the properties of the Cobb-Douglas DPM.

3.1 Quadratic

A simple example is sufficient to illustrate the properties of the quadratic production function which are important to dynamic models. Let

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[1]The standard linear-quadratic-gaussian equivalence between deterministic closed loop solutions and open loop with feedback solutions (see Norman) can be utilized when the model exhibits only input dynamics, as in the Hansen and Sargent model (example 1.1). More generally, this result will not apply with output dynamics since the production is not usually linear in Q. (Even in the Cobb-Douglas case analyzed below, Q enters the criterion while the logarithm of Q appears in the constraint.)

(14) $Q_t = \alpha_0 + \alpha_1 x_t + \alpha_2 Q_{t-1} + \alpha_3 x_t^2 + \alpha_4 Q_{t-1}^2 + \alpha_5 x_t Q_{t-1} + \varepsilon_t$

which is a production function with output dynamics. Q_t is quadratic in x_t and Q_{t-1}, but recursive substitution shows that it is not quadratic in $x_{t-1}, x_{t-2}, \ldots$. Rather, it is a higher degree polynomial in the inputs beyond the first period. Moreover, (14) implies that the distribution of Q_t is evolving over time as a higher degree polynomial in $\varepsilon_t, \varepsilon_{t-1}, \ldots$. This means that the distribution of Q_t evolves as a complex nonlinear convolution of the error distributions. The only conditions under which (14) is quadratic in all inputs and linear in $\varepsilon_t, \varepsilon_{t-1}, \ldots$ is when Q_t is linear in past outputs, i.e., when $\alpha_4 = \alpha_5 = 0$ for all t in (14). This amounts to imposing additive separability across each period's inputs and past outputs.

Replacing Q_{t-1} with x_{t-1} in (14) shows input dynamics are generally admissible with quadratic production functions. Therefore, we can conclude that only processes with input dynamics alone can be used to construct a control model in which profit is a quadratic function of inputs and the output distribution evolves in a tractable form.

3.2 Cobb-Douglas

Since output dynamics characterize many production processes (see the examples above), a tractable alternative to the quadratic functional form is desirable. We shall find that a dynamic Cobb-Douglas production function, though nonlinear and nonquadratic, admits a closed form solution which is a loglinear analog of the well known static Cobb-Douglas model. Thus it provides a dynamic linear equation system for estimation. Although the Cobb-Douglas model restricts elasticities of substitution to unity, it has the attractive properties of constant demand elasticities, self duality, and parameter parsimony, making it an attractive alternative to the quadratic for applied research, especially when micro data are used.

To illustrate, define the production functions as

$$(15) \quad Q_t = \alpha_t \, x_t^{\beta_t} \, Q_{t-1}^{\gamma_t} \, e^{\varepsilon_t} , \qquad \gamma_1 = 0, \ t = 1, 2, \ldots, T.$$

Assume ε_t is $N(0, \sigma_t^2)$ and that input and output prices are independently distributed. For simplicity (without loss), we also do not update with respect to prices, e.g., $E_t p_{t+j} = \bar{p}_{t+j}$ for all t, $j > 0$. Consider the multistage problem (i.e., only terminal output is sold) given in (11). In the T^{th} period, the firm solves the decision problem

$$\max_{x_T} \quad E_T \, [\delta_T p_T Q_T - \sum_{t=1}^{T} \delta_t w_t x_t \,|\, x^{T-1}, \, Q^{T-1}].$$

The Cobb-Douglas model has the optimal solution (see Antle 1983b)

$$\begin{aligned} \ln x_T^* &= \frac{1}{1 - \beta_T} \ln \left(\alpha_T \beta_T + \frac{\sigma_T^2}{2}\right) - \ln \left(\frac{\bar{w}_T}{\bar{p}_T}\right) + \gamma_T \ln Q_{T-1} \\ (16) \qquad & \\ &= \psi_{0T} + \psi_{1T} \ln Q_{T-1} + \pi_{TT} \ln \left(\frac{\bar{w}_T}{\bar{p}_T}\right) . \end{aligned}$$

In period T-1 the decision problem is

$$\max_{x_{T-1}} \quad E_{T-1} [\delta_T p_T Q_T - \sum_{t=1}^{T} \delta_t w_t x_t \,|\, x^{T-2}, \, Q^{T-2}, \, x_T = x_T^*]$$

The first-order condition for X_{T-1} is

$$\delta_T \bar{p}_T \frac{\partial E_{T-1}[Q_T | x^{T-2}, Q^{T-2}, x_T = x_T^*]}{\partial x_{T-1}} - \delta_{T-1} w_{T-1}$$

$$- \delta_T \frac{\partial E_{T-1}[w_t x_t | x^{t-2}, Q^{T-2}, x_T = x_T^*]}{\partial x_{T-1}} = 0$$

Using (15) and (16), and simplifying, we obtain

$$\ln x_{T-1}^* = \psi_{0,T-1} + \psi_{1,T-1} \ln Q_{T-2} + \pi_{T-1,T-1} \ln(\frac{w_{T-1}}{\bar{p}_T}) + \pi_{T,T-1} \ln(\frac{w_T}{\bar{p}_T})$$

with the ψ's and π's defined as the implied combinations of individual stage production function parameters. Further application of the dynamic algorithm using the open loop feedback solution shows the general solution in t is

$$\ln x_t^* = \psi_{0t} + \psi_{1t} \ln Q_{t-1}$$

(17)

$$+ \pi_{tt} \ln(\frac{\bar{w}_t}{\bar{p}_T}) + . . . + \pi_{Tt} \ln(\frac{\bar{w}_T}{\bar{p}_T}), \quad t = 1, 2, . . ., T,$$

where the forward relative price terms ($\overline{w}_t/\overline{p}_T$, t=1, 2, . . ., T) are unrevised (in this case) forecasts based on the parameter vector μ_t. Combining (15) in logarithmic form with the system (17) results in a recursive system of loglinear equations. An analogus result holds in the multiperiod problem.

It is an open question whether or not other nonlinear models also have closed form solutions. We speculate that there may well exist a more general class of functions that do. The important Cobb-Douglas case shows that the quadratic model is not the only functional form commonly used in production analysis with an analytical solution.

4. Estimating Dynamic Production Models

Research objectives, data availability, and data aggregation all play important roles in guiding the researcher in model formulation. While many hypotheses of interest require estimates of both the production function parameters and the input demand equation parameters, only one or the other parameter set may be directly involved in hypothesis tests. For example, the research objective may be to test the productivity effects of some exogenous variable, requiring estimates only of the production function parameters and not estimates of the factor demand functions. Similarly, alternative hypotheses may directly involve only the parameters of the factor demands. Under some conditions, these parameter sets can be separately estimated, while in other cases both the production function and input demand system must be jointly specified and estimated due to indirect requirements stemming from the unique nature of the problem. In this section, we investigate a number of special cases of the DPM and show how the appropriate econometric procedures depend both on the formulation of the model and the research objectives.

Some general properties of DPM's can be identified which are important to estimation. First, as shown in section 2 above, DPM's generally are composed of a recursive system of production function and input demand functions, i.e., there is no feedback from dependent variables to regressors. It follows that DPM's are not, and cannot be, true simultaneous equations systems with jointly dependent endogenous variables. This fundamental property has important implications for estimation.

Second, at least at the decision maker level (although perhaps not to the observing econometrician), since the input demand functions are the solutions to an optimization problem they are <u>exact</u>, <u>nonstochastic</u> equations, as in (13). This is especially important when the research objectives require estimation of the production function only, because it implies that the system of input demand functions may not be required for consistent and efficient estimation.

A third and related point concerns the use of duality. In a deterministic model, it is well known from duality that knowledge of the production function parameters is sufficient to infer the cost or profit function, and hence, input demand equations. However, under uncertainty this duality result does not follow. Equation (13) shows that with uncertainty the input demand functions depend on the parameters of future prices and output distributions. Therefore, estimation of the production functions generally is not adequate to identify the input demand functions.

Fourth, because the firm's decision problem is modeled as a stochastic control problem, there may not be an analytical solution for the input demand equations. Therefore, the issue of whether or not the input demand functions need be estimated is again important. It is unnecessary to solve the control problem to derive the input demand equations if the research objectives require only production function estimates and these estimates can be obtained separately. When the input demand functions are required, either to test hypotheses or, in some cases, to obtain parameter estimates of the production function with desirable properties, models without analytical solutions pose a special estimation problem that cannot be solved using conventional econometric procedures. The Generalized Method of Moments (GMM) procedure devised by Hansen (1982) can be adapted to estimate the parameters of a dynamic model using the first order conditions ("stochastic Euler equations") of the model, as in Hansen and Singleton (1983). Another approach, suggested by Fair and Taylor (1983), uses a numerical solution to the control problem in conjunction with maximum likelihood estimation.

A fifth issue concerns the type of data used and the level of aggregation. Using firm-level data, multistage models are often appropriate, as the examples of section 1 show. However, it is uncommon to have data on intermediate products, raising the problem of identification and estimation of multistage models with only final output observed by the econometrician. Aggregate production relations, on the other hand, need not satisfy the structure of a firm's optimization problem, even though this assumption is often used in the literature to derive aggregate structural models. Nevertheless, the structure of firm-level production systems can be used to deduce the statistical properties of aggregate variables.

When the various dimensions of the DPM are considered, we obtain a general classification of models according to the headings of Table 1. The research objective may involve estimating production functions or input demands; the objective function may be multiperiod or multistage; and the production process may exhibit input dynamics or output dynamics (it will become apparent that estimators valid for output dynamics also are valid for combined input and output dynamics); intermediate outputs may not be observed; the model may have across-equation restrictions or correlated errors; and the model may or may not have an analytical solution. Table 1 summarizes the results of this variety of combinations. In the sections below we consider estimation of production functions or input demands for three major cases: models with (i) input dynamics; (ii) output dynamics and observable intermediate products; (iii) output dynamics with unobservable intermediate products. The discussion first assumes firm-level data in the form of pooled time series of cross-sections of observations (in which the time-dated variable is a vector of cross-sectional observations), with specializations to unpooled cases where appropriate.

TABLE 1. Attributes of the DPM and Efficient Estimators

Research Goal		Model Characteristics							Efficient Estimator*		
Estimate		Decision Criterion[1]		Dynamics[1]		Observable Intermediate Output	Restrictions or error Correlation	Analytic Solution Exists	Single Equation	Multiple Equations	"Simultaneous" Equations
Production Function	Input Demands	Multi-stage	Multi-period	Input	Output						
X		X	X	X	X	X			OLS/NLS		
X		X	X	X	X	X	X			ZA[2]	
X			X	X			X		A	ZA[2,3]	
X			X		X		X		A[2]	ZA[2,3]	
X	(X)[4]	X			X			X			LI/IV
X	(X)[4]	X			X		X	X			FI/IV
(X)[4]	(X)[4]	X			X		X				FI/GMM-FT
	X	X	X	X	X	X		X		ZA	
	X	X	X	X	X	X	X	X		ZA[2]	
	X	X	X	X	X		X				FI/GMM-FT

*Estimators:
- OLS/NLS = Linear or nonlinear least squares
- ZA = Zellner-Aitken
- IV = Instrumental variable
- GMM-FT = Generalized method of moments or Fair-Taylor
- A = Aitken
- LI = Limited information (single equation)
- FI = Full information (multiple equations)

[1] The X indicates either or both.
[2] Error correlation requires iteration to convergence for either consistency or efficiency.
[3] Across-equation restrictions require multiple equation estimation.
[4] (X) = Estimation required even if not the primary research goal

4.1 Input Dynamics

4.1.1 Production function estimation

Consider a model in the form of equation (1), incorporating only input dynamics with output observable every period, i.e.,

$$
\begin{aligned}
Q_1 &= f_1(x_1,\ x_0,\ \varepsilon_1) \\
Q_2 &= f_2(x_2,\ x_1,\ x_0,\ \varepsilon_2) \\
&\ \vdots \\
Q_T &= f_T(x_T,\ x_{T-1},\ \ldots,\ x_0,\ \varepsilon_T),
\end{aligned}
\tag{18}
$$

These input dynamics are in a "growing" moving average form in which the production function in period t is an MA(t), as would be appropriate when, e.g., x_t's represent investments in new technology begun at t=0 and continuing over time. Input dynamics might also take the form of a moving average of fixed order in each period, as might be appropriate for an aggregate production model. Observe that all inputs are predetermined relative to the corresponding output in each production function in (18). Therefore, the production functions can be directly estimated as a system of equations without estimating the input demand equations. If there are restrictions across equations (e.g., $f_1 = f_2 = \ldots = f_T$) or correlated errors, there are efficiency gains from pooling. Otherwise, with multiple cross-sections, each production function can be estimated separately, or, if the functions are identical or their differences can be parameterized, time series observations alone are sufficient. It is perhaps worth noting that the frequency of observation of the inputs need not match the output frequency, or even each other when there are multiple inputs; each input in each time period is logically a different input (the model may be thought of as being in state-space form), so there is no missing observation problem.

The case of input dynamics is the dynamic analogue of the Zellner, Kmenta, Dreze (1966) estimation procedure for static models. Since the input demand equations are unnecesssary for production function estimation, any linear or nonlinear functional form may be used with an appropriate linear or nonlinear estimator.

4.1.2 Factor demand estimation

If an analytic solution exists, each factor demand as observed is of the form

(19) $x_t = g(x_t^*[x^{t-1}, w_t, \mu_t], u_t)$, $t=1, 2, \ldots, T$,

where the stochastic error u_t represents the randomly distributed differences between the observed input quantity x_t and the theoretical decision rule x_t^*. We assume u_t and the production function errors ε_t are independent. It is not necessary to jointly estimate the input demands and the production functions in this case, but the input demand functions are often of interest themselves.

The parameter vector of the decision maker's subjective probability distribution of future outputs and prices, μ_t, appears in each factor demand, requiring its prediction. Since these estimates affect the system (19) and are not affected by it (under the usual atomism assumptions), they are predetermined variables to the factor demand block. Thus, so long as their values are available for estimation of (19), it is irrelevant whether they were generated by, for example, an (unspecified) econometric model, or a vector time series process, or even a survey. One obvious set of forecast estimates is the actual values, although with these estimates--as well as with estimates from any other scheme--errors of observation will enter to the extent that they are not the values actually used by the individual decision maker.

When the decision problem does not have an analytic solution, the input demand functions (19) cannot be derived explicitly. The Hansen GMM or Fair and Taylor maximum likelihood procedures can be utilized in this case. It is useful to consider the Cobb-Douglas case to gain insight into the structure of these models.

For the multiperiod,[2] growing moving average Cobb-Douglas function, e.g., $O_t = \alpha_t \, x_t^{\beta tt} \, x_{t-1}^{\beta tt-1} \, . \, . \, ., \, x_0^{\beta t0} \, e^{\varepsilon t}$ the factor demands take the specific form

$$(20) \quad \ln x_t = \psi_{tt} + \psi_{t-1,t} \ln x_{t-1} + \, . \, . \, . + \psi_{0,t} \ln x_0$$

$$+ \pi_{tt} \ln\left(w_t/E_t p_T\right) + \pi_{t+1,t} \ln\left(E_t w_{t+1}/E_t p_T\right) + \, . \, . \, .$$

$$+ \pi_{Tt} \ln\left(E_t w_T/E_t p_T\right) + u_t, \quad t=1,2,. \, . \, .,T,$$

with the appropriate relative price ratios substituted for μ_t. Even though the T inputs are all determined within the system, the equations are recursive so that x_{t-1} is predetermined to all later demand equations, and thus should not be treated as jointly dependent. (Whenever the forecasts are revised in succeeding time periods, i.e., $E_{t-1}p_t \neq E_t p_t$, estimation efficiency is lost if the equations determining x_{t-1}, x_{t-2}, . . ., are substituted for these values, since additional parameters--on the expectations--are unnecessarily introduced.) If the observer errors u_t are serially uncorrelated, so that there is no correlation across the errors of the T equations of (19), <u>and if there were no across-equation restrictions implied by the production function parameters inbedded in the ψ's and π's</u>, then the factor demands could be

[2]This section (4.1) is concerned with input dynamics only. Recall that single and multistage models are observationally equivalent in the absence of output dynamics.

efficiently estimated one at a time. When the errors u_t are not serially independent, the lagged inputs and the equation errors are necessarily correlated; in this case, with the error covariances unknown and thus requiring estimation, the Zellner-Aitken seemingly unrelated regressions estimator must be iterated to convergence (and thus equivalence with the maximum likelihood estimator) to obtain even consistent estimates.

It is easily demonstrated that for the growing MA(t) Cobb-Douglas case (equation 18) there are necessarily across-equation restrictions on the ψ's and π's due to fewer underlying production function parameters. Observe that in (20) the t^{th} equation contains t+1 parameters ψ_{ij} and T−t+1 parameters π_{ij}, giving T+2 total parameters per equation and thus T(T+2) parameters in the system (20). But the system of production functions contains fewer parameters, only t+2 in the t^{th} equation (for the Cobb-Douglas form) for a total of $3 + 4 + \ldots + T + (T+1) + (T+2) < T(T+2)$ parameters. Since the factor demand parameters are derived from fewer elementary production parameters, they are not free; efficient estimation requires specification of the production function and imposition of the implied side relations between the factor demand parameters. Thus the factor demand equations cannot be efficiently estimated independent of the production function.

Similarly, if the input dynamics in (18) are specified as a fixed length, moving average process over τ periods [MA(τ)], then each Cobb-Douglas production function contains τ+2 parameters and it is readily verified that the input demand equations contain $\tau + 2 + h$ parameters, where the planning horizon extends h periods into the future. There are then h restrictions to be imposed for efficient estimation.

Whenever the number of elementary (production) parameters equals or exceeds the number of factor demand parameters, demand equation estimation can be separated from (further) knowledge of the production process. In the Cobb-Douglas case this occurs under static expectations, i.e., when all the expected relative prices in (20) are the same. Also, other production functions may have more elementary parameters than input demand parameters. For example, in the quadratic model with input dynamics of order MA(τ), each production function has $(\tau^2+\tau)/2$ parameters, while the input demand functions are known to be linear with $\tau + 2 + h$ parameters. In general, however, efficient estimation of the parameters of factor demand equations cannot be separated from a complete specification of the production process.

4.2 Output Dynamics, Observable Intermediate Products

Production functions of the growing autoregressive [AR(t)] form

$$
\begin{aligned}
&Q_1 = f_1(x_1, Q_0, \varepsilon_1) \\
&Q_2 = f_2(x_2, Q_1, Q_0, \varepsilon_2) \\
&\cdot \\
&\cdot \\
&\cdot \\
&Q_T = f_T(x_T, Q_{T-1}, Q_{T-2}, \ldots, Q_0, \varepsilon_T)
\end{aligned}
\tag{21}
$$

as well as systems of a fixed AR form constitute a recursive system with the related estimation considerations discussed in the previous section. Although some of the regressors $(Q_1 \ldots Q_{T-1})$ are determined within the system, none are jointly dependent,[3] and each equation is in fact already in reduced form. [Again, replacing the regressor Q_1 in the Q_2 equation with $f_1(x_1, Q_0, Q_{-1}, \varepsilon_1)$, for example, is inefficient since it introduces additional coefficients to estimate unnecessarily.] If there are no restrictions across the functions $f_1, f_2, \ldots, f_T$ and if the errors are serially uncorrelated (so there are no

[3]Since each affects the dependent variables below it but is not affected by them, there is no simultaneity.

covariances between the errors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T$), then pooling is not necessary for efficiency and the equations can be estimated separately from multiple cross-sections in each period. With uncorrelated errors, estimation is possible from time-series observations alone with the variations in $f_1, f_2, \ldots, f_T$ parameterized. When the errors are serially correlated, the intermediate output regressors are correlated with the equation errors so that ordinary least squares estimates are biased and inconsistent. In this case, the Zellner-Aitken estimator iterated to convergence gives consistent and asymptotically efficent estimates.

With output dynamics and analytic solutions the input demand equations are generally

$$(22) \quad x_t^* = x_t^*[w_t, Q^{t-1}, \mu_t].$$

When the research objectives require their estimation, the discussions of identification and estimation in the preceding section are valid. If the errors are independent, there are no across-equation parameter restrictions, and an expectations model is assumed, the input demands can be efficiently estimated singly using least squares. However, if the input demand function errors and the production function errors are contemporaneously or serially correlated, the iterated GLS estimator is required for consistency and efficiency, as discussed in section 4.1 above. Similarly, if the production function parameters impose restrictions across equations, multiple equation estimators are required for efficiency.

4.3 Output Dynamics, Unobservable Intermediate Products

It is frequently the case that observations on intermediate outputs are not available for use in estimation despite being utilized by the decision maker. Since the (observed) optimal input demands depend on these intermediate outputs, the system of input demand relations (when analytic) can be used in place of the unobserved quantities. In a multistage model like equation (9) above, terminal output Q_T depends on intermediate outputs Q^{T-1} observable to the decision maker but not to the econometrician. These intermediate outputs can be written as a function of the inputs used in the different periods, which, for profit maximizers, take the general form (13). The result is a recursive system of input demands culminating in a production function in terms of observables in the form of equation (10).

4.3.1 A simplified version

Let us first consider the particularly tractable multistage Cobb-Douglas production of (15) and its optimal factor demands, with two additional simplifications: (i) we shall ignore any relationships between the factor demand parameters (ψ's and π's) and the more basic production parameters (α, β, and γ), and (ii) we shall again [as we did in deriving equation (17)] assume that the predicted relative prices are not revised, so that $E_t(w_{t+j}/p_T) = E_{t+1}(w_{t+j}/p_T)$ for $j > 1$, for example. Under these conditions, the factor demand equations and the production function, after substituting for the unobserved intermediate products,

$$\ln x_1 = \psi_{01} + \pi_{11} \ln(\overline{w_1/p_1}) + \ldots + \pi_{T1} \ln(\overline{w_T/p_T})$$

$$\ln x_2 = \psi_{02} + \psi_{12} \ln\bar{x}_1 + \phi_{12}\varepsilon_1 + \pi_{22} \ln(\overline{w_2/p_2}) + \ldots + \pi_{T2} \ln(\overline{w_T/p_T})$$

$$\ln x_3 = \psi_{03} + \psi_{13} \ln\bar{x}_1 + \psi_{23} \ln\hat{x}_2 + \phi_{13}\,\varepsilon_1 + \phi_{23}\,\varepsilon_2$$
$$+ \pi_{33} \ln(\overline{w_3/p_3}) + \ldots + \pi_{T3} \ln(\overline{w_T/p_T})$$

$$\vdots$$

$$(23)\quad \ln x_t = \psi_{0t} + \psi_{1t}\ln\bar{x}_1 + \psi_{2t}\ln\hat{x}_2 + \ldots + \psi_{t-1,t}\ln\hat{x}_{t-1}$$

$$+ \phi_{1t}\,\varepsilon_1 + \ldots + \phi_{t-1,t}\,\varepsilon_{t-1} + \pi_{tt}\ln(\overline{w_t/p_t})$$

$$+ \ldots + \pi_{Tt}\ln(\overline{w_T/p_T})$$

$$\vdots$$

$$\ln x_T = \psi_{0T} + \psi_{1T}\ln\bar{x}_1 + \psi_{2T}\ln\hat{x}_2 + \ldots + \psi_{T-1,T}\ln\hat{x}_{T-1}$$

$$+ \phi_{1T}\,\varepsilon_1 + \ldots + \phi_{T-1,t}\,\varepsilon_{T-1} + \pi_{TT}\ln(\overline{w_T/p_T})$$

$$\ln O_T = \psi_{0q} + \psi_{1q}\ln\bar{x}_1 + \psi_{2q}\ln\hat{x}_2 + \ldots + \psi_{Tq}\ln\hat{x}_T$$

$$+ \phi_{1q}\,\varepsilon_1 + \ldots\ \phi_{T-1,T-1}\,\varepsilon_{T-1} + \varepsilon_T,$$

where the predetermined variables have been identified with an overbar and the jointly dependent variables with a hat.[4] Note that the joint dependency of the regressors marked with hats does not arise from equation simultaneity (since x_2, x_3, . . ., x_{t-1} are predetermined when x_t is to be set, i.e., the system is recursive) but rather from the increasing string of production function disturbances ε_2, ε_3, . . ., ε_{t-1} that result from the intermediate outputs being unobservable, akin to an errors-in-measurement model. We shall see that the unique structure of the system (23) permits use of simultaneous equations estimators to obtain consistent and asymptotically efficient parameter estimates, however.

Even if the production function--the last equation in the system (23)--is the only equation of interest, direct estimation using observed inputs as

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[4]The overbars and hats are simply indicators (and not newly defined quantities) designed to aid in the discussion that follows.

regressors results in biased and inconsistent parameter estimates since $\ln x_t$, $t = 2, \ldots, T$, contains $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{t-1}$ which are also included in the production function disturbance $\phi_{1q}\varepsilon_1 + \ldots + \phi_{Tq}\varepsilon_T$. A set of instruments correlated with inputs but uncorrelated with the equation disturbance is required, and the logical choice is of course the set of sample predictions of the input demand functions.

The identity determining $\ln x_1$ is irrelevant to estimation except that it establishes $\ln x_1$ as predetermined, permitting consistent estimation of the $\ln x_2$ equation by ordinary least squares. Then the sample prediction of $\ln x_2$, denoted $\widehat{\ln x_2}$, is orthogonal to ε_1 (by the property of least squares) and to all later ε_t's so that $\widehat{\ln x_2}$ is a suitable instrument for all equations below it in the recursive triangle. Then the $\ln x_3$ equation can be estimated using $\widehat{\ln x_2}$ as an instrument, resulting in predictions $\widehat{\ln x_3}$ suitable for all equations further down the recursive system, and so on until, finally, instruments are available for all of the jointly dependent variables in the production function and its parameters can be estimated.

Further, since there is always just one more excluded predetermined variable and one more included jointly dependent variable in each successive equation[5]--e.g., the $\ln x_3$ equation includes $\ln \hat{x}_2$ but excludes $\ln \left(\overline{w_1/p_1}\right)$, the $\ln x_4$ equation includes $\ln \hat{x}_3$ in addition to $\ln \hat{x}_2$ but excludes $\ln \left(\overline{w_2/p_2}\right)$ as well as $\ln \left(\overline{w_1/p_1}\right)$, and so forth--the system is just identified. Thus, the sample forecasts from the instrumental variable technique described above are

[5]When there is more than one input in each period, the system remains just identified as multiple included jointly dependent and excluded predetermined variables are added, given the two simplifications noted at the beginning of this section.

identical to those resulting from the reduced form equations (obtained by substituting the ln x_2 equation into the ln x_3 demand, etc., to the ln x_T equation): the instrumental variable estimator is equivalent to the two-stage least squares estimator and hence to the three-stage least squares estimator, the iterated three-stage least squares estimator, the limited information maximum likelihood estimator, and the full information maximum likelihood estimator, among others. Estimates obtained in this manner thus enjoy the properties of the full system estimators, in particular, consistency, asymptotic efficiency, and asymptotic normality under the usual assumptions.

The unique structure of the system (23), with different time periods resulting in different equations, means that estimates with and without serially correlated errors are observationally equivalent. When the disturbances are serially correlated, there is an additional source of correlation between the T + 1 equation errors besides that due to the elimination of unobserved intermediate products; since the system is just identified, however, any form of across-equation error covariance is irrelevant.

4.3.2 The general Cobb-Douglas system

Estimation without the two simplifications--regarding the basic parameters and the forecast updates--underlying the above discussion is slightly more complicated, resulting in an overidentified system (requiring utilization of error covariances for efficiency) that is nonlinear in the parameters to be estimated. Nevertheless, the estimation problem remains manageable.

Dropping the last simplifying assumption first, when the price forecasts are revised (typically each period), the forecast relative price terms in (23) reflect this with time-dated expectations. Thus a typical factor demand becomes

$$\ln x_t = \psi_{0t} + \psi_{1t} \ln\bar{x}_1 + \psi_{2t} \ln\hat{x}_2 + \ldots + \psi_{t-1,t} \ln\hat{x}_{t-1}$$

$$+ \phi_{1t}\varepsilon_1 + \ldots + \phi_{t-1,t}\varepsilon_{t-1} + \pi_{tt} \ln(w_t/E_t p_T)$$

$$+ \pi_{t+1,t} \ln(E_t w_{t+1}/E_t p_T) + \ldots + \pi_{Tt} \ln(E_t w_T/E_t p_T),$$

rather than the comparable (t^{th}) equation in (23). Since the set of predetermined variables is now vastly augmented (the expectations are different for each period), the equations are all overidentified, with all the implied ramifications (e.g., for full asymptotic efficiency, error covariance information must be explicitly incorporated, perhaps by three stage least squares or full information maximum likelihood techniques).

Dropping the other simplification, i.e., recognizing the dependence of the factor demand parameters on the elementary production function parameters, adds parameter nonlinearity to the considerations above. The form of the production function usually implies a set of nonlinear restrictions on the parameters of the factor demand equations. For example, in the Cobb-Douglas case, the ψ's and π's of (23) are composed of more basic production parameters analogous to those of equation (16). When there are fewer production parameters than input demand parameters, efficient estimation requires imposition of these (typically nonlinear) restrictions. This can be accomplished by carrying these parameters through the derivation of the input demands and using a nonlinear estimation algorithm that differentiates with respect to these elementary parameters.

5. Conclusion

In this paper we define dynamic stochastic production models in terms of the structure of the production functions (input dynamics versus output dynamics) and the type of decision problem (multistage versus multiperiod). This classification of production models admits a broad spectrum of dynamics due to technological phenomena such as "time to build," biological processes, and learning, as well as the cost of adjustment phenomenon. Dynamic production models are shown generally to be recursive systems of both production functions and input demand functions. Although the general solution for optimal input demand functions is complicated, the important Cobb-Douglas model is tractable. It can be used to represent both input and output dynamics, while the quadratic production function model can only be used to represent input dynamics.

We show that, in general, dynamic production functions can be estimated with standard linear or nonlinear ordinary least squares or generalized least squares procedures when all outputs are observed. The input demand functions need not be estimated in this case, and therefore it is not necessary to solve the control problem to estimate the production functions. However, when the research objective is to analyze input demand behavior, it may be necessary to explicitly specify the production functions to obtain efficient input demand estimates. This is because the factor demand parameters are composed of production function parameters. If there are fewer production function parameters than input demand parameters, the implied restrictions must be imposed on the input demand parameters for efficient estimation.

When outputs are not observed, as is usually the case with firm-level multistage models or aggregate multiperiod models, it is necessary to solve the multiperiod optimization problem for a set of input demands to provide instruments for consistent estimation. Although the resulting equation system is recursive, the time-aggregated errors cause an estimation problem identical to equation simultaneity. Remarkably, when two simplifications are imposed, the Cobb-Douglas model is exactly identified so that single equation estimators are equivalent to and thus as efficient as full system estimators. In the absence of these simplifying conditions, estimation requires simultaneous equation algorithms suitable for problems nonlinear in the parameters.

These results show that researchers must carefully evaluate their research objectives, model structure, and data availability in formulating dynamic, stochastic production models and in selecting appropriate estimators. The prevalence of both input dynamics and output dynamics in production systems, combined with the difficulty in observing intermediate products, suggest that dynamic, stochastic production systems should be formulated and estimated as interrelated recursive systems of production functions and factor demand functions.

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