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EXPECTATIONS AND ADJUSTMENTS IN QUADRATIC PROGRAMMING

by

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INTRODUCTION

The traditional mean-variance framework as suggested by Markovitz and Freund can be justified in terms of expected utility of income, where utility is exponential and prices are normal random variables. This approach takes the form of a quadratic program where mean prices and their variance are known parameters. The maximization is carried out with respect to quantities of commodities subject to a linear technology. The pair of dual problems associated with this formulation can be stated as

(1) Primal $\quad$ Maximize $\{\bar{p}'x - (\phi/2)x'\Sigma_p x\}$

subject to $\quad Ax \leq b \;,\; x \geq 0$

(2) Dual $\quad$ Minimize $\{b'y + (\phi/2)x'\Sigma_p x\}$

subject to $\quad A'y + \phi\Sigma_p x \geq \bar{p} \;,\; y \geq 0,\; x \geq 0$

where $\bar{p}$ is a (n x 1) vector of mean prices, Σ_p is a (n x n) matrix of their variances, ϕ is a risk aversion coefficient, A is an (m x n) matrix of technical coefficients, y is an (m x 1) vector of dual variables and b is an (m x 1) vector of resources. The conventional interpretation of the two problems can be briefly outlined as follows. In problem (1) the economic agent maximizes the difference between expected revenue and the risk premium, subject to technological constraints. In problem (2), a competitor will want to minimize the cost of buying the firm's resources ($b'y$) as well as the risk premium associated with the operation. The dual constraints stipulate that an equilibrium is obtained when marginal activity costs ($A'y$), adjusted by a marginal risk premium ($\phi\Sigma_p x$), are greater or equal to mean prices.

In problem (1) $\bar{p}'x$ is the expected money value (EMV) of the risky revenue, while $(\phi/2)x'\Sigma_p x$ is the risk premium (RP). Hence, the primal objective function can be reinterpreted as the maximization of the certainty equivalent (CE), or the amount of sure money which makes the entrepreneur indifferent between accepting such an amount and undertaking the risky prospect of operating his firm under uncertain conditions. Formally, this proposition is stated as $U(CE) = EU(p'x)$, where $U(\cdot)$ is a suitable utility function, implying that the certainty equivalent is equal to the expected money value of the risky prospect minus the risk premium, or CE = EMV − RP. At an optimal solution, therefore, the dual objective function minimizes the imputed value of resources (IVR) as well as the risk premium. Imputed value of resources can thus be expressed as IVR = CE − RP. In other words, the imputed value of resources is equal to the difference between the certainty equivalent and the risk premium.

AN EXPECTATION AND ADJUSTMENT MODEL

The conventional formulation outlined above can be extended in a number of directions. In a planning context, for example, the economic agent must formulate expectations about prices as well as adjust quantities to those expectations. Obviously, the process of expectation formation can take many forms. One which is appealing in the present context suggests that, when prices and quantities are uncertain, an economic agent generates his expectations by solving a quadratic program (representing his utility of money) subject to the relevant technological and market constraints.

To formalize the above discussion, suppose $p = (p_1, p_2, \ldots p_n)$ and $x = (x_1, x_2 \ldots x_n)$ are multivariate normally distributed random vectors of prices and quantities, that is, $p \sim N(P, \Sigma_p)$ and $x \sim N(X, \Sigma_x)$, where P and X are expected prices and quantities, respectively, while Σ_p and Σ_x are their associated covariance matrices. The two distributions are subjective distributions developed by the economic agent. Prices and quantities are not necessarily independent.

Given this set-up, revenue, $R = p'x$, is also a random variable with expectation

$$(3) \quad E(R) = P'X + u'\Sigma_{px}u$$

where Σ_{px} is the covariance matrix between p and x and $u = (1,1, \ldots 1)$ is a vector of ones. The variance of revenue (Bohrnstedt and Goldberger) is

$$(4) \quad VAR\ (R) = P'\Sigma_x P + X'\Sigma_p X + 2X'\Sigma_{xp}P + u'\Sigma_{xp}\Sigma_{px}u + u'\Sigma_x\Sigma_p u.$$

In a planning context and under a mean-variance approach, we assume that an economic agent will want to maximize U[E(R), Var(R)] subject to a linear technology. More explicity, the relevant primal problem is that of

$$(5) \quad \text{Max} \quad \{P'X - (\phi/2)[X'\Sigma_p X + P'\Sigma_x P + 2X'\Sigma_{xp}P]\} + K$$

$$\text{subject to} \quad AX \leq b \ , \ P \geq 0 \ , \ X \geq 0.$$

The constant K does not depend on either P or X but only on the variances and covariances of p and x; ϕ is a risk aversion coefficient. Problem (5) can be interpreted as the process of searching for those expectations about prices and the associated quantities which will maximize revenue under risk while satisfying the relevant technological constraints. With knowledge of Σ_p, Σ_x, Σ_{px}, A and b, the problem corresponds to finding the location parameter of the subjective distributions of prices and quantities. In more conventional terms, the objective function of (5) is the maximum difference between expected revenue and the approximate risk premium (Pratt). This objective function is neither concave nor convex because the term $P'X$ makes the relevant quadratic form indefinite. The problem, however, can be solved by means of an algorithm such as MINOS, written by Murtagh and Saunders.

The dual problem corresponding to (5) can be stated as

$$(6)\quad \text{Min}\ \{y'b + [(\phi/2)\ (X'\Sigma_p X + P'\Sigma_x P + 2X'\Sigma_{xp}P) - P'X]\}$$

$$\text{subject to}\qquad \begin{aligned} &P - \phi\Sigma_p X - \phi\Sigma_{xp}P - A'y \leq 0 \\ &X - \phi\Sigma_{px}X - \phi\Sigma_x P \leq 0 \\ &X \geq 0,\ P \geq 0,\ y \geq 0 \end{aligned}$$

where y is the vector of dual variables corresponding to the primal constraints. The objective function of the dual problem stipulates the minimization of the total imputed value of resources ($y'b$) minus the (approximate) certainty equivalent corresponding to the risky primal problem. This implies that the imputed value of the resources is equal to twice the approximate certainty equivalent. In the dual objective function, the certainty equivalent is approximated because the constant K does not appear in it and higher moments of the distribution of revenue are disregarded.

The first set of dual constraints, rewritten as $P \leq A'y + \phi\Sigma_p X + \phi\Sigma_{xp}P$ for convenience, indicates that an equilibrium solution is achieved when marginal activity costs ($A'y$) adjusted for uncertain prices and quantities by marginal risk premia are greater or equal to expected prices. Except for the covariance term $\phi\Sigma_{xp}P$, this set of constraints is similar to the traditional quadratic programming formulation.

The second set of dual constraints constitutes a novel relation which places an upper bound on the expected equilibrium quantities. This upper bound can be explicitly formulated as

$$(7)\quad 0 \leq X \leq (I/\phi - \Sigma_{px})^{-1}\Sigma_x P.$$

Because of the risky price environment, the range of the expected equilibrium quantities is no longer the positive orthant but a subset of it, as determined by (7). It is of interest to notice that the dual variables associated with the dual constraints of (6) are quantities and prices, respectively (see Appendix.) This important criterion may be used to verify the correct formulation of the problem and the accuracy of computations. Problems (5) and (6) are of interest because of their generality. Solution of this formulation was obtained for a number of numerical examples using the MINOS package.

ALTERNATIVE FORMULATIONS

Two alternative formulations are possible if one assumes knowledge of either the expected prices or quantities.

Suppose, in fact, that one is willing to assume that the subjective expected prices are equal to the actual mean prices, say $P \equiv \overline{p}$. In this case, problem (5) can be reformulated as

$$(8)\quad \text{Max} \quad \{(\overline{p}' - \phi\overline{p}'\Sigma_{px})X - (\phi/2)X'\Sigma_p X\} + K$$
$$\text{subject to} \quad AX \leq b \;,\; X \geq 0.$$

This problem is now a concave quadratic program. It retains, however, some features of the more general formulation via the covariance matrix Σ_{px} of prices and quantities which appears in the linear part of the objective function. For the rest, problem (8) resembles a conventional quadratic program. The constant K does not depend on X.

A more interesting formulation is obtained if one assumes knowledge of long-run equilibrium quantities and solves for the expected equilibrium prices. This programming version is not unusual when considered in a planning context. The planning board may wish to choose a priori the expected quantity targets and find out the associated expected prices. In this case, letting $X \equiv d$, the appropriate specification obtained from problem (6) is as follows:

$$(9)\quad \text{Min} \quad \{y'b + [(\phi d'\Sigma_{xp} - d')P + (\phi/2)P'\Sigma_x P]\} + K$$

$$\text{subject to} \quad P - \phi\Sigma_{xp}P - A'y \leq \phi\Sigma_p d$$

$$-\phi\Sigma_x P \leq (\phi\Sigma_{px} - I)d$$

$$P \geq 0,\ y \geq 0.$$

Problem (9) is also a concave program and follows from problem (6) after replacing the expected quantities X with the known levels d. The constant K does not depend on P. This version can give, indirectly, a measure of the wisdom of fixing a priori the expected quantity targets. In fact, dual variables for the two sets of constraints in problem (9) are to be regarded as expected quantities and prices, respectively (as in problem (6)). Dual quantities which are very different from the pre-assigned levels of quantities, X, correspond to an inefficient allocation that inevitably will manifest itself also in a difference between expected prices and dual prices. Economic problems formulated according to the structure of model (9) are very useful for analyzing the impacts of administered prices within a sector, a region, or the entire economy. Since prices appear explicitly in the constraints, they can be further restricted to the range desired by the administering board.

A RATIONAL EXPECTATION MODEL

The above discussion has allowed the gradual introduction and analysis of a novel quadratic programming model incorporating expectations about prices and long-run adjustments of quantities. Following Muth, one can define as rational expectations those processes of expectation formation which are based on the relevant economic theory and the most complete information set. Accordingly, an expectation model which incorporates supply and demand conditions and technological relations as well as information about the risky prospects can be regarded as a rational expectation model. To this purpose, let X_D and X_S represent expected long-run demand and supply quantities. Then, the following model is a rational expectation model in the form of a quadratic program.

$$\text{(10)} \quad \text{Max} \quad \{P'X_D - (\phi/2)[X_D'\Sigma_p X_D + P'\Sigma_x P + 2X'_D\Sigma_{xp}P]\}$$

$$\begin{array}{lrl} \text{subject to} & AX_S \leq b & \text{Production technology} \\ & X_S - SP = f & \text{Supply functions} \\ & X_D + DP = c & \text{Demand functions} \\ & X_D - X_S + V^+ - V^- = 0 & \text{Market clearing conditions.} \\ & P \geq 0,\ X_D \geq 0,\ X_S \geq 0,\ V^+ \geq 0,\ V^- \geq 0, & \end{array}$$

where V^+ and V^- are nonnegative slack vectors.

The objective function has the same meaning as in problem (5). The first set of constraints characterize the production technology. The second set represents a system of a linear supply functions with S being the matrix of slopes and f the vector of intercepts. The third set of constraints represents a system of linear demand functions. The corresponding slopes are grouped into the D matrix while the intercepts are the elements of the c vector. Notice that this formulation allows for either excess demand or excess supply or for equality between the two quantities. The empirical implementation of a model such as (10) would require the econometric estimation of the demand and supply systems. The inclusion of technological constraints is optional. In general, supply functions incorporate (implicitly) the relevant technological information. It is suggested that while the supply functions are aggregate relations, the production technology expressed by the A matrix is a detailed, regional description of the production processes capable of transmitting at the local level the impacts of administered prices and general market equilibria.

AN ADAPTIVE EXPECTATION MODEL

When price expectations are assumed to be adaptive, their analytical expression can be stated as

$$(11) \quad P_t = P_{t-1} + B(P_{At-1} - P_{t-1}) = B\, P_{At-1} + (I - B)\, P_{t-1}$$

where B is a diagonal matric of known expectation coefficients bounded by zero and unity, P_{At-1} is the vector of actual equilibrium prices at time t-1, and P_t is the vector of expected prices in period t. The coefficients in B can be estimated econometrically. Prices P_{At-1} are known since, at time t, markets have revealed prices at time t-1. Furthermore, when the interest is to use model (5) and scheme (11) for a period of years, it is convenient to consider the vector of expected prices P_{t-1} as unknown. This implies that the first time model (5) is solved, P_{t-1} will appear explicitly as a variable while in subsequent periods the corresponding price expectations will be computed according to (11). The "first-period" adaptive expectation model can readily be obtained by substituting relation (11) into model (5) and rearranging terms. This substitution results in the following model

$$(12) \quad \text{Max } \{C'_{t-1}X_t - D'_{t-1}P_{t-1} - (\phi/2)[X'_t\Sigma_p X_t + P'_{t-1}QP_{t-1} + 2X'_tRP_{t-1}]\}$$

$$\text{subject to } \; AX_t \leq b_t \;,\; X_t \geq 0 \;,\; P_{t-1} \geq 0$$

where $C'_{t-1} \equiv (P'_{At-1}B - \phi P_{At-1}\Sigma_{px})$, $D'_{t-1} \equiv \phi\, P'_{At-1}(I - B)\Sigma_x B$, $Q \equiv (I - B)\Sigma_x(I - B)$, $R \equiv \Sigma_{xp}(I - B)$.

In subsequent periods, the problem of finding expected long-run quantities is

$$(13)\quad \text{Max } \{[C'_t - \phi P'_t R']X_{t+1} - (\phi/2)X'_{t+1}\Sigma_p X_{t+1}\}$$

$$\text{subject to } \quad AX_{t+1} \leq b_{t+1}\ ,\ X_{t+1} \geq 0.$$

The recursive nature of this expectation model is well suited for forecasting optimal quantities one step ahead since it incorporates all the available information on expected and actual prices as it becomes available.

A LONG-RUN ADJUSTMENT MODEL

Nerlove has suggested that a long-run adjustment model can be stated as

$$(14)\quad X_{At} = X_{At-1} + \Gamma(X_t - X_{At-1})$$

where X_t is the vector of long-run equilibrium outputs, X_{At} is the vector of short-run equilibrium outputs, and Γ is a (n x n) diagonal matrix of known adjustment coefficients bounded by zero and unity, $0 < \gamma_i \leq 1$, $i = 1,\ldots n$. From (14), expected long-run equilibrium quantities can be expressed as

$$(15)\quad X_t = GX_{At} + (I - G)X_{At-1}$$

where $G = \Gamma^{-1}$. The Nerlovian adjustment hypothesis expresses the long-run adjustment quantities as a weighted average of short-run equilibrium quantities in the two most recent periods.

Substituting relation (15) into model (6), a quantity-price adjustment model is obtained which exploits the recursive nature of (15):

$$(16)\quad \text{Min}\ \{y'_t b_t + f'_{t-1}X_{At} + g'_{t-1}P_t + (\phi/2)[X'_{At}G\Sigma_p GX_{At} + P'_t\Sigma_x P_t + 2X'_{At}G\Sigma_{xp}P_t] - P'_t GX_{At}\}$$

$$\text{subject to}\quad P_t - \phi\Sigma_{xp}P_t - \phi\Sigma_p GX_{At} \le \phi\Sigma_p(I - B)X_{At-1}$$

$$-\phi\Sigma_x P_t + (G - \phi\Sigma_{px}G)X_{At} \le [\phi\Sigma_{px}(I - G) - (I - G)]X_{At-1}$$

$$P_t \ge 0\ ,\ X_{At} \ge 0.$$

where $f'_t \equiv \phi x'_{At-1}(I - G)\Sigma_p G$ and $g'_t \equiv [\phi X'_{At-1}(I - G)\Sigma_{xp} - X'_{At-1}(I - G)]$. In model (16) the quantity vector X_{At-1} is presumed known. The unknowns, therefore, are the expected prices P_t and the short-term equilibrium quantities X_{At}. To the extent that the level of X_{At} obtained by solving problem (16) differs from the levels of quantities realized in the economy, the expectations about prices, P_t, will not be exactly those desired. However, the recursive character of the model allows for the updating of the information in the vectors g and f as soon as it becomes available. Also in this case, a one-period-ahead prediction of equilibrium prices and quantities is conveniently formulated.

CONCLUSIONS

Traditional quadratic programming models can be extended to include the determination of expectations about prices and quantities. This requires casting the empirical problem in a planning framework. Several versions of the expectation model were discussed. The most satisfactory seem those which incorporate the maximum amount of technological and economic information about production and marketing conditions.

From a planning viewpoint, the price expectation model has the distinct advantage of allowing the explicit imposition of constraints on prices. This practice may be very useful when prices are administered as in many agricultural and infant industry situations.

An interesting perspective is the possibility of using the adjustment and the adaptive expectation models in tandem to minimize the inconsistency of estimated price and quantity levels over a period of years. The price vector, P_t, estimated from model (16) can be substituted into model (13) to obtain the expected quantity levels X_{t+1} which, in turn, can be used in the adjustment equation (14) to derive an estimate of the actual quantity levels X_{At+1} to be used again in model (16), and so on. This procedure has the advantage of using all the price and quantity information as produced by the economy to verify and correct the prediction of expected prices and long-run equilibrium quantities.

APPENDIX

To obtain the dual of problem (5) it is necessary to set up the relevant Lagrangean function, to derive the appropriate Khun-Tucker conditions and, then, to simplify the dual function. That is, the Lagrangean function is

(A.1) $L = P'X - (\phi/2)[P'\Sigma_x P + X'\Sigma_p X + 2X'\Sigma_{xp}P] + y'(b - AX)$.

The Khun-Tucker conditions are

(A.2) $(\partial L/\partial X) = P - \phi\Sigma_p X - \phi\Sigma_{xp}P - A'y \leq 0$

(A.3) $X'(\partial L/\partial X) = X'P - \phi X'\Sigma_p X - \phi X'\Sigma_{xp}P - X'A'y = 0$

(A.4) $(\partial L/\partial P) = X - \phi\Sigma_x P - \phi\Sigma_{px}X \leq 0$

(A.5) $P'(\partial L/\partial P) = P'X - \phi P'\Sigma_x P - \phi P'\Sigma_{px}X = 0$

(A.6) $(\partial L/\partial y) = b - AX \geq 0$

(A.7) $y'(\partial L/\partial y) = y'b - y'AX = 0$.

The objective function of the dual problem, represented by the minimization of Lagrangean function, can be simplified using conditions (A.3) and (A.5) as follows:

$$\text{Min } L = (\phi/2)X'\Sigma_p X - (\phi/2)P'\Sigma_x P + y'b \qquad \text{using (A.3)}$$

$$= (\phi/2)X'\Sigma_p X - \phi P'\Sigma_x P + (\phi/2)P'\Sigma_x P + y'b$$

$$= (\phi/2)[X'\Sigma_p X + P'\Sigma_x P + 2X'\Sigma_{xp}P] - P'X + y'b \qquad \text{using (A.5).}$$

The constraints of the dual problem are relations (A.2) and (A.4).

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