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A NOTE ON SLUTSKY'S THEOREM AND THE ASYMPTOTIC EQUIVALENCE OF FEASIBLE AND TRUE GLS ESTIMATORS

by

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A NOTE ON SLUTSKY'S THEOREM AND THE ASYMPTOTIC EQUIVALENCE OF FEASIBLE AND TRUE GLS ESTIMATORS

The usefulness of Aitken's generalized least squares (GLS) estimation technique for the linear regression model is hampered by the fact that the true covariance matrix of the regression disturbances is rarely known. Consequently, researchers typically use "feasible" GLS estimators which are based on an estimated covariance matrix, and must appeal to the asymptotic properties of such estimators. While very general sufficient conditions for the asymptotic equivalence of the feasible and true GLS estimators are well known, researchers are faced with the often difficult task of checking whether these sufficient conditions are satisfied for each feasible estimator. The necessity of checking the convergence of each feasible estimator to the true GLS estimator is demonstrated by Schmidt (1976, p. 68-71) who shows that the feasible and true GLS estimators need not have the same asymptotic distribution even when a "consistent" estimator of the error covariance matirx is used in the feasible estimator. One set of "regularity conditions" under which the feasible GLS estimator always converges in distribution to the true GLS estimator has been devised by Fuller and Battese (1973), although these conditions are difficult to verify in many cases because they involve the properties of derivatives taken with respect to the inverse of the covariance matrix. The importance of the problem of asymptotic equivalence is underlined by the confusion in the literature over this issue; for example, some econometrics textbooks state or imply that the feasible GLS estimator generally converges in distribution to the true GLS estimator (Kmenta, 1971, p. 507; Dhrymes 1974, pp. 152-153).

In this paper it is shown that under some easily verifiable conditions, namely if the elements of the consistently estimated covariance matrix in the feasible GLS estimator are functions of a fixed finite parameter vector and if these functions satisfy the conditions of Slutsky's Theorem, the feasible GLS estimator always does converge in distribution to the true GLS estimator. However, if this condition is not satisfied in terms of Slutsky's Theorem, as it is not in many cases, then the asymptotic equivalence may fail, as Schmidt has shown, and it is still true that one must prove the convergence holds in each case. Thus, there are actually three classes of feasible GLS estimators: one class for which the asymptotic equivalence necessarily holds because Slutsky's Theorem is satisfied; one class for which Slutsky's Theorem is violated but the convergence in distribution nevertheless holds; and one class for which convergence to the true GLS estimator does not hold. Consequently, the results of this paper show that the task of devising and applying feasible GLS estimators can be greatly simplified in those cases for which the estimated covariance matrix satisfies Slutsky's Theorem.

The paper begins with a proof of the theorem for the asymptotic equivalence of the feasible and true GLS estimators. The proof is followed by a discussion of the importance of Slutsky's Theorem to the proof and by examples to show how the asymptotic equivalence may fail when Slutsky's Theorem is not satisfied. Also the theorem by Fuller and Battese is discussed.

I. A Theorem for the Asymptotic Equivalence of Feasible and True GLS Estimators

Let the standard linear regression model be given by

$$y = X\beta + \varepsilon,$$

where y is a (T x 1) vector of dependent variables, X is a (T x k) matrix of nonstochastic exogenous variables, β is a conformable coefficient parameter vector, and ε is a (T x 1) vector of error terms such that

$$E(\varepsilon\varepsilon') = \Sigma$$

is a positive definite matrix for all T. In addition we make the standard assumption that $\lim (T \to \infty) [X'\Sigma^{-1}X]/T$ is positive definite. Slutsky's Theorem, which we use below, is stated as follows:

Theorem 1: If g is a continuous function independent of T, and if z_T converges in probability to z, then $g(z_T)$ converges in probability to g(z).

Proof: See Rao (1973), p. 124.

It is important to note that the function $g(\cdot)$ in the above Theorem must be independent of T, the sample index; otherwise, it may not be true that

$$\text{plim } g(z_T) - g(z) = 0.$$

For example, letting $z_T = \frac{1}{T}$, $z = 0$, and $g(z) = Tz$ illustrates this point. As we shall see below, it is exactly this phenomenon which may cause the asymptotic equivalence of the feasible and true GLS estimators to break down.

The next Theorem uses Slutsky's Theorem to prove the asymptotic equivalence of feasible and true GLS estimators for a broad class of estimated error covariance matrixes.

Theorem 2: Assume there exists a finite dimensional parameter vector θ and a consistent estimator $\hat{\theta}_T$ of θ based on a sample of T. Also assume there exist continuous functions g_{ij} which do not depend on T except through their arguments, such that $\text{plim } g_{ij}(\hat{\theta}_T) = \sigma_{ij}$, where $\Sigma = [\sigma_{ij}]$. Letting the estimated covariance matrix for the feasible GLS estimator be $\hat{\Sigma}_T = [g_{ij}(\hat{\theta}_T)]$, it follows that the feasible GLS estimator

$$\tilde{\beta}_T = (X'\hat{\Sigma}_T^{-1}X)^{-1}X'\hat{\Sigma}_T^{-1}y.$$

converges in distribution to the true GLS estimator

$$\bar{\bar{\beta}} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y.$$

<u>Proof</u>: First we define

$$\Delta\sigma_T = \sup\{|\hat{\sigma}_T^{ij} - \sigma^{ij}|\}$$

where $\hat{\sigma}_T^{ij}$ and σ^{ij} are the (i,j) elements of $\hat{\Sigma}_T^{-1}$ and Σ^{-1}.

Since

$$\text{plim}\, g_{ij}(\hat{\theta}_T) = g_{ij}(\theta) = \sigma_{ij}$$

and since $\hat{\sigma}_T^{ij}$ is a continuous function of the $g_{ij}(\hat{\theta}_T)$ which satisfies Slutsky's Theorem, it follows that plim $\Delta\sigma_T = 0$. The sufficient conditions for the convergence of $\tilde{\beta}$ in distribution to $\bar{\bar{\beta}}$ are:

(1) $$\text{plim}\, \frac{X'\hat{\Sigma}_T^{-1}X}{T} = \text{plim}\, \frac{X'\Sigma^{-1}X}{T}$$

(2) $$\text{plim}\, \frac{X'\hat{\Sigma}_T^{-1}\varepsilon}{\sqrt{T}} = \text{plim}\, \frac{X'\Sigma^{-1}\varepsilon}{\sqrt{T}}$$

The (i,j) element of $(X'\hat{\Sigma}_T^{-1}X)/T$ is

$$\sum_k \sum_\ell x_{ki}\, x_{\ell j}\, \hat{\sigma}_T^{k\ell}$$

To prove that (1) holds we show the above term converges in probability to the corresponding element of $[X'\Sigma^{-1}X]/T$, as follows:

$$\text{plim} \quad \left[\frac{1}{T}\sum_k\sum_\ell x_{ki}\, x_{\ell j}\, \hat{\sigma}_T^{k\ell} - \frac{1}{T}\sum_k\sum_\ell x_{ki}\, x_{\ell j}\, \sigma^{k\ell}\right]$$

$$\leqslant \text{plim} \left[\frac{1}{T}\sum_k x_{ki}\, x_{kj}\, \Delta\sigma_T\right]$$

$$= \text{plim} \left[\frac{1}{T}\sum_k x_{ki}\, x_{kj}\right] \text{plim}\, \Delta\sigma_T = 0.$$

To prove (2) we note that the ith element of $[X' \hat{\Sigma}_T^{-1} \varepsilon]/\sqrt{T}$ is

$$\frac{1}{\sqrt{T}}\sum_k\sum_\ell x_{ki}\, \hat{\sigma}_T^{k\ell}\, \varepsilon_\ell$$

To show the above term converges in probability to its corresponding element on the r.h.s. of (2) we have

$$\text{plim} \quad \left[\frac{1}{\sqrt{T}}\sum_k\sum_\ell x_{ki}\, \varepsilon_\ell\, \hat{\sigma}_T^{k\ell} - \frac{1}{\sqrt{T}}\sum_k\sum_\ell x_{ki}\, \varepsilon_\ell\, \sigma^{k\ell}\right]$$

$$\leqslant \text{plim} \left[\frac{1}{\sqrt{T}}\sum_k x_{ki}\, \varepsilon_k\, \Delta\sigma_T\right] = \text{plim} \left[\frac{1}{\sqrt{T}}\sum_k x_{ki}\, \varepsilon_k\right] \text{plim}\, \Delta\sigma_T = 0$$

Therefore, (1) and (2) hold and it follows that

$\text{plim}\, \sqrt{T}[\tilde{\beta} - \bar{\bar{\beta}}] = 0.$ <u>Q.E.D.</u>

II. The Importance of Slutsky's Theorem

The critical role of Slutsky's Theorem in the above proof can be illustrated by an example provided by Schmidt (1976, p. 69-70). Schmidt's example shows that some feasible GLS estimators with "consistently" estimated covariance matrixes may fail to converge in distribution to the true GLS estimator. In the example we let

$$X = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \lambda & & & & \\ & \lambda^2 & & & \\ & & \ddots & & \\ & & & & \lambda^T \end{bmatrix}.$$

Also let $\hat{\lambda} = \lambda + 1/T$ and assume $\lambda = 1$. It follows that plim $\hat{\lambda} = \lambda$, and in addition for all $i < \infty$, plim $\hat{\lambda}^i = \lambda^i$, so that

$$\hat{\Sigma}_T = \begin{bmatrix} \hat{\lambda} & & & & & \\ & \hat{\lambda}^2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot\ \hat{\lambda}^T \end{bmatrix}$$

would appear to be a "consistent" estimator of Σ. Schmidt shows, however, that whereas the asymptotic distribution of $X'\Sigma^{-1}\varepsilon/T$ converges to N(0,1), the asymptotic distribution of $X'\hat{\Sigma}_T^{-1}\varepsilon/T$ is $N(0, \frac{e^2 - 1}{2(e - 1)^2})$. The explanation for this result is the fact that the estimates of the elements of the covariance matrix in the feasible GLS estimator violate the conditions of Slutsky's Theorem. Because of this violation, in this particular case we have

$$\lim \hat{\lambda}^T = \lim (1 + 1/T)^T = e \neq 1.$$

To show that this result need not hold if Slutsky's Theorem is not violated, note that if the covariance matrix were the diagonal matrix of λ^j, for a given j for all T, then

$$\lim \hat{\lambda}^j = \lim (1 + 1/T)^j = 1$$

and one could easily show that the feasible GLS estimator converges in distribution to the true GLS estimator as Theorem 2 requires.

While the results of the above example are interesting, one may ask in what practical applications in econometrics these results are relevant. One can readily see that models with autocorrelated error terms typically will tend to violate the conditions of Theorem 2, as the simple regression model with first-order autocorrelation shows; in that case

$$\hat{\Sigma}_T = \begin{bmatrix} 1 & \rho^2 & \cdots & \rho^{T-1} \\ \rho^2 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ \rho^{T-1} & & \cdots & 1 \end{bmatrix}$$

and therefore replacing ρ by $\hat{\rho}$ would give a "consistent" estimate of Σ which violates Slutsky's Theorem and hence the results of Theorem 2 are not applicable. Of course it can nevertheless be shown that if plim $\hat{\rho} = \rho$ then $\tilde{\beta}$ converges in distribution to $\overset{=}{\beta}$ (Theil 1971, pp. 403-404). However, other models in the econometrics literature do have estimated covariance matrixes which satisfy Slutsky's Theorem. For example, in a heteroscedastic regression model developed by Amemiya (1977),

$$\hat{\Sigma}_T = \begin{bmatrix} (z_1\hat{\alpha})^2 & & 0 \\ & \ddots & \\ 0 & & (z_T\hat{\alpha})^2 \end{bmatrix}$$

where z_i is a vector of exogenous variables and plim $(z_i\alpha)^2 = \sigma_{ii}$, the ith diagonal element of Σ. By Theorem 2, it follows immediately that the feasible GLS estimator based on this estimated covariance matrix converges in distribution to the true GLS estimator.

III. The Fuller-Battese Theorem

Fuller and Battese have shown that $\tilde{\beta}$ converges in distribution to $\bar{\bar{\beta}}$ under the following conditions:

(i) the elements σ_{ij} of Σ are functions of paramenters $\theta = (\theta_1,\ldots,\theta_r)$ such that the matrices

$$G_j = \frac{\partial \Sigma^{-1}}{\partial \theta_j}, \ j = 1,\ldots,r$$

are continuous functions of θ in an open sphere of the true value of θ.

(ii) $\lim \frac{1}{T} X' \Sigma^{-1} X$ is positive definite.

(iii) $\lim \frac{1}{T} X' G_j X$ is a matrix whose elements are continuous functions of θ.

Clearly, conditions (i) and (iii) are more difficult to verify than the conditions of Theorem 2, especially when Σ is not a diagonal matrix, since it is a trivial matter to know if Slutsky's Theorem is satisfied but a very nontrivial matter to check for the continuity required in the above conditions. However, the Fuller-Battese Theorem can be applied to cases which do violate Slutsky's Theorem, and is clearly of interest in such cases.

In the case of Schmidt's example discussed above, condition (i) can be verified for any T, but condition (iii) requires that

$$\lim \frac{1}{T} \sum_{i=1}^{T} i \lambda^{-i-1}$$

be a continuous function of λ. For the case of $\lambda = 1$ considered above the function diverges and thus condition (iii) is violated. It is easy to verify that if the elements of Σ were λ^{j}, for fixed j, then condition (iii) would be satisfied and the asymptotic results would go through.

IV. Conclusion

In this paper it has been shown that one may define three classes of feasible GLS estimators. In the first class the asymptotic equivalence of feasible and true GLS estimators necessarily holds because Slutsky's Theorem is satisfied, as proved by Theorem 2. In the second class, Slutsky's Theorem is violated but the asymptotic equivalence nevertheless holds; for this class of estimators, the theorem of Fuller and Battese may help one verify the asymptotic properties of the feasible GLS estimator. The third class consists of those feasible GLS estimators which do not converge in distribution to the true GLS estimator. Schmidt's example provides an illustration of one such member of this third class. Therefore, for those models which do satisfy Slutsky's Theorem as set out in Theorem 2 of this paper, one can immediately attribute the asymptotic properties of the true GLS estimator to the feasible estimator.

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