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THEORY AND MEASUREMENT OF OUTPUT DISTRIBUTIONS

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ABSTRACT

A production model is developed based on the postulate that the firm's economic behavior depends on the form of the probability distribution function of output which is defined as the firm's output distribution. In general all of the moments of the output distribution may be functions of the inputs and the parameters which characterize the firm's technology. After showing how this approach is related to the literature, it is shown that the "standard" stochastic production function specifications impose generally unacceptable restrictions on the relationship between the inputs and the output distribution's moments. Next, a "linear moment model" is proposed which is free of such restrictions, and large sample econometric techniques are devised which provide parameter estimators which are asymptotically equivalent to Aitken estimators. The paper concludes with a comparison of the proposed econometric techniques to others that have been proposed for estimation of output distribution and frontier production models.

THEORY AND MEASUREMENT OF OUTPUT DISTRIBUTIONS

In this paper I set out a production model based on the thesis that the firm's decision maker views output, Q, as a random variable conditionally distributed on the input vector x and the parameter vector α which characterize the firm's technology. Therefore, the firm's economic behavior depends on the form of the probability distribution function of Q which is denoted by $f(Q|x, \alpha)$ and referred to as the firm's output distribution. This interpretation of a firm's stochastic technology is of interest because it suggests that, in general, all of the moments of the output distribution may be functions of the inputs and the parameters which characterize the firm's technology. In this paper I discuss the theoretical foundations and propose some empirical techniques for the study of production models based on the output distribution concept.[1]

The first section of the paper is devoted to various results from the statistics, econometrics, and economics literature which serve to motivate the output distribution approach and illustrate how the output distribution's moments are important in terms of specifying a firm's technology and understanding the firm's behavior under technological uncertainty. The second section discusses several of the stochastic production function specifications in the literature and shows that they all impose generally unacceptable restrictions on the relationship between the inputs and the output distributions's moments. In section 3, large sample econometric techniques are devised which produce consistent, and asymptotically normal estimators of the model's parameters. The paper concludes with a comparison of these econometric techniques to others that have been proposed for estimation of output distributions and frontier production models.

1. Statistical and Economic Foundations of the Output Distribution Model

Our first task is to explore the relationship between the moments of the output distribution and the characteristics of a firm's stochastic technology. Can we use these moments to uniquely characterize a firm's technology? To demonstrate that the answer is affirmative I draw upon some results from the "problem of moments" in the statistics literature.[2] Kendall and Stuart (1976, Ch. 4) prove the following theorem:

<u>Theorem 1</u>: Let μ_i denote the moments of a distribution calculated about any origin. The moments uniquely determine the distribution if the series

$$\sum_{i=0}^{\infty} \mu_i t^i / i!$$

converges for some real non-zero t.

For the study of output distributions the following corollary to Theorem 1 is particularly important:

<u>Corollary 1</u>: The moments uniquely determine the distribution if it is defined over a finite interval of the real line.

Since we know that an output distribution <u>must</u> be defined over a finite interval of the real line, it is uniquely determined by its moments. In practice we may often need to approximate a firm's output distribution with relatively few moments because the distribution may have many (or even an infinity) of moments. Kendall and Stuart also suggest a method for such an approximation. They let a distribution be approximated over some interval, say [0, Q*], by:

$$\sum_{i=1}^{n} b_i z^i$$

where the b_i are found by minimizing the squared approximation error:

$$\int_0^{Q^*} [f(z) - \sum_{i=1}^{n} b_i z^i]^2 \, dz$$

Minimization of the above expression gives:

$$\int_0^{Q^*} f(z) z^i dz = \mu_i' = \int_0^{Q^*} \sum_{j=1}^{n} b_j z^{i+j} \, dz$$

Thus, Kendall and Stuart (p. 90) conclude that "if two distributions have moments up to order n then they must have the same least squares approximation, for the coefficients b_i are determined by the moments."

These results have great importance for our interpretation of output distribution models, for they tell us that we can uniquely identify, or approximate to a desired degree, a firm's stochastic technology with the output distribution's moments. Consequently the firm's optimal input choices depend on these moment functions, as we shall now demonstrate. We let product and input prices be given by p and r and for simplicity assume they are not random variables (assuming they are random would complicate the analysis without altering the qualitative conclusions concerning the output distribution model). Expressing profit as $\pi = pQ - rx$, and letting $U(\cdot)$ denote the firm's concave utility function, the firm's objective is: $\max_x EU(\pi) = \int U(\pi) f(Q|x, \alpha) \, dQ$. The input choice which maximizes the expected utility of profit satisfies:

$$\frac{\partial EU(\pi)}{\partial x} = \int U' \frac{\partial \pi}{\partial x} f(Q|x, \alpha) dQ + \int U(\pi) \frac{\partial f}{\partial x} dQ = 0$$

$$\text{or:} \quad r \int U' f(Q|x, \alpha) dQ = \int U(\pi) \frac{\partial f}{\partial x} dQ$$

As usual, the optimal input choice equates the expected marginal cost to the expected marginal benefit.[2] The first term above is the cost term, the expected marginal utility per dollar times the input price. The second term, the expected marginal benefit, is expressed in terms of the change in expected utility due to the shift in the mass of the distribution which is induced by the input change. Since we know that:

$$\int \frac{\partial f}{\partial x} dQ = 0$$

we can interpret the marginal benefit as a weighted average about zero of the utility of each possible profit level. If on average the mass of the distribution increases for higher utility levels and decreases for lower utility levels, the marginal benefit is positive.[3]

The connection between the output distribution moments and the firm's behavior can be introduced explicitly by expanding the utility function about expected profit before taking the expectation of utility (see Anderson, Dillon, and Hardaker, Ch. 6). Letting $E\pi = \overline{\pi}$, and letting U_i denote the ith derivative of U, we obtain:

$$EU(\pi) = U(\overline{\pi}) + \sum_{i=2}^{\infty} \frac{U_i}{i!} p^i \mu_i$$

Differentiation gives:

$$\frac{\partial EU(\pi)}{\partial x} = [U_1 + \sum_{i=2}^{\infty} \frac{U_{i+1}}{i!} p^i \mu_i] \frac{\partial \overline{\pi}}{\partial x} + \sum_{i=2}^{\infty} \frac{U_i}{i!} p^i \frac{\partial \mu_i}{\partial x}$$

Now recalling that the firm's technology is uniquely characterized by the output distribution's moments, we can see that the firm's optimal input choices depend on the effects of inputs on the technology as represented by the functional relationship between the μ_i and x.

A further understanding of the behavioral implications of the output distribution model can be gained by interpreting it as a frontier production function model (for references to the literature see Forsund, et al. 1980). To do this we note that a frontier output distribution model must account for two factors: the effects of inputs on the relative position of the mass of the distribution function, and the effects of inputs on the location of the production frontier, denoted as Q_F. These two elements of the frontier model can actually be considered as one and the same by defining the distribution function over the positive real line, by assuming it is unimodal, and by defining Q_F as that $Q > 0$ which satisfies $f(Q|x, \alpha) = 0$. Viewing the frontier model this way is useful because it shows that the firm's choice of inputs determines the position of the mass of the distribution. The different ways the various inputs and production techniques shift the distribution's mass, together with the form of the firm's objective function, determine the benefits the firm obtains from the inputs. For example, an increase in an input could shift the entire mass in the positive direction (see Figure 1a); other inputs may have little effect on the frontier while shifting the mass rightward towards the frontier (see Figure 1b); still other inputs may shift the frontier rightward without greatly altering the position of the mass (see Figure 1c).

The frontier interpretation of the model has several interesting implications. First, frontier output distributions will not generally be symmetric distributions. Clearly, therefore, odd moments of the third and possibly higher order are likely to play an important role in determining the shape of the output distribution. For example, in Day's (1965) study of yield distributions of field crops, it was found that as fertilizer applications increased the distributions systematically shifted from a positive skew shape to a negative skew. Essentially, the increased fertlizer input shifted much of the mass rightward towards the production frontier, with more and more of the mass "piling up" near the frontier as fertilizer input increased. Secondly, the frontier model suggests that output distributions are not likely to be well approximated by normal or other symmetric distributions; consequently, the mean-variance criterion may be very inadequate in analyzing behavior under technological uncertainty. As an example of how the mean-variance model could lead to exactly the opposite conclusions from a more general expected utility model, consider a case in which an input increase has a small effect on the mean but substantially increases the distribution's variance and skewness. In this case the mean-variance criterion could indicate a reduction in expected utility whereas a more general utility model (say, with mean, variance, and skewness) which attached positive utility to skewness could show that the input change would increase utility. As an example of how this might occur, consider a third-order approximation of the utility function. The firm's decision would depend on the mean, variance, and third moment of the output distribution. A risk averse individual has $U_2 < 0$, and if the individual is downside risk averse as defined by Menezes, Geiss, and Tressler (1980) then $U_3 > 0$. Therefore, the decision maker prefers output distributions with a positive skew over those with a negative skew. This issue of skewness is not only of academic interest, as shown by research on the behavior of subsistence-level farmers in developing agricultures. It seems likely that these farmers are very concerned with down-side risk and may choose techniques accordingly; moreover, this attitude may differ substantially from the attitudes of wealthy farmers. The output distribution model could be used to determine the relationships between production inputs and the shape of the alternative technologies available to these farmers, and thus improve our understanding of their technology choices.

We can also relate the output distribution model to stochastic dominance theory. Hadar and Russell (1969) show that if distribution g is preferred to f by all agents with monotonic increasing utility functions, that is if g is greater than f in terms of first order stochastic dominance, all of the odd moments of g about the mean are greater than the odd moments of f. It also follows that all of the positive moments about zero are greater for g than for f.[4] Therefore, first order stochastic dominance implies a set of testable restrictions for the effects of inputs on the output distribution's moments. If an input has a positive effect on all moments, then we know that more of the input is preferred to less in the sense of first order stochastic dominance. However, in the case of second order stochastic dominance (the case of risk averse agents) there is no such systematic relationship between moments and preferred distributions. This does not mean that studying the moments of output distributions is not a worthwhile endeavor, for as we have shown above we can use the moments to approximate distributions, and using the local approximation of the utility function leads to direct interpretations of the moments' effects on firm behavior. In addition, if we were interested in studying second order stochastic dominance, the moments could be used to approximate the distribution function and determine whether the stochastic dominance condition holds.

2. Interpreting Stochastic Production Models as Output Distributions

Two stochastic specifications of production functions have dominated the econometrics literature. These are:

$$Q_k = m(x_k, \beta) + u_k, \; Eu_k = 0, \; Eu_k^i \text{ independent of x for all i} \qquad (1a)$$

$$Qk = m(x_k, \beta)\, u_k, \; Eu_k > 0, \; Eu_k^i \text{ independent of x for all i} \qquad (1b)$$

where $m(x, \beta)$ is a function obeying the usual regularity conditions. The first specification implies that only the mean of the output distribution is a function of the inputs. The second implies that the relationship between the inputs and the ith moment is determined by the parameters β, since:

$$E(Q_k^i) = m(x_k, \beta)^i \, E(u_k^i)$$

Clearly, both of these specifications are unacceptable for a theory of production based on the hypothesis that the moments may be general functions of the inputs and not arbitrarily restricted as in these models.

In order to study the variance of output as a function of the inputs, Just and Pope (1978) specified a model which does not restrict the relationship between inputs and variance as model (1b) does. Their model is:

$$\begin{aligned} Qk &= m(x_k, \beta) + h(x_k, \gamma)\varepsilon_k, \ \varepsilon_k \text{ i.i.d.} \qquad (2) \\ &= m(x_k, \beta) + u_k \end{aligned}$$

Just and Pope show that this model allows inputs to have separate effects on the mean and variance of output. They propose an estimation algorithm which involves: (i) least squares regression of Q_k, on $m(x_k, \beta)$ to produce consistent (but inefficient) estimates of β; (ii) linear regression or the log residual $\ln |\hat{u}_k|$ on $\ln h(x_k, \gamma)$ to produce consistent, asymptotically normal estimates of γ [$\ln h(x_k, \gamma)$ is assumed to be linear in the paramters]; reestimation of β using weighted least squares, with $h(x_k, \hat{\gamma})^{-.5}$ as weights to obtain asymptotically efficient estimates; an interative estimation algorithm to obtain asymptotically efficient estimates of γ.

Although the Just and Pope model is somewhat more general than the models in (1), I will now show that their specification imposes generally undesirable restrictions on the second and higher order moments of the output distribution, just as model (1b) imposes such restrictions on all of the moments. To show this we simply note that, from (2),

$$u_k^i = h(x_k, \gamma)^i \varepsilon_k^i$$

so that

$$E(u_k^i) = h(x_k, \gamma)^i E(\varepsilon_k^i)$$

Clearly, then, for i, j $\geq$ 2 the parameters of the ith moment are directly related to the parameters of the jth moment; if ln $h(x_k, \gamma)$ is linear in the parameters then the parameters of the jth moment are equal to j/i times the parameters of the ith moment.[5] It is also clear that if the ith moment of ε_k exists, then the ith moment of Q_k exists and is a function of x_k with these restrictive properties.

If we interpret the Just and Pope criticism of the standard models in (1) as saying that the effects of inputs on the second moment should not be constrained to be zero or to have the same sign as their effects on the first moment, then the consideration of higher order moments suggests the following generalization of their principle: the effects of inputs on each moment of the output distribution should not necessarily be determined by their effects on lower order moments. Each of the models discussed above fails to satisfy this "principle of the output distribution." In the following section I propose an alternative model which does satisfy this principle, and then discuss estimation procedures for it.

3. Specification and Estimation of Output Distributions

In this section I develop estimators for the parameters of the following linear moment model:

$$Q_k = m(x_k, \beta) + u_k, \; Eu_k = 0, \; Eu_k u_{k'} = 0, \; k \neq k' \tag{3}$$

$$u_k^i = x_k \gamma_i + v_{ik}, \; Ev_{ik} = 0, \; i \geq 2. \tag{4a}$$

Note that (4a) implies :

$$Eu_k^i = \mu_{ik} = x_k \gamma_i, \; E(v_{ik}^2) = \mu_{2i,k} - \mu_{ik}^2, \; E(V_{ik} V_{ik'}) = 0, \; k \neq k' \tag{4b}$$

This model clearly satisfies the output distribution principle defined above, as there are no a priori restrictions on the γ_i across moments. For the asymptotic results derived below we let X be the (N x k) matrix of the x_k, u^i and v_i are (N x 1) vectors of the u_k^i and v_{ik}, and we assume that X'X/N converges to a positive nonsingular matrix as N approaches infinity and plim $X'u_i/N = 0$. Under the conditions in (3) and other standard assumptions, least squares estimation [either linear or nonlinear, depending on the function $m(x, \beta)$] produces a consistent estimate $\hat{\beta}$ of β (Malinvaud, 1970). We then have the residuals:

$$\hat{u}_k = Q_k - \hat{Q}_k = m(x_k, \beta) + u_k - m(x_k, \hat{\beta}) \tag{5}$$

Using the binomial theorem, (5) implies:

$$\hat{u}_k^i = [u_k + m(x_k, \beta) - m(x_k, \hat{\beta})]^i = \sum_{j=0}^{i} \binom{i}{j} [u_k]^{i-j} [m(x_k, \beta) - m(x_k, \hat{\beta})]^j$$

$$= u_k^i + \sum_{j=1}^{i} \binom{i}{j} [u_k]^{i-j} [m(x_k, \beta) - m(x_k, \hat{\beta})]^j \tag{6}$$

From (6) it is clear that $E\hat{u}_k^i \neq \mu_{ik}$. However, in the limit the bias vanishes because plim $\hat{\beta} = \beta$, and using a well known limit theorem (Rao 1971, pp. 120-124) it follows that:

$$\text{plim } \hat{u}_k^i = [u_k + m(x_k,\beta) - \text{plim } m(x_k,\hat{\beta})]^i = u_k^i \tag{7}$$

Therefore, $\hat{u}_k^i$ converges in distribution to u_k^i (Dhrymes 1974, p. 93), that is:

$$\hat{u}_k^i \xrightarrow{\text{i.d.}} u_k^i \tag{8}$$

Next I construct the estimator:

$$\gamma_i^* = (X'X)^{-1} X'u^i = \gamma_i + (X'X)^{-1} X'v_i \tag{9}$$

which is obtained by regressing u_k^i on $x_k\gamma_i$. It follows from (8) that:

$$\hat{\gamma}_i = (X'X)^{-1} X'\hat{u}^i \xrightarrow{\text{i.d.}} \gamma_i^* \tag{10}$$

and hence

$$\text{plim } \gamma_i^* = \gamma_i = \text{plim } \hat{\gamma}_i \tag{11}$$

Using (11) and (4) we have:

$$\text{plim } x_k\hat{\gamma}_{2i} - \text{plim } (x_k\hat{\gamma}_i)^2 = \mu_{2i,k} - \mu_{ik}^2 \tag{12}$$

Now define:[6]

$$\Omega_i = \text{diag } [\mu_{2i,k} - \mu_{ik}^2] \text{ and } \hat{\Omega}_i = \text{diag } [x_k\hat{\gamma}_{2i} - (x_k\hat{\gamma}_i)^2], \tag{13}$$

$$k = 1, \ldots, N$$

We are now prepared to prove the following Theorem:

<u>Theorem 2</u>: Suppose $\lim_{N \to \infty} \frac{X'\Omega_i^{-1}X}{N}$ is finite and nonsingular and that μ_{ik} and

$\mu_{2i,k}$ exist for $i = 2, \ldots, M$. Then the feasible generalized least squares estimator:

$$\hat{\gamma}_i^{GLS} = (X'\hat{\Omega}_i^{-1}X)^{-1}X'\hat{\Omega}_i^{-1}u^i, \quad i = 2, \ldots, M$$

has the same asymptotic distribution as the Aitken estimator:

$$\gamma_i^{GLS} = (X'\Omega_i^{-1}X)^{-1}X'\Omega_i^{-1}u^i, \quad i = 2, \ldots, M$$

Proof: It is sufficient to show that:

$$\text{plim}\,\frac{X'\hat{\Omega}_i^{-1}X}{N} = \text{plim}\,\frac{X'\Omega_i^{-1}X}{N} \tag{14}$$

and

$$\text{plim}\,\frac{X'\hat{\Omega}_i^{-1}\hat{u}^i}{\sqrt{N}} = \text{plim}\,\frac{X'\Omega_i^{-1}u^i}{\sqrt{N}} \tag{15}$$

Equations (12) and (13) show that (14) is satisfied for each element. (15) holds by the fact that the elements of $\hat{\Omega}_i$ converge in probability to nonstochastic term, and by the fact that $\hat{u}^i$ converges in distribution to u^i. Therefore, each element of $\hat{\Omega}_i^{-1}\hat{u}^i$ converges in distribution to $\Omega_i^{-1}u^i$ (Rao 1971, pp. 116-124). Thus (14) and (15) hold and we can assert that:

$$\text{plim}\,\sqrt{N}(\hat{\gamma}_i^{GLS} - \gamma_i^{GLS}) = 0 \text{ and hence } \hat{\gamma}_i^{GLS} \xrightarrow{i.d.} \gamma_i^{GLS}. \quad \text{Q.E.D.}$$

Although the estimator $\hat{\gamma}_i^{GLS}$ converges in distribution to γ_i^{GLS}, it is not a fully efficient estimator because the errors v_{ik} are correlated across equations. To see this, note that:

A GLS estimator for the parameter vector β can also be devised. Let Q be the (N x 1) vector of the Q_k, let:

$$\Sigma_1 = \text{diag}\,[\mu_{2k}], \quad k = 1,\ldots,N$$

and define $\hat{\Sigma}_1$ accordingly. By the same arguments as in the above proof we have the following result:

<u>Theorem 3</u>: Under the conditions of Theorem 2 the feasible GLS estimator:

$$\hat{\beta}^{GLS} = (X'\hat{\Sigma}_1^{-1}X)^{-1}X'\hat{\Sigma}_1^{-1}Q$$

has the same asymptotic distribution as the Aitken estimator:

$$\beta^{GLS} = (X'\Sigma_1^{-1}X)^{-1}X'\Sigma_1^{-1}Q.$$

In addition to the heteroscedasticity of the errors in the moment functions, it is easy to show that the errors are contemporaneously correlated across equations. Therefore, efficiency may be gained by utilizing a multiple equation GLS estimator. In the discussions to this point the same input matrix X has been used in each moment equation, although this need not be the case. For example, restrictions on the parameters of the moment functions would essentially break the identical regressor situation. Therefore, I shall now consider the joint GLS estimation problem under the assumption that the regressors are not identical across moments; the identical regressor situation is obviously a special case of this model.

The cross equation correlations are given as follows:

$$\begin{aligned} E(u_k v_{ik}') = E(u_k^i u_k' - u_k x_k' \gamma_i) &= \mu_k^{i+1}, \quad k = k' \\ &= 0 \quad\quad\;, \quad k \neq k' \end{aligned}$$

$$E(v_{ik}v_{jk}) = E(u_k^i - x_k\gamma_i)(u_k^j - x_k\gamma_j) = \mu_{i+j,k} - \mu_{ik}\mu_{jk}, \text{ all } i,j$$

$$E(v_{ik}v_{jk'}) = 0, \ k \neq k'$$

Now let

$$\Sigma_i = \text{diag}[\mu_k^{i+1}], \ i = 1,\dots,M$$

$$\Omega_{ij} = \text{diag}[\mu_{i+j,k} - \mu_{ik}\mu_{jk}], \ i,j = 2,\dots,M$$

$$\hat{\Sigma}_i = \text{diag}[x_k\hat{\gamma}_{i+1}], \ i = 1,\dots,M$$

$$\hat{\Omega}_{ij} = \text{diag}[x_k\hat{\gamma}_{i+j} - x_k\hat{\gamma}_i x_k\hat{\gamma}_j], \ i,j = 2,\dots,M$$

and let

$$\Omega = \begin{bmatrix} \Sigma_1 & \cdot & \cdot & \cdot & \cdot & \Sigma_M \\ \cdot & \Omega_{22} & \cdot & \cdot & \cdot & \Omega_{2M} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \Sigma_M & \Omega_{M2} & \cdot & \cdot & \cdot & \Omega_{MM} \end{bmatrix}$$

and $\hat{\Omega}$ is defined accordingly. Finally, define X_i as the matrix of regressors for the ith moment function, and define:

$$X^* = \text{diag}[X_i]$$

$$y' = (Q, u^2,\dots,u^M), \ \hat{y}^1 = (Q,\hat{u}^2,\dots,\hat{u}^M)$$

$$\delta' = (\beta, \gamma_2,\dots,\gamma_M)$$

$$\hat{\delta}^{GLS} = (X^{*\prime}\hat{\Omega}^{-1}X^*)^{-1}X^*\hat{\Omega}^{-1}\hat{y}$$

The following theorem is a direct generalization of Theorem 2:

<u>Theorem 4</u>: Let $\Omega^{-1} = [\Omega^{ij}]$, and assume $\lim X_i'\Omega^{ij}X_j/N$ is finite and nonsingular for $i,j = 1,\ldots,M$. Then $\hat{\delta}^{GLS}$ converges in distribution to the joint GLS estimator:

$$\delta^{GLS} = (X^{*\prime}\Omega^{-1}X^{*})^{-1}X^{*\prime}\Omega^{-1}y.$$

<u>Proof</u>: By (11) we know that the elements of $\hat{\Omega}^{-1}$ converge in probability to the elements of Ω^{-1}. Therefore, as in Theorem 2 we have that:

$$\text{plim } X^{*\prime}\hat{\Omega}^{-1}X^{*}/N = \lim X^{*\prime}\Omega^{-1}X^{*}/N$$

$$\text{plim } X^{*\prime}\hat{\Omega}^{-1}\hat{y}/\sqrt{N} = \text{plim } x^{*\prime}\Omega^{-1}y/\sqrt{N}$$

and it follows that $\hat{\delta}^{GLS}$ $\xrightarrow{i.d.}$ δ^{GLS}. Q.E.D.

Based on these results, an estimation algorithm for the moments of the output distribution proceeds as follows:

(i) use least squares to obtain unbiased or consistent estimates of β;

(ii) use least squares to obtain consistent estimates of the moment function parameters γ_i which are of interest to the analysis;

(iii) use the $\hat{\gamma}_i$ to form estimators of the covariance matrixes;

(iv) use the estimated covariance matrixes in GLS regressions.

4. Applications and Extensions of the Linear Moment Model

The econometric models in equations (3) and (4) and the proposed estimation techniques have several important advantages over the models and methods used in the literature. One major advantage is that only one observation per individual is required when cross section data is used, or only one observation per time period is needed in the case of time series data. Repetitions for each individual are not required as in the "method of moments" estimation approach used by Day (1965). Another advantage is that standard test statistics can be employed to test hypotheses about the effects of inputs on the moment functions. A third advantage is that these linear functions and associated test statistics can be computed with readily available software at a low cost.

In terms of approximating the shape of the distribution, at least two approaches are available. One is use of a polynomial approximation as discussed in Section 1; however, it is not clear whether this approach would produce useful approximations. An attractive method would appear to be to use the first four moment functions in combination with a Pearson-type distribution, as a means of achieving an approximation. Day (1965) found Pearson's Type-I distribution performed well and it would appear to be suitable for other agricultural or nonagricultural applications.

Recalling the interpretation of the output distribution model as a frontier production model, it is interesting to note that the production frontier is obtained as a by-product of the distribution approximation. It is quite remarkable to observe that production frontiers are apparent in the distributions Day studied, although he did not discuss the frontier production function interpretation. As an empirical technique for estimation of frontier production models it is also worth noting that because the linear moment model can be estimated with standard linear techniques, it may be an attractive alternative to the "stochastic frontier production function" proposed by Aigner, Lovell, and Schmidt (1978). The stochastic frontier model requires the use of nonlinear optimization techniques to obtain maximum likelihood estimates of the parameters. Furthermore, by using the output distribution model it is not necessary to make the stochastic frontier assumption that was introduced largely to facilitate the use of the maximum likelihood procedure. Although recent research has introduced distributions into the stochastic frontier framework which are more plausible than the half-normal model of Aigner, Lovell and Schmidt (see the papers in Annals of Applied Econometrics, 1980), a more flexible approach might well be the use of approximations based on the estimated moment function introduced in this paper. Future research will investigate this possibility.

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Footnotes

1. The concept of the output distribution is, of course, implicit in any stochastic production function model. However, most specifications are designed to measure mean productivity and typically ignore the possibility that other moments may be functions of the inputs. The first explicit recognition of the broader implications of the output distribution concept appears to be Day's (1965) seminal study of yield distributions in agriculture. Assuming the yield distributions were in the class of Pearson distributions, Day found standard measures of skewness and kurtosis to be functions of fertilizer input. Also Anderson has studied the relationship of output distribution moments to input use (see Anderson, Dillon, and Hardaker, 1977, Ch. 6), and Just and Pope (1978) have developed a two moment production model and discussed its economic interpretation.

2. Of course this approach presumes that the moments of the output distribution exist. If they do not exist, all of the standard econometric formulations of production models would be misspecified.

3. In the "usual" expected utility model Q is written as $Q = m(x)u$ or $Q = m(x) + u$, where $m(x)$ is a deterministic part and u is a stochastic part independent of x. Then the optimality condition is expressed as:

$$\frac{\partial EU(\pi)}{\partial x} = \int U' \frac{\partial \pi}{\partial x} f(u)du = 0$$

Using this model we could perform a change of variable and rewrite $EU(\pi)$ as:

$$EU(\pi) = \int U(\pi)\, f(Q|x, \alpha)dQ$$

Then the optimality condition becomes:

$$\frac{\partial EU(\pi)}{\partial x} = \int U' \frac{\partial \pi}{\partial x} f(Q|x, \alpha)dQ + \int U(\pi) \frac{\partial f}{\partial x} dQ = 0$$

as shown in the text.

4. Note that we must have $\partial^2 EU/\partial x^2 < 0$ to ensure concavity of the objective function. When f(Q) depends on x, concavity of the utility function is not sufficient to ensure this condition. It can be shown that $\partial^2 f/\partial x^2 < 0$ and $\int U' \frac{\partial f}{\partial x} dQ > 0$ are sufficient conditions for concavity; only the former condition is needed if the firm is risk neutral.

5. Hadar and Russell show that first order stochastic dominance implies g dominates f if:

$$\int u(v)\, g(v)\, dv \geq \int u(v)\, f(v)\, dv,$$

where $u' > 0$. Since u is a monotonic increasing function for odd moments about the mean (or zero) and even moments about zero, these moments are all greater for g than for f.

6. Due to this condition, it follows that the standard measures of skewness and kurtosis, $S = \mu_3/\mu_2^{3/2}$ and $K = \mu_4/\mu_2^2$, are <u>independent</u> of the inputs, regardless of the form of the $h(x, \gamma)$ function.

7. Diag $[Z_{kk}]$, K = 1, . . ., N, is the (N x N) diagonal matrix with k^{th} diagonal element Z_{kk}.

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