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MULTIPLE OPTIMAL SOLUTIONS IN LINEAR PROGRAMMING MODELS

by

Q. Paris

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ABSTRACT

MULTIPLE OPTIMAL SOLUTIONS IN LINEAR PROGRAMMING MODELS

Quirino Paris

Large scale linear programming models of empirical problems are likely to exhibit degeneracy of either primal and/or dual solutions. In these events multiple optimal solutions occur. In the empirical literature, however, no paper can be found discussing this problem which is especially crucial when the study intends to make efficiency judgments. The presence of multiple optimal solutions brings forth explicitly the problem of validating linear programming models. In this connection, two alternative procedures are suggested, based upon the criteria of minimizing either the sum of least-squares or the sum of absolute deviations between the actual levels of the activities and those revealed optimal by the linear programming model. In this way, the empirical analysis of economic reality based upon a linear programming framework acquires a positive connotation.

Key Words: multiple optimal solutions, linear programming, least-squares, model validation, positive analysis.

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MULTIPLE OPTIMAL SOLUTIONS IN LINEAR PROGRAMMING MODELS

During the past 30 years, empirical studies cast in the framework of linear programs (LP) have all neglected the consequences of one important aspect of mathematical programming. Simply stated, the polyhedral nature of the solution set in LP models may be responsible for the existence of multiple optimal solutions if some plausible conditions are realized. If and when empirical problems possess alternative optimal solutions, it is natural to ask why only one of them is usually selected for presentation in final reports which--often--make also efficiency judgements and prescribe sweeping policy changes.

The conditions under which multiple primal and dual optimal solutions can occur in linear programming problems[1/] are related to the phenomenon of degeneracy, rather plausible in empirical studies. Intuitively, degeneracy of the primal solution surfaces when one activity happens to employ inputs in exactly that proportion which exhausts completely two or more available resources. Analogously, degeneracy of the dual solution is encountered when given an optimal plan, the accounting loss for some nonbasic activity happens to be the same (that is, zero) as that of the activities included in the optimal plan. Hence, the likelihood of either primal or dual degeneracy increases rapidly with the size of the model. Baumol (p. 315) asserts that "computational experience indicates that such cases (primal and dual degeneracy) are encountered more frequently than might be expected in advance." In view of this fact, it would appear that a correct and informative report of empirical results generated by LP should include complete information about the problem's size and an explicit statement of whether the primal and dual optimal solution presented are indeed unique.

Unfortunately, an exhaustive search of the empirical literature has revealed that a majority of papers fails to disclose even the number of columns and rows which constitute the matrix of constraints. No paper was found to mention whether the reported solution is unique.

The discussion of multiple optimal solutions brings forth in an explicit way the problem of validating linear programming problems. Of course, validation of mathematical programming models ought to be performed also when their solution is unique. As argued in this paper, while validation of these models has been overlooked very often in the past, it is an unavoidable step when multiple optimal solutions are present. The manner in which validation is carried out suggests also that the normative versus positive dichotomy with which mathematical programming and econometric models are often set apart becomes irrelevant and meaningless.

In this paper, the problem of multiple optimal solutions in LP models is addressed by first reviewing a theorem associated with it; secondly, by discussing two numerical examples; thirdly, by analyzing the consequences of multiple optimal solutions upon relations derived by parametric linear programming; fourthly, by reviewing some empirical studies using the LP framework which are likely to admit multiple optimal solutions; and finally, by proposing alternative methodologies for dealing with the problem.

A Theorem of Linear Programming

Consider the following LP problem:

(1) $\underset{x}{\text{maximize}}\ c'x$

subject to

(2) $Ax \leq b, \quad x \geq 0$

where A is an m x n matrix of known coefficients and the other elements of the problem are conformable to it. It is well known that, if the system of inequalities (2) is consistent, the set of feasible solutions corresponds to a convex polyhedron, that is, a geometric configuration with facets, edges and vertices (or extreme points). If, furthermore, the maximization operation places the objective function (1) parallel to (and hence, coincident with) one of the facets (or edges) of the feasible solution set, then, the problem possesses multiple primal optimal solutions. Figure 1 illustrates these propositions for a simple, two activity model. The extreme points $\bar{x}$ and $\hat{x}$ represent two alternate optimal solutions and the line segment connecting the two solutions constitutes an edge of the convex set of feasible solutions. Each point of this line segment, in turn, constitutes an alternate optimal solution. An analogous figure could be drawn to represent the multiplicity of dual optimal solutions. The above discussion can be cast in more rigorous terms by the following theorem:

Theorem 1. The objective function of a linear programming problem assumes its optimum (maximum for problem (1) and (2)) at an extreme point of the convex set of feasible solutions.

If it assumes its optimum at more than one extreme point, then it takes on the same optimum value for every convex combination of those extreme points.

Proof of this theorem can be found in any linear programming textbook. The multiplicity of optimal solutions does not concern only the primal side of a linear programming problem. In general, it is likely encountered also in the dual solution.

Illustration of Alternate Optimal Solutions

The discussion of three numerical examples illustrates the occurrence of primal and dual optimal solutions.

Example 1.

$$\max g = (53/22)x_1 + (3/2)x_2 + 5x_3 + 2x_4 + x_5$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 + x_3 + 4x_4 - 2x_5 &\leq 6 \\ 2x_1 + 3/4x_2 + 5x_3 + x_4 + x_5 &\leq 4 \\ x_1 + 3x_2 - 2x_3 + 4x_4 &\leq 0 \\ x_i \geq 0,\ i=1,\dots,5. \end{aligned}$$

This LP example possesses three extreme point primal optimal solutions. They are

Primal Solutions

	1	2	3
x_1	0	8/9	0
x_2	0	0	16/33
x_3	8/11	4/9	8/11
x_4	4/11	0	0
x_5	0	0	0

The dual optimal solution is unique and corresponds to $y_1 = 0$, $y_2 = 12/11$, $y_3 = 5/22$. The optimal value of the objective function is $g = 48/11$.

According to theorem 1, the three optimal solutions can be combined in convex combinations to give an infinite number of optimal solutions. By arbitrarily choosing weights $w_1 = 1/4$, $w_2 = 1/2$ and $w_3 = 1/4$, for example, we can obtain the additional optimal solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ 8/11 \\ 4/11 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 8/9 \\ 0 \\ 4/9 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 \\ 16/33 \\ 8/11 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/9 \\ 4/33 \\ 16/33 \\ 1/11 \\ 0 \end{bmatrix} \geq 0.$$

The peculiarity of this optimal solution is that it exhibits four nonzero components while there are only three constraints. Hence, the generally held idea that in a LP problem the optimal nonzero activity levels cannot exceed the number of constraints must be confined to those LP problems which possess unique primal solutions.

Example 2. The following example illustrates the possibility of multiple dual optimal solutions:

$$\max r = 4x_1 + 1x_2 + 5x_3$$

$$\text{subject to} \quad \begin{aligned} 3x_1 + 2x_2 - x_3 &\leq 2 \\ -x_1 + x_2 + 5x_3 &\leq 4 \\ x_1 + x_3 &\leq 2 \\ x_i \geq 0,\ i=1,2,3. \end{aligned}$$

The unique primal optimal solution is $x' = (x_1 = 1,\ x_2 = 0,\ x_3 = 1)$. The optimal value of r is 9. There are two extreme point dual optimal solutions. The first one is $y_1' = (y_{11} = 5/16,\ y_{12} = 3/8,\ y_{13} = 55/16)$ while the second one is $y_2' = (y_{21} = 25/14,\ y_{22} = 19/14,\ y_{23} = 0)$. Again, one can easily verify that any convex combination of the two solutions y_1 and y_2 is also a solution of the problem. As an illustration of this point let $w_1 = 1/5$ and $w_2 = 4/5$. Then a vector y constructed as $y = w_1y_1 + w_2y_2$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5/16 \\ 3/8 \\ 55/16 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} 25/14 \\ 19/14 \\ 0 \end{bmatrix} = \begin{bmatrix} 167/112 \\ 65/56 \\ 11/16 \end{bmatrix} \geq 0.$$

is also a dual optimal solution because it is nonnegative, feasible, and generates a value of the dual objective function equal to nine which corresponds to the optimal value of the primal objective function.

Example 3. The third example of multiple optimal solutions is taken from Hadley (p. 15) and is remarkable for the fact that although it exhibits only one optimal extreme point, the number of optimal solutions is nonetheless infinite:

$$\max \quad g = -x_1 + 2x_2$$

$$\text{subject to} \qquad x_1 - x_2 \geq -1$$

$$-(1/2)x_1 + x_2 \leq 2$$

$$x_1 \geq 0_1\ x_2 \geq 0.$$

Figure 2 reproduces this problem graphically. It is easy to verify that $(x_1 = 2,\ x_2 = 3)$ constitutes the unique extreme point optimal solution but also that any $(x_1,\ x_2)$ combination satisfying the relation $2x_2 - x_1 = 4$, constitutes an optimal solution. To exemplify: (2,3), (4,4), (8,6), (14,9), etc. are all (x_1,x_2) optimal solutions.

Parametric LP Analysis with Multiple Optimal Solutions

The existence of multiple optimal solutions affects the results of the conventional analysis of parametric programming. It is well known that within the framework of activity analysis, the parametric variation of the resource constraint vector traces a step function interpreted as a marginal productivity schedule. The parametric variation of the objective function coefficients (assumed to be prices of final products), traces step supply schedules of those products. If the LP problem has a unique solution, the supply schedule for the jth activity is discontinuous and looks as in Figure 3. In this case, the optimal quantities of x_j, at the various prices, are 0, x_{1j}, x_{2j}, and x_{3j}. The quantities falling in between zero and x_{1j}, x_{1j} and x_{2j}, x_{2j} and x_{3j}, are inaccessible to the entrepreneur who wishes to follow the optimal prescriptions of the linear programming model.

On the contrary, when multiple optimal solutions occur, those segments of the supply schedule falling between the optimal quantities of extreme point solutions become accessible via convex combinations. To illustrate this point, parametric analysis was carried out for activity 3 of example 1. The results are graphically reproduced in Figure 4. In this case, the supply schedule is continuous except for the quantities of x_3 falling in the open interval (0, 4/9). The two configurations of supply schedules obviously make a difference when researchers intend to infer further behavior of economic agents from parametric analysis. Some authors in the field of resource economics for example, have used parametric techniques to define step demand schedules of limiting resources to be used subsequently, in a quadratic programming formulation of an optimal control problem. Hence, the choice between continuous and discontinuous schedules will both affect the quantitative results and their interpretation.

<u>Necessary and Sufficient Conditions for Unique Optimal Solutions</u>

The numerical examples have provided some evidence about why and how multiple optimal solutions occur in linear programming problems. The results can be summarized as follows: (a) multiple primal optimal solutions occur whenever the dual solution is degenerate, that is when the objective function coefficient of a nonbasic activity (expressed in terms of the activities in the optimal basis) takes on a zero value; (b) multiple dual optimal solutions occur whenever the primal solution is degenerate, that is when some of the primal activities are present in the optimal basis at zero level.

To state these conclusions more formally it is convenient to rewrite problem (1) and (2) in partitioned form as follows:

(3) $\max g = c_B'x_B + c_{NB}'x_{NB}$

subject to

(4) $Bx_B + A_{NB}x_{NB} \leq b$

$x_B \geq 0, \; x_{NB} \geq 0$

where B is assumed to be an optimal basis. Hence, the subscript (B) indicates basic activities while the subscript (NB) denotes nonbasic activities. Following the assumption that B is an optimal basis, the original and final simplex tableaux are

$$
\begin{array}{ll}
 & \begin{array}{ccccc} g & x_B' & x_{NB}' & s' & \end{array} \\
\begin{array}{l}\text{Original}\\\text{Simplex}\\\text{Tableau}\end{array} & \begin{bmatrix} 1 & -c_B' & c_{NB}' & 0 & 0 \\ 0 & B & A_{NB} & I & b \end{bmatrix} \\
\begin{array}{l}\text{Optimal}\\\text{Simplex}\\\text{Tableau}\end{array} & \begin{bmatrix} 1 & 0 & (c_B'B^{-1}A_{NB}-c_{NB}') & c_B'B^{-1} & c_B'B^{-1}b \\ 0 & I & B^{-1}A_{NB} & B^{-1} & B^{-1}b \end{bmatrix}
\end{array}
$$

The components of the optimal tableau can be easily recognized as

$\bar{g} = c_B'B^{-1}b$, the optimal value of the objective function,

$\bar{x}_B = B^{-1}b$, the primal optimal solution,

$\begin{bmatrix} c_B'B^{-1} \\ c_B'B^{-1}A_{NB}-c_{NB}' \end{bmatrix}$ the reduced cost coefficients.

Notice that $y = B'^{-1}c_B$ is the dual solution vector. Finally, let us denote the first components of the reduced cost vector as z, that is define z as the partition vector $z' = (c_B'B^{-1}, c_B'B^{-1}A_{NB})$, as it is customary in LP textbooks. Then, we can readily observe that when any of the quantities $(z_j - c_{NBj})$, for j not in B, the (set of basic activities), is equal to zero, the LP problem possesses multiple primal optimal solutions.

If, on the other hand, any component of the primal optimal solution, $\bar{x}_B = B^{-1}b$, is equal to zero, the LP problem possesses multiple dual optimal solutions.

It is possible to conclude, therefore, with the observation that to primal degeneracy there corresponds multiplicity of dual optimal solutions, while to dual degeneracy there corresponds multiplicity of primal optimal solutions.

We can now state the following

Proposition. Given a consistent LP (maximization) problem, the necessary and sufficient conditions for the existence of unique primal and dual optimal solutions are

$\bar{x}_{Bi} > 0$, for all indexes i in the optimal basis

$z_j - c_{NBj} > 0$, for all indexes j of nonbasic activities.

Conversely, when an optimal basis B exists and some $z_j - c_{NBj} = 0$, for j not in B, one of the following two situations occurs: either it is possible to replace a basic activity with the jth one and obtain an alternate extreme point optimal solution, or this is not so (because no positive pivot is found in the jth column) and, therefore, we encounter the situation illustrated by example 3 and Figure 2.

Multiple Optimal Solutions and Empirical Studies

No empirical study using LP as the analytical framework was found to discuss the problem of uniqueness (or multiplicity) of the optimal solutions reported. In fact, only a few papers provide scant information about the size of the model. The preceding discussion, on the other hand, leaves no doubt about the crucial role of the problem's size. As the size increases, the polyhedral solution set acquires more facets, edges, and extreme points. This fact increases the possibility that the objective function's hyperplane will have a slope parallel to some facet (edge) of the solution set located in the direction of optimization. Furthermore, the more constraints defining the problem, the more likely the primal optimal solution will be degenerate which, in turn, opens the way to multiple dual optimal solutions.

In a practical, empirical sense, however, it is important to be more complete in reporting the structure of the problem and its solutions. When dealing with information which is not exact (like input-output coefficients, prices, input quantities), the notion of alternate "almost" optimal solution becomes relevant. We have seen that an alternate primal optimal solution exists if any nonbasic activity is associated with a zero reduced cost, that is, if $z_j - c_{NBj} = 0$ for some $j \notin B$. Similarly, an alternate dual optimal solution occurs if any basic variable is equal to zero, that is, if $\bar{x}_{Bj} = 0$ for any j, j=1, ... , m, in the optimal basis.

But what if $0 < z_j - c_{NBj} \leq \varepsilon$ or $0 < \bar{x}_{Bj} \leq \delta$, where both ε and δ are arbitrarily small positive real numbers? In this case it would seem appropriate to conclude that the problem possesses "alternate almost optimal solutions". In fact, a small change in the coefficient of the objective function, c_{NBj}, and of some component of the constraint vector b would generate exact alternate optimal solutions. This amounts to use the perturbation technique so often quoted in the literature for generating multiple solutions rather than for avoiding their complications.

The problem of multiple optimal solutions in linear programming becomes then analogous to that of multicollinearity in econometrics. It is well known that exact linear dependence of regressors produces multiple estimates (solutions) of regression coefficients. The difficulties of parameter estimation and hypothesis testing, however, are even more pronounced when multicollinearity (or almost exact linear dependence among regressors) is present. As there are various suggestions for detecting and reporting the presence of a multicollinearity and its consequences in regression analysis, the thoughtful and correct researcher using LP ought to detect and report the presence of "alternate almost optimal solutions." To leave this information out of an LP report can be likened to omitting standard errors of the estimates and t ratios in an econometric study. Considering the widespread use of LP models for making "efficiency judgements" and formulating "efficient policies", the cost of ignoring the existence of multiple optimal solutions and their consequences may be high.

Consider any LP regional study of production location so abundant in the literature. The typical procedure is to compute <u>one</u> optimal solution of the problem and then attempt a rationalization of why producers (in the aggregate) do not behave as indicated by the reported LP program. Sometimes the discrepancy is so large that the researcher does not hesitate to point out "<u>obvious inefficiencies of production location</u>." But, if there are alternate optimal or quasi-optimal solutions, this type of judgement is unwarranted. The noncomputed and unreported optimal solutions may indeed be closer to the actual regional production pattern. Furthermore, a convex combination of either some or all optimal and quasi-optimal solutions may be even closer than any individual one.

Consider now the case of multiple dual optimal solutions in which the dual variables are interpreted as ~~inputed~~ imputed prices of resources. In general, any one of such solutions is quite different from the vector of market prices of the same resources. However, a convex combination of the dual solutions, judiciously constructed, might be sufficiently close to those market prices.

In summary, in order to approach the problem of multiple optimal solutions in a sensible and informative way, it is necessary (a) to be aware of it, (b) to check for the existence of alternate primal and dual optimal solutions and, finally, (c) to report them all, together with the almost alternate optimal solutions.

To give the reader some flavor of the potential problem of multiple optimal solutions we present a brief survey of papers dealing with "large" LP problems. These papers were selected mainly because they present some information about the structure and size of the problems described in them.

In 1964 Heady and Egbert published a paper on the spatial allocation of crop production in the United States which was destined to become the model for studies of this type. The analytical framework is a LP model. The number of structural activities (excluding slacks) is clearly indicated to be 710, while the number of constraints is either 127 or 128, depending on the various versions of the model. There are 122 producing regions and either 5 or 6 demand constraints. Hence, the simplex tableau exhibits 127(128) rows and 832 columns (710 activities plus 122 slacks. The other 5 constraints are equalities). Three variants of the basic model were solved. The authors do not mention the problem of possible alternate primal and/or dual optimal solutions, whereas extensive efforts are directed towards rationalizing the results obtained vis-a-vis the contrasting pattern of the traditional production location and levels. We conjecture that with a model of dimensions 127 x 832, the likelihood of alternate primal and/or dual solutions is high.

Some agricultural economists have shown little reservation about constructing and solving gigantic LP models. Heady and Dvoskin must surely be affiliated with that group. Their paper summarizes the research efforts by Dvoskin and Heady and by Dvoskin, Heady and English, in constructing a national programming model for analyzing various scenarios of energy policy. Suprisingly, in their paper and technical reports they never explicitly state what kind of programming model they did construct (linear, quadratic, nonlinear), although, from their presentation it seems fair to conclude that they dealt with a linear programming model. Its size is staggering: 938 constraints and 12,000 activities. Yet, no assurance is given that the reported primal and dual solutions, associated to the various runs of the model are unique. In the absence of an explicit statement, it is hard to believe that this model has no alternate optimal solution.

Stretching the credibility of solution uniqueness is the even larger analysis of Taylor and Frohberg. Their LP model, constructed to analyze welfare effects of erosion and pesticide control in the Corn Belt comprises 14,821 columns (excluding slacks). Unfortunately, while they give (in a footnote) an interesting report on computer time and cost of running this model, they do not say exactly how many constraints there are. In reality, their model was originally a quadratic program (with positive <u>semidefinite</u> quadratic form)[2/] which the authors elected to transform into a LP problem by segmenting each demand function in about 75 steps. (This procedure has been strongly recommended by Duloy and Norton, Hazell and Scandizzo, and Simmons and Pomareda, to whose works, therefore, the conclusions of this paper should also apply.) Of course, Taylor and Frohberg fail to consider the possibility of existence of multiple optimal solutions in their very large LP model.

The same issue of the Journal where Taylor and Frohberg's extraordinary model can be found contains another large LP model by Wade and Heady which deals with nonpoint pollution issues. Unfortunately, the authors do not give the exact information about the problem's size. They state, however, that it is a large scale LP model. No mention is made of solutions' uniqueness.

McCarl, Candler, Doster and Robbins have presented evidence of their extensive experience with LP models. They designed and managed the "Purdue Automatic Cropping Budget Model B9" which is used in conjunction with "Purdue's Top Farmer Workshop". The basic LP model used in assisting farmers in their decisions has 116 constraints and 294 columns (including slacks). The model is adapted to an individual farmer's specifications especially by changing the vector of constraints describing the resource availabilities and, possibly, the coefficients of the objective function. The authors assert that they have personally helped over 4,000 farmers using this model. This means running the model at least 4,000 times with different objective function coefficients and right hand side vectors. With such an impressive computational experience, the burden of declaring that all 4,000 runs have produced unique optimal solutions rests with the authors.

The list of large scale LP models appeared in the published literature could continue without modifying the conclusion: authors do not seem to be aware of the problem of multiple optimal solutions and, therefore, may have completely ignored its consequences.

Dealing with Multiple Optimal Solutions

The obvious question looms on the horizon: how can a LP study be properly concluded when alternate (almost) optimal solutions occur? In other words, how should one choose among all the optimal and quasi-optimal solutions? Remaining in tune with the paper's theme, this question has no unique answer.

Two suggestions will be discussed. They are based, respectively, on the criteria of minimizing either a quadratic or an absolute deviations loss function.

Criterion 1: A Quadratic Loss Function. Suppose problems (1) and (2) represents a regional production location problem and possesses k alternate primal optimal solutions, $k \leq n$. Let P be the matrix whose column vectors are such k extreme point optimal solutions. Let x_A be a vector of activity levels actually operated in the region. Then, recalling theorem 1, the following problem

$$(5) \quad \underset{w}{\text{minimize}} \quad (x_A - Pw)'(x_A - Pw)$$

subject to

$$(6) \quad s'w = 1, \qquad w \geq 0$$

defines a procedure to estimate the weights w in some optimal sense. The components of the vector s are all unitary. The optimality criterion is the least-squares problem of choosing that optimal LP solution, $P\hat{w}$ (a convex combination of all extreme point optimal solutions), which is closest to the activity levels actually operated in the region.

The formulation of problem (5) and (6) is appealing for a number of reasons. First of all, the objective function can be interpreted as a measure of the loss incurred by the region for not producing according to those optimal activity levels which require minimum deviations from the present practices. Secondly, the matrix P is of full column rank since the optimal solutions which define it are associated with extreme points and are, therefore, independent of each other. Hence, the solution vector of weights $\hat{w}$ is unique and, in turn, the projection $P\hat{w}$ constitutes an optimal LP solution which itself is unique in the sense defined by criterion 1. Thirdly, if the vector x_A is considered to be a random sample of activity measurements—as it is usually the case when dealing with econometric studies--it is possible to apply the standard statistical procedures to verify hypotheses of how closely the LP model comes to explain the behavior of economic agents. This suggestion is discussed more fully in the following section.

The formulation of the quadratic loss function can be refined as the complexity of the problem studied requires. For example, if x_A is regarded as a random sample of n commodities in T time periods, it is likely that the n commodities have different variances and, possibly, nonzero covariances. In this case, a generalized least-squares approach will be more appropriate.

A problem analogous to (5) and (6) can be formulated to deal with multiple dual optimal solutions.

Criterion 2: An Absolute Deviations Loss Function. An alternative approach for dealing with the multiple optimal solutions' problem is to minimize a loss function defined as the sum of absolute deviations from the vector x_A

of realized activity levels. The criterion is formally stated as the following linear programming problem:

$$(7)\quad \underset{w,\, d^+,\, d^-}{\text{minimize}} \quad (s_n'd^+ + s_n'd^-)$$

subject to

$$(8)\quad Pw + d^+ - d^- = x_A$$

$$(9)\quad s_k'w = 1$$

$$w \geq 0,\ d^+ \geq 0,\ d^- \geq 0,$$

where d^+ and d^- are vectors of vertical deviations "above" and "below" the optimal LP solution $P\hat{w}$, and s_n, s_k are sum vectors of n and k unitary components, respectively.

One advantage of the minimum sum of absolute deviations criterion is that large deviations will not be given undue importance as contributors to the sum. As it is well known, the opposite is true in the least-squares criterion. To understand clearly this property it is convenient to study the dual problem of (7), (8) and (9), which is

$$(10)\quad \underset{\pi,\, \psi}{\text{maximize}} \quad \{\pi'x_A + \psi\}$$

subject to

$$(11)\quad P'\pi + s_k\psi \leq 0$$

$$(12)\quad -s_n \leq \pi \leq s_n,\ \psi \text{ unrestricted},$$

where π and ψ are dual variables associated with primal constraints (8) and (9), respectively.

Notice that the dual variables π are bounded to be less than or equal to one, in absolute value. The interpretation of this property of the least absolute deviations estimator may be given as follows: any sample observation x_{Ai} will cause an increment (or decrement) of the corresponding sum of absolute deviations which is (in absolute value) less than or equal to the value of any observation in the sample. This corresponds to avoiding giving outliers more weight than other observations. An increasingly larger and larger body of literature has discussed the statistical properties of the least-absolute deviations estimator. It is, therefore, more and more feasible to utilize this criterion for hypothesis testing.

When a vector of optimal weights, $\hat{w}$, is computed with either the first or the second criterion, the resulting convex combination of the extreme point optimal solutions is itself an optimal solution. This follows from theorem 1, that is:

$$c'\bar{x}_1 = c'\bar{x}_2 = \dots = c'\bar{x}_k = c'P\hat{w} = \text{optimum},$$

where the $\bar{x}_i$ vectors, $i=1, \dots, k$, are the optimal solutions defining the P matrix.

In general, therefore, (that is, when there are multiple optimal solutions) finding a plausible optimal solution of LP problems involves a two-stage optimization process. In the first stage all the extreme point optimal solutions are generated. In the second stage, optimal weights are computed to collapse such solutions into an optimal convex combination.

The Validation of LP Models

A survey of the empirical literature reveals that the problem of validating LP models vis-a-vis the economic reality they intended to articulate, was not an explicit and formal objective for a large majority of such studies. This, of course, is in sharp contrast with econometric and even simulation studies, in which the validation process, through hypothesis testing of either parametric or nonparametric nature, is widely recognized as an essential step of the analysis.

It is difficult and surely beyond the scope of this study to conjecture plausible reasons for the different scientific treatment accorded studies which utilize a mathematical programming framework. The analysis of the multiple optimal solutions problem has clearly pointed toward the necessity of including model validation in a formal and well structured manner within the LP analysis of economic realities. The proposals advanced in the previous section are only a first step in this direction. In order to develop a more complete and satisfactory procedure it seems logical to parallel as much as possible the inferential methodology associated with econometric studies. In spite of seemingly radical differences, in fact, there is a considerable analogy between the logical structure of econometric and mathematical programming studies.

In econometric studies, the structural form of the model is always the result of an <u>implicit</u> optimization process assumed to underlie the behavior of the economic agent involved in the analysis. That is, the economic agent is assumed to maximize his utility, profit, expected utility, etc. In linear programming studies, such an optimization process is always explicit and corresponds, in fact, to the LP model itself. The LP constraints can be regarded as structural form relations.

In econometrics, the inferential process and validation methodology relies upon two fundamental steps: identification and hypothesis testing based on the reduced form equations. Linear programming analysis, especially when multiple optimal solutions occur, faces the same two-fold articulation. To be sure, the large size which, in general, characterizes empirical models tends to reduce the importance of the identification problem. Formally, such a problem can be stated as follows: given the LP structural form relations $Ax \leq b$, is it possible to find a matrix T such that the system $TAx \leq Tb$ is equivalent (possesses the same optimal solutions) to $Ax \leq b$? If the T matrix exists and is not an identity matrix, the LP structural forms are not identified.

The reduced form equations of a LP model are given, by the relations of the quadratic loss criterion (5) and (6). In this case, $x_A = Pw + u$, $1 = s'w$, $w \geq 0$ constitute the LP reduced forms. Hypothesis testing associated to these reduced forms can be interpreted as a validation procedure of the LP specification. If the LP model is identified and the difference between the sample observations x_A and the projection $P\hat{w}$ is not statistically significant (measured by some appropriate statistical test), we can conclude that the LP specification "explains" the behavior of the economic agents whose decisions are revealed through the vector x_A. In other words, the LP model is validated. If the difference $(x_A - P\hat{w})$ is statistically significant, the indication is that the LP model is not appropriate and an alternative formulation ought to be sought. This is in no way different from the conclusion to be derived in econometric studies.

Multiple Optimal Solutions and Economic Equilibrium

The existence of multiple optimal solutions may serve, in principle and by itself, as a validation of economic models analyzed by linear programming.[3/] Consider, for example, the problem of profit maximizing entrepreneurs who are assumed to operate in a perfectly competitive environment using homothetic technologies. If the assumptions hold, it is well known that first order conditions imply that inputs are proportional to each other or, in other words, that the expansion path is a ray. When encountered in the empirical data, these necessary conditions (expressed in the form of either exact or quasi-linear dependence) generate a problem of multicollinearity, usually disliked by the econometrician. Doll, however, in a lucid and perceptive re-examination of the problem, correctly pointed out that "users of the(se) ... model(s) who are dismayed to find multicollinearity among the 'independent' variables should be pleased because the presence of multicollinearity serves as a verification of their economic model" (p. 558). This penetrating observation (which has not yet received proper attention) has its analogous counterpart in linear programming models. The reader may recall that, in a previous, section the problem of multiple (quasi) optimal solutions was compared to that of multicollinearity in econometrics. Now, the analogy can be further elaborated in economically meaningful terms. If the firms analyzed by linear programming are assumed to operate under perfectly competitive conditions, and if the LP model so constructed is really capable of reflecting such an environment, one would expect a large number of nondominated activities to be equally profitable, whether they are included or not into the optimal basis. This is so because zero profitability is a long run necessary condition for competitive markets and, in general, commodities are available in these markets in larger number than the LP constraints of a single firm. Let us reformulate this idea as follows: in a competitive situation, where there are many firms using essentially the same technology, one observes that similar firms produce different product mixes. In the absence of uncertainty considerations, this occurrence is explainable when all the activities are equally profitable (at zero level) and, therefore, it just happens that one firm chooses a particular combination of activities while others select a different mix. In LP terminology this situation is characterized by zero relative loss not only for the optimal basic activities but also for those not in the basis. This situation, as we well know, gives rise to the multiple optimal solution phenomenon. Hence, an extensive dual degeneracy may be interpreted as a validation of an LP economic model based upon the assumption of perfect competition under certainty conditions. Of course, it is improbable that any large scale LP model will ever exhibit zero profitability for all the non-basic activities. For this reason, validation of the model must take the form of a probabilistic statement as argued above.

Normative Versus Positive Analysis

This discussion suggests another relevant issue. It has been customary, in some circles, to label linear programming as a normative analytical framework, while the character of positive analysis was reserved for econometric procedures. This black and white dichotomy has always been suspicious since both approaches are crucially dependent upon either the implicit or the explicit optimization of some structural model.

The present analysis makes that dichotomy irrelevant since it clearly points out that the normative character of linear programming is not an intrinsic property of the technique but, rather, it is the consequence of incomplete understanding and inappropriate interpretation of the programming approach. If econometric analysis is regarded as a positive procedure, the proposal of solving the multiple optimal solutions' problem by criterion 1 introduces such a positive aspect also into the solution of linear programming problems.

Any empirical analysis exhibits both positive and normative aspects in varying degrees and proportions. If a LP study is confined exclusively to the specification of technological constraints according to engineering recommendations (as it has been the traditional procedure in mathematical programming studies), it is not difficult to admit a prevalence of the normative character of the analysis. But, if the same study is validated following the scientific method requirements and the procedure exemplified in previous sections, it is difficult to distinguish between the normative and the positive aspects of the analysis.

Unfortunately, since the pronouncement of Lionel Robbins, it has been peculiar of economists to emphasize normative versus positive analysis. In this connection, it seems interesting to point out that such a distinction can hardly be found in the writings of physical scientists. They simply do not talk or write about normative and positive aspects of their research, although both components are certainly present.

A purely normative analysis is clearly unsatisfactory because it does not lend itself to scientific prediction. This has been the fate of almost the totality of mathematical programming studies. No wonder, therefore, that even the results of very large, detailed and complex models have not found a credible audience among policy makers. This paper suggests that this state of affairs is not a necessary consequence of the nature of mathematical programming models but, rather, of the simplistic use of them--without validation procedures--made by the great majority of researchers.

It is also entirely possible that, validation of LP models may turn out to be more complex and difficult than that of econometric models. But this possibility cannot be an acceptable excuse for conducting and proliferating analyses which extract a minimal rather than a maximal set of information from the specified LP structure.

Conclusions

This paper has accomplished several tasks. It has drawn attention to a neglected but important problem arising in empirical studies using an LP framework. It has outlined a correct way of reporting LP results which includes not only the exact problem size but also an explicit statement about the uniqueness (or multiplicity) of primal and dual solutions. It has suggested two plausible criteria for dealing with the (embarrassing) choice created by the existence of multiple optimal and quasi-optimal solutions. It has argued that the difference between econometric and programming approaches, based on a misunderstood dichotomy of positive versus normative characteristics, is not as clear cut as generally thought.

In summary, it has suggested that the analysis of empirical problems using a linear programming framework should not be conducted in the simplistic and mechanical way of the past 30 years, especially when the bottom line of the study is either an efficiency judgement or the recommendation of a new policy orientation.

dcb
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FOOTNOTES

1/ Multiple optimal solutions may occur also in quadratic programming models.

2/ Semidefiniteness of the matrix is a necessary condition for the presence of multiple optimal solutions in quadratic programming models.

3/ I am indebted to R. A. Collins for this suggestion.

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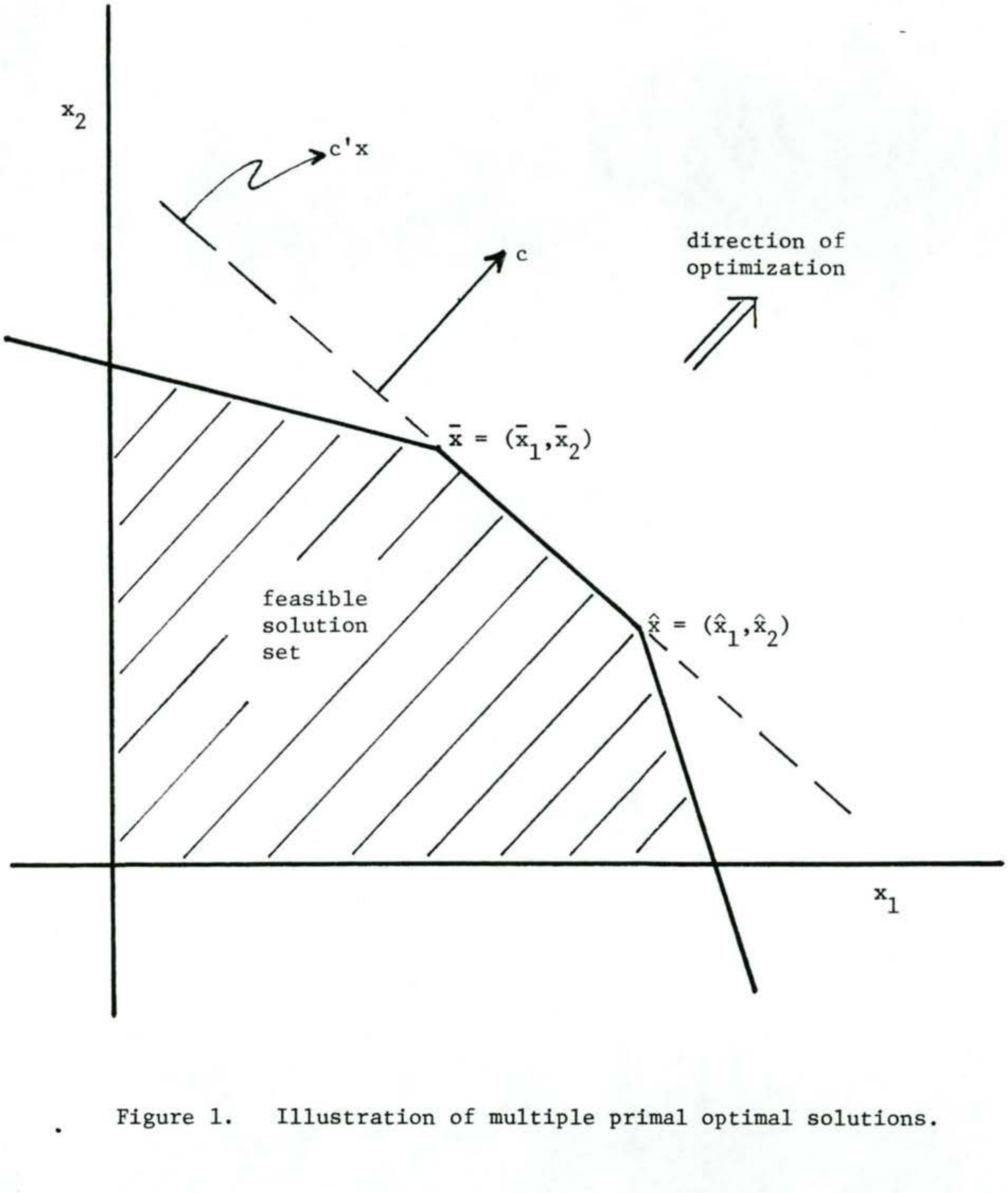

Figure 1. Illustration of multiple primal optimal solutions.

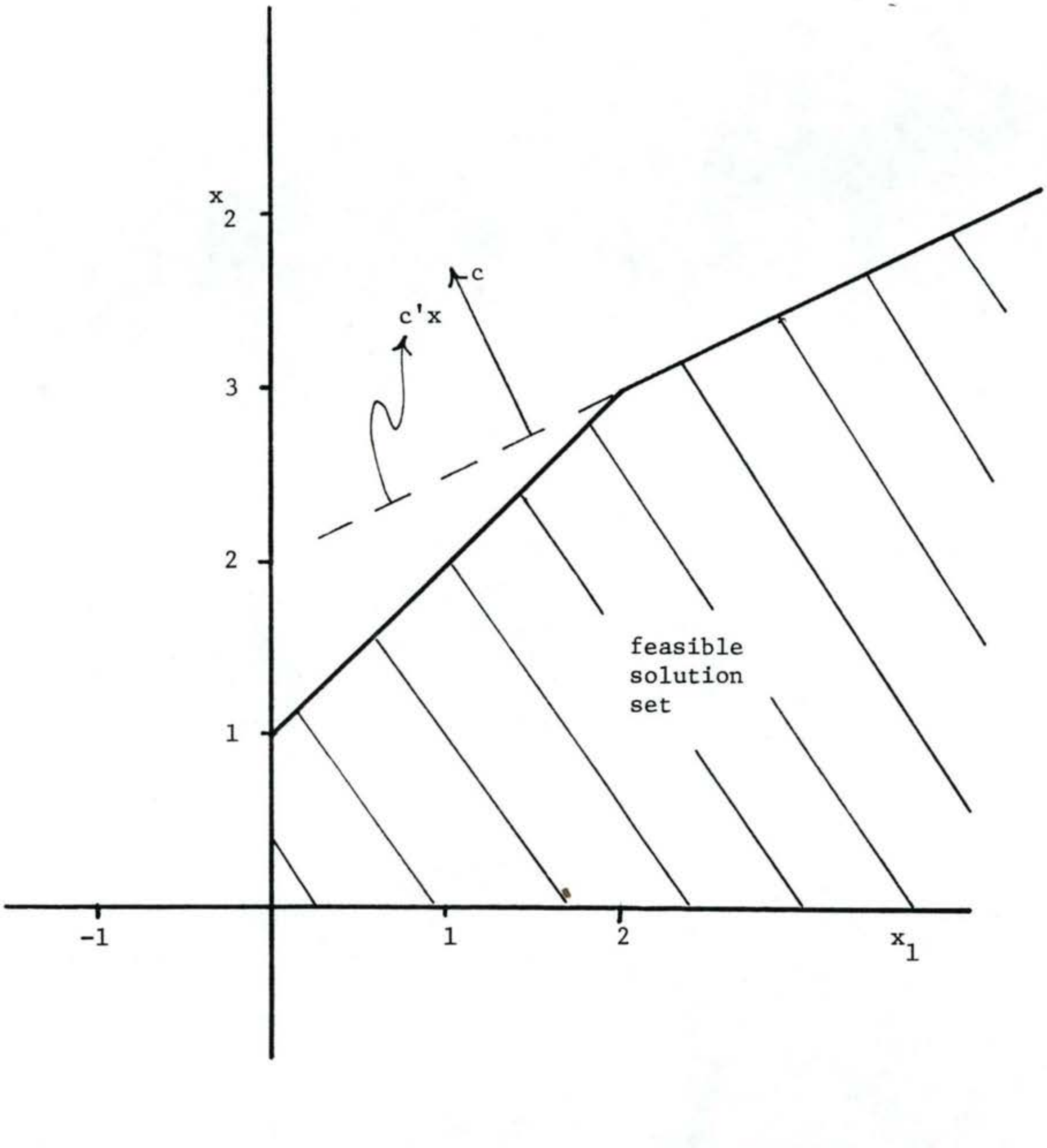

Figure 2. Multiple optimal solutions with a unique optimal extreme point.

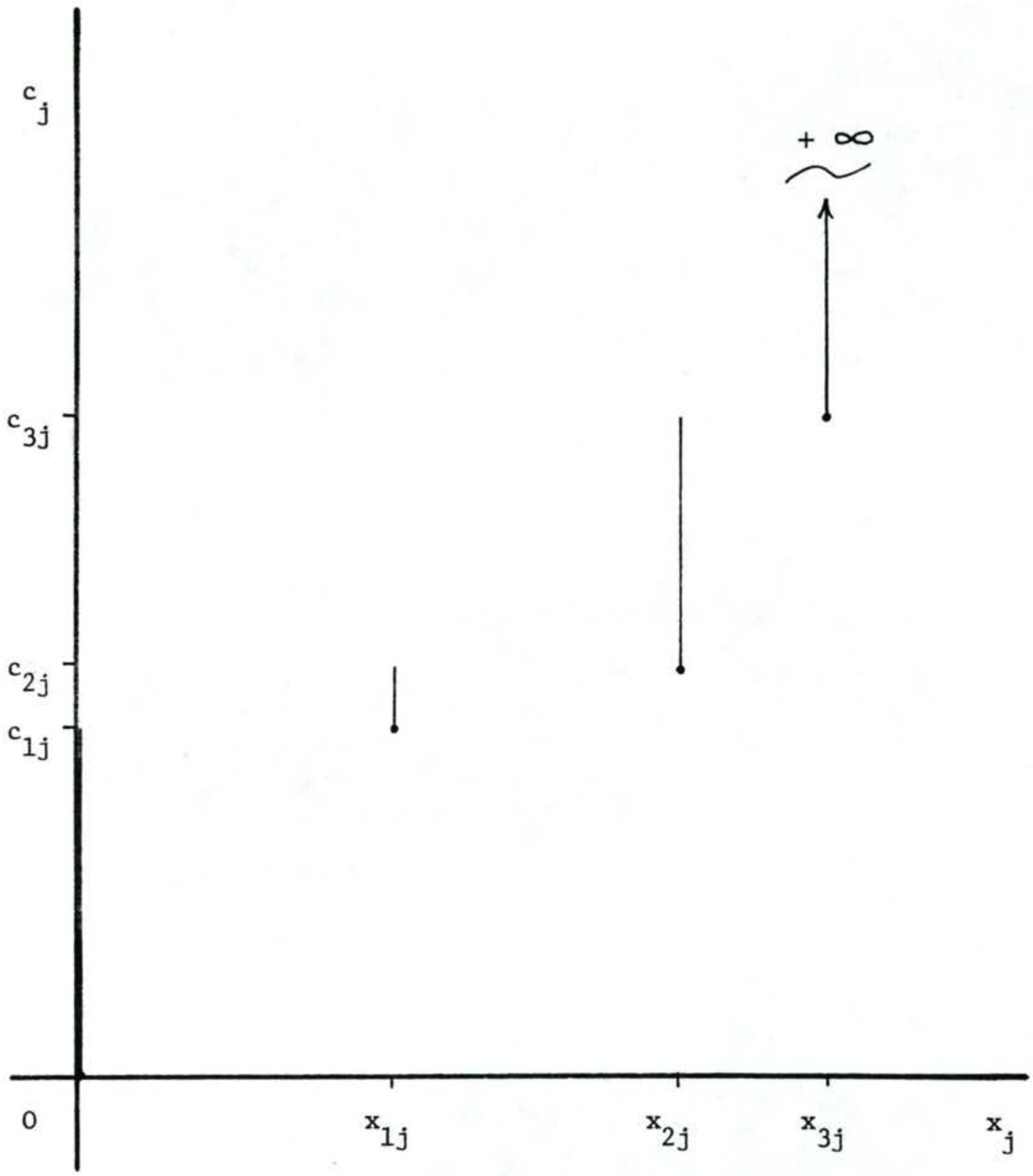

Figure 3. Illustration of parametric analysis with unique optimal bases.

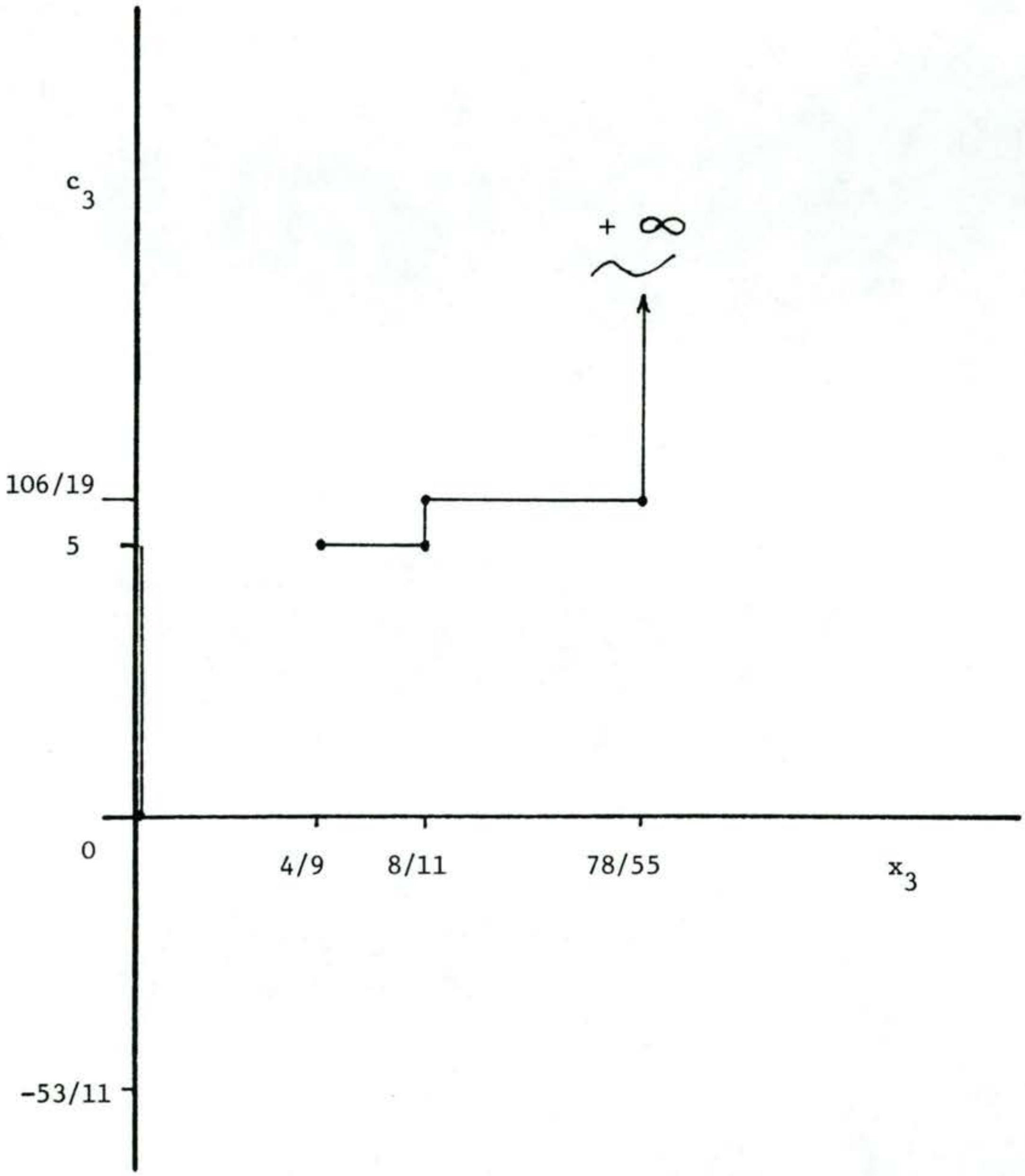

Figure 4. Illustration of parametric analysis with multiple optimal bases.