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THE LINEAR PROGRAMMING APPROACH TO CONVEX QUADRATIC PROGRAMMING

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ABSTRACT

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The solution of convex quadratic programs is shown to be accessible by means of conventional linear programming methods. The principal idea is to construct the supporting hyperplane to the convex set defined by the objective function of the quadratic programming problem. The optimal hyperplane is obtained by a maximization operation. The dimensionality of the resulting linear program is the same as that of the quadratic programming procedures. The solution of several classes of empirical problems requiring a quadratic programming specification is, thus, made easier.

A LINEAR PROGRAMMING APPROACH TO CONVEX QUADRATIC PROGRAMMING

Every convex quadratic programming (QP) problem can be solved by conventional linear programming(LP) techniques. This seemingly paradoxical proposition has gone unnoticed for more than two decades. In 1959 Wolfe published his seminal paper on the simplex method for quadratic programming. As is well known, Wolfe's [7] method requires that the simplex rules for linear programming be applied with an important qualification: no pair of complementary (slack and dual) variables with the same index can ever appear in the solution.

In the linear programming method discussed in this paper, no such rule about complementary variables is needed. Thus, every reliable standard computer code for solving linear programming problems can directly be used to obtain the solution of any convex quadratic program.

1. The Linear Programming Formulation of Quadratic Programs

Consider the following quadratic programming problem:

$$\text{minimize} \quad [c'x + x'Qx], \qquad Ax \geq b, \quad x \geq 0 \tag{P}$$

where c and x are n x 1 vectors of known coefficients and unkown variables, respectively; Q is a known n x n positive semidefinite matrix; A is a known m x n matrix and b is an m x 1 vector.

It is well known that, under the above specification, the necessary and sufficient conditions for a solution of problem P are obtain ed by differentiating the associated Lagrangean function with respect to primal and dual variables. Let y be an m x 1 vector of Lagrange multipliers. Then, the Lagrangean function of problem P is

$$L = c'x + x'Qx + y'[b - Ax] \tag{1}$$

The corresponding Kuhn-Tucker conditions are $x \geq 0$, $y \geq 0$ and

$$\partial L/\partial x = c + 2Qx - A'y \geq 0 \tag{2}$$

$$x'(\partial L/\partial x) = x'c + 2x'Qx - x'A'y = 0 \tag{3}$$

$$\partial L/\partial y = b - Ax \leq 0 \tag{4}$$

$$y'(\partial L/\partial y) = y'Ax - y'b = 0. \tag{5}$$

The Lagrangean function (1), when maximized with respect to y for any given x, constitutes the objective function of the dual problem. Notice, however, that by appropriate substitution of relations (3) and (5) into (1), the same Lagrangean function can be expressed as a hyperplane. Indeed,

$$\begin{aligned} \text{maximize } L &= c'x + (x'A'y - x'c)/2 && \text{using (3) and (5)} \\ &= [c'x + b'y]/2 && \text{using (5).} \end{aligned} \tag{6}$$

Therefore, a solution (if it exists) of the following linear programming problem

$$\begin{aligned} &\text{maximize } [c'x + b'y]/2 && && \text{(P*)} \\ &\text{subject to} && A'y - 2Qx \leq c \\ &\text{and} && -Ax \leq -b, \quad y \geq 0, \quad x \geq 0 \end{aligned}$$

represents also a solution of the original quadratic problem P.

2. The Legendre Transformation of problem P

The results of the previous section are related to the duality structure of problem P. To see this, it is convenient to study the Legendre transformation associated with it. Recall that the Legendre transformation of a differentiable function $f(z)$ is defined as $\phi(d) = f(z) - d'z$, where $d \equiv \partial f/\partial z$ is the gradient vector of $f(z)$. In our case, the function $f(z)$ corresponds to the Lagrangean function (1), $z' = (x', y)$, $d' = (d_1', d_2')$ where $d_1 \equiv c + 2Qx - A'y$ and $d_2 \equiv b - Ax$. Therefore, the Legendre transformation of problem P is

$$\begin{aligned} \phi(d) &= c'x + x'Qx + y'(b - Ax) - x'(c + 2Qx - A'y) - y'(b - Ax) \\ &= y'Ax - x'Qx \\ &= y'b + x'c/2 - x'A'y/2 \\ &= (x'c + y'b)/2. \end{aligned} \tag{7}$$

In the development of (7) we have made use of relations (3) and (5). Hence, one concludes that the maximization of $\phi(d)$ subject to the condition $d \geq 0$, is equivalent to the solution of problem P.

The Legendre transformation can be given the following interpretation: since problem P is convex, $\phi(d)$ is the intercept of the tangent plane to the convex set defined by (1). The member of the family of tangents to the convex set of P which satisfies $d \geq 0$, also generates the optimal value of the intercept $\phi(d)$. Alternatively, $(c'x + y'b)/2$ represents the supporting hyperplane to the convex set defined by $c'x + x'Qx$.

3. A Numerical example

To illustrate the linear programming method for solving convex quadratic programs discussed in previous sections we choose the following numerical example

$$Q = \begin{pmatrix} 13 & -4 & 3 \\ -4 & 5 & -2 \\ 3 & -2 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} -.5 \\ 1 \\ 2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The matrix Q is positive semidefinite; yet, this fact is of no consequence in the computations.

The solution of problem P* is $\bar{x}' = (3.0, 2.3, 0.0)$, $\bar{y}' = (59.1, 0.0)$, $(c'\bar{x} + b'\bar{y})/2 = 89.05$ and it is identical to the solution of problem P, where $c'\bar{x} = .8$ and $\bar{x}'Q\bar{x} = 88.25$. Problem P was solved using a quadratic programming computer code written by Cutler and Pass [3] and based on Dantzig's [4] variant of Wolfe's [7] simplex algorithm for quadratic programming. Problem P* was solved using a commercial linear programming subroutine called "Tempo" and copyrighted by Burroughs Corporation [1].1/

4. Conclusions

In principle, the use of linear programming methods for solving quadratic programming problems simplifies both the algorithmic computations and the problem of searching for and maintaining reliable quadratic programming computer codes. As it should be evident from the discussion, the LP approach to QP requires exactly the same dimensionality of the tableau as Wolfe's [7] QP method.

Convenient applications of the approach described in this paper include (a) the solution of generalized least-squares problems subject to deterministic and stochastic inequalities [5], (b) the solution of the linear-quadratic control problem with bounded policy instruments, (c) the solution of the portfolio selection problem, and (d) the verification of LP computer codes' reliability [6]. Of course, the same LP method can be used for solving the symmetric quadratic programming originally specified by Cottle [2]. We conjecture that other useful applications of the procedure will soon be found.

FOOTNOTES

1. Although Burroughs' "TEMPO" subroutine is unreliable, it is possible to make it work properly by inputing some information characterizing the optimal solution. Such information was available via the solution of the problem by means of the QP subroutine of Cutler and Pass [3]. For a more detailed discussion of the unreliability of TEMPO (and other LP codes) consult [6].

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