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PRODUCTION UNCERTAINTY AND FACTOR DEMANDS FOR THE COMPETITIVE FIRM

by

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Production Uncertainty and Factor Demands
for the Competitive Firm

Though the notion that production is random is an old one (Knight, 1921), most economic theory has concentrated on random demand (price). Perhaps, such focus has occurred because relatively greater importance is placed on the effects of random price. However, for many sectors of an economy (particularly those involving biological growth), production uncertainty may have relatively greater impact than market uncertainty. Further, forward contracting, hedging, etc., often can reduce price uncertainty, but few, if any, instruments exist which alleviate production uncertainty (e.g., crop insurance is not widespread in agriculture).

Many development economists have turned to production uncertainty as an important analytical tool for explaining technique of production and adoption of technologies (e.g., Roumasset (1971), de Janvry (1972), Anderson (1974)). In particular, the marginal effects of input use on the probability distribution of output is crucial to an understanding of the relationship of input use, risk and risk aversion. After positing a definition of inputs which marginally increase or decrease risk, it is argued that many empirical and seemingly all theoretical inquiries have suffered a major deficiency by not explicitly allowing for the fact that many factors of production have a risk reducing marginal effect on output.[1/] Here, a simple functional form is utilized which can describe this phenomenon and yet is empirically tractable (Just and Pope (1978)). Further, the marginal rate of factor substitution is random. The commonly used multiplicative stochastic specification is seen to be a special case of the new function. Finally, comparative statics are analyzed in terms of the marginal impact of inputs on the probability distribution of production. It is established that these marginal risk effects are crucial in the determination of comparative static results.

Motivation and Definitions

Definition: an input is said to be marginally risk reducing (increasing) if under risk aversion the expected value of the marginal product is less (greater) than factor price at the optimum - maximum expected utility.

As we shall see below, the above definition can be made more intuitive by comparing quantities of factor uses under risk aversion and risk neutrality.

Consider an expected utility maximizer, optimizing $E[U(\pi)] \equiv E[U]$, where E denotes expectation and U = utility. It will be assumed throughout that risk aversion prevails, or $U'' < 0$; the risk neutral results ($U'' = 0$) will be transparent. Profit is given by

$$\pi = Pq - C'X = Pq - \sum_{j=1}^{M} c_j x_j,$$

where,

P = price of output

q = output quantity

C = vector of input prices

X = vector of input levels.

The production function is given by

$$(1)\ q = F(X, \varepsilon),$$

where ε is a random disturbance. First order conditions (which are assumed sufficient) can be written as (following Horowitz (1970)):

$$E\ [U'(\pi)\ (PF_j\ (X, \varepsilon) - c_j)] = 0 \qquad j=1\ .\ .\ .\ M,$$

or,

$$(2)\ P\ E\ [F_j\ (X, \varepsilon) + P\ \mathrm{Cov}\ [U'(\pi), F_j\ (X, \varepsilon)] = c_j \qquad j=1\ .\ .\ .\ M.$$

Using (2) and the above definition, x_j is said to be marginally risk increasing (decreasing) as $\mathrm{Cov}\ [U'(\pi), F_j\ (X, \varepsilon)] < (>)\ 0$.

If (2) is viewed in a ceteris paribus context, (i.e., all other inputs are fixed), then assuming concavity of F in X, the definition can be restated:

Definition: an input is said to be marginally risk reducing (increasing) if the risk averse firm utilizes a larger (smaller) quantity of the input than the corresponding risk neutral firm.

Example 1. Let $F(X, \varepsilon)$ in (1) be written in the multiplicative form

$$(3)\ q = f(X)\ g(\varepsilon),\ f_j > 0;\ f_{jj} < 0 \text{ at the optimum.}$$

In this case, $\text{cov}\ [U'(\pi), F_j(X, \varepsilon)] = f_j\ \text{cov}\ [U'(\pi), g(\varepsilon)]$. Adopting Sandmo's (1971) and Batra's (1974) results, it follows that $\text{cov}\ [U'(\pi), g(\varepsilon)] < 0$ as $g'(\varepsilon) > 0$ (i.e., high "draws" of ε are associated with high "draws" of output). Hence, the commonly used functional form in (3) implies that all inputs are marginally risk increasing.

Note also that given (3), the variance of q is given by

$$(4)\ V(q) = f^2(X)\ V(g(\varepsilon)) \text{ where } V \equiv \text{variance.}$$

From (4), the marginal change in dispersion is given by

$$(5)\ \frac{\partial V(q)}{\partial x_j} = 2\ f(X)\ f_j(X)\ V(g(\varepsilon)) > 0,$$

where the inequality in (5) holds because all terms are presumed positive.

It is not the intention here to provide a detailed analysis of commonly used stochastic specifications of production technology, but a further example is illustrative (for further justification of the functional form, see Just and Pope).

Example 2. Consider the production function

$$(6)\ q = F(X, \varepsilon) = f(X) + h(X)\varepsilon;\ f_j > 0,\ f_{jj} < 0,\ E(\varepsilon) = 0.$$

It is assumed that $h > 0$ such that high "draws" of ε are associated with high values of q. For this function, note that $\text{cov}\ [U'(\pi), F(X, \varepsilon)] = h_j\ \text{cov}\ [U'(\pi), \varepsilon]$.

Hence, assuming risk aversion the covariance term is negative and

$$E(Pf_j) \gtrless c_j \text{ as } h_j \gtrless 0.$$

Thus, an input is marginally risk increasing or decreasing as h_j is positive or negative.

Note also that

$$(7)\ V(q) = h^2\ (X)\ V(\varepsilon)$$

and from (7) the marginal change in dispersion is given by

$$(8)\ \frac{\partial V(q)}{\partial x_j} = 2\ h(X)\ h_j(X)\ V(\varepsilon) \gtrless 0 \text{ as } h_j \gtrless 0.$$

It is seen that variance increases (decreases) as h_j is positive (negative).

In the case of a single input it is clear that a risk averter would use more (less) of an input which marginally decreases (increases) risk. However, in the case of the multiplicative form (3), the risk averter always uses less of the input than the risk neutral firm, ceteris paribus. In other words, there is no motive for a risk averter to have increased precautionary use of the input because the input marginally increases risk.

Consider several agricultural examples. Research indicates that at least in the short-run, firms may view the marginal dose of pesticide as insurance against production uncertainty (e.g., Turpin and Maxwell (1976), Roumasset (1977)). A priori, one might suppose many factors potentially could decrease risk. Increases in the capital-labor ratio often have the effect of reducing the effects of weather variability on agricultural production (Fuller). A similar effect may occur in manufacturing: particularly in situations where the flow of labor services is predominantly random, risk may be reduced through capital intensification. Finally, one expects generally that risk is decreasing in attention, fire fighting, or supervisory inputs. Yet the multiplicative stochastic form in conventional usage would not be able to describe these conditions.

Given the above discussion, it would appear that the functional form in (6) may be capable of describing more general forms of behavior than the multiplicative form. It should also be noted that if $h = f$ in (6), then the multiplicative form, (3), is obtained where $g(\varepsilon) = 1 + \varepsilon$. Hence, because of its simplicity and generality, the functional form proposed in (6) merits further investigation as a tool for economic analysis. Here we consider static analysis of a classical two factor ex ante model: the marginal impact of risk, risk aversion and price changes are examined.

A Two Factor Competitive Model

Let $U(\pi)$ again denote the preference function, and

(9) $\pi = \text{profit} = Pq - rk - w\ell$

where,

r = cost per unit of capital

k = quantity of capital

w = wage rate

ℓ = quantity of labor,

and P and q are as defined earlier. It will be assumed that $U' > 0$, $U'' < 0$ and $R_a'(\pi) = d/d\pi(-U''/U') < 0$.[2/] The latter assumption indicates decreasing absolute risk aversion.

The production function corresponding to (6) is

(10) $q = F(k, \ell, \varepsilon) \equiv F \equiv f(k, \ell) + h(k, \ell)\varepsilon, \quad E(\varepsilon) = 0.$

First order conditions are:

(11) $E[U'(PF_\ell - w)] = 0$ or, $Pf_\ell + Ph_\ell t = w$

(12) $E[U'(PF_k - r)] = 0$ or, $Pf_k + Ph_k t = r$

where $t = \dfrac{\text{cov}(U', \varepsilon)}{E(U')}$, cov denotes the covariance function and subscripts on functions denote partial derivatives.[3/] Note that at the optimum $k = k^*$, $\ell = \ell^*$

$$(13)\quad \frac{Pf_\ell - w}{Ph_\ell} = -t = \frac{Pf_k - r}{Ph_k}.$$

(13) implies $Pf_\ell - w = \frac{h_\ell}{h_k}(Pf_k - r)$ and hence,

$$(14)\quad \pi_\ell = \delta\pi_k, \qquad \delta = \frac{h_\ell}{h_k},$$

where $\pi_\ell = \partial\pi/\partial\ell$ and $\pi_k = \partial\pi/\partial k$.

Second order conditions imply that the principal minors of the Hessian, H, be alternating in sign for a unique maximum, where

$$(15)\quad H = \begin{bmatrix} E[U''\pi_k^2 + U'PF_{kk}] & E[U'PF_{k\ell} + \delta\pi_k^2U''] \\ E[U'PF_{k\ell} + \delta\pi_k^2U''] & E[\delta^2U''\pi_k^2 + U'PF_{\ell\ell}] \end{bmatrix}$$

and, e.g., $F_{kk} = \partial^2F/\partial k^2 = f_{kk} + h_{kk}\varepsilon$. Second order conditions indicate that $H_{11} = E[U''\pi_k^2 + U'PF_{kk}]$, $H_{22} = E[\delta^2U''\pi_k^2 + U'PF_{\ell\ell}] < 0$; this does not necessarily constrain the sign of h_k, h_ℓ, h_{kk}, or $h_{k\ell}$. However, given risk aversion $E(U''\pi_k^2) < 0$; under concavity of F, $E(U'PF_{kk}) < 0$, and the sufficiency condition involving H_{11} is satisfied.[4/] It can further be verified that $|H| > 0$ does not necessarily restrict signs of derivatives of h or the sign of $F_{k\ell}$.[5/] Though concavity of production need not be assumed to satisfy second order conditions, it appears to be a reasonable assumption based upon empirical evidence (that is, for all random draws of ε, marginal products are diminishing). Hence, throughout this paper the following assumptions regarding F are maintained:

$$(16)\quad F_{kk},\ F_{\ell\ell} < 0, \text{ or } E[U'PF_{kk}],\ E[U'PF_{\ell\ell}] < 0.$$[6/]

Lemmas Used in Comparative Static Analysis

Several Lemmas are helpful in determining comparative static results. These Lemmas are extensions of results derived by Batra and Ullah (1974), Sandmo (1971) and Feder (1978). These results are stated here and proved in the Appendix.

Lemma 1: Assuming $h > 0$, and $U'' \leq 0$, then $E[U'\varepsilon] \leq 0$.

Lemma 2: Given the assumptions in Lemma 1, and nonincreasing absolute risk aversion, then

$$E(U''\pi_\ell) \geq (\leq)\ 0 \text{ as } h_\ell \geq (\leq)\ 0$$

$$E(U''\pi_k) \geq (\leq)\ 0 \text{ as } h_k \geq (\leq)\ 0.$$

Lemma 3: Given the assumptions of Lemma 2, $E(U''\pi_\ell h_\ell \varepsilon)$, $E(U''\pi_k h_k \varepsilon) < 0$ regardless of the signs of h_k, h_ℓ.

Several comments concerning the Lemmas are in order. For each of the above Lemmas it is assumed that $f > 0$, $h > 0$ and derivatives for f and h are singularly signed (nonzero) near the optimum. The strict inequality holds in Lemma 1 if $U'' < 0$: the strict inequalities hold in Lemmas 2 and 3 if it is additionally assumed that absolute risk aversion is decreasing. Finally, note that the implications of Lemma 2 are that $h_\ell E[U''\pi_\ell]$, $h_k E[U''\pi_k] \geq 0$. For convenience only, discussion of comparative static results will focus on the implications of the strict inequality versions of Lemmas 1-3.

Comparative Statics

Mean Preserving Spread

Following Sandmo (1971), a mean preserving spread is given by defining

$$(17) \quad \varepsilon^* = \gamma\varepsilon$$

and writing (11) and (12) in terms of ε^*. Differentiating (11) and (12) with respect to γ (evaluating at $\gamma = 1$) yields:

$$(18)\quad \begin{bmatrix} \partial k/\partial\gamma \\ \partial \ell/\partial\gamma \end{bmatrix} = -H^{-1} \begin{bmatrix} E(U''Ph\varepsilon\pi_k + U'Ph_k\varepsilon) \\ E(U''Ph\varepsilon\delta\pi_k + U'Ph_\ell\varepsilon) \end{bmatrix}$$

where,

$$(19)\quad H^{-1} = \frac{1}{|H|} \begin{bmatrix} E(\delta^2\pi_k^2U'' + U'PF_{\ell\ell}) & -E(U'PF_{k\ell} + \delta\pi_k^2U'') \\ -E(U'PF_{k\ell} + \delta U''\pi_k^2) & E(U''\pi_k^2 + U'PF_{kk}) \end{bmatrix}.$$

Utilizing (19), $\partial k/\partial\gamma$ becomes

$$(20)\quad \partial k/\partial\gamma = -\frac{P}{h_k}\frac{1}{|H|}\{E(U'F_{\ell\ell} - U'F_{k\ell}\delta)[E(U''Ph\varepsilon\pi_k h_k) + E(U'Ph_k^2\varepsilon)]\}.$$

By Lemmas 1 and 3, the square bracketed term is negative given decreasing absolute risk aversion. Given h_k, $h_\ell > 0$, then $\delta > 0$. Given the concavity assumptions made (see (16)), $|H| > 0$ and $E(U'F_{\ell\ell}) < 0$. Further, if stochastic complementarity is assumed, $F_{k\ell} > 0$ and $E(U'F_{k\ell}) > 0$. Hence, $\partial k/\partial\gamma < 0$. We note that $F_{k\ell} < 0$ and $h_\ell < 0$, $h_k > 0$ yields an identical result. $\partial k/\partial\gamma$ is also negative if $h_k < 0$, $h_\ell > 0$, $F_{k\ell} > 0$ and $|EU'F_{\ell\ell}| < E(U'F_{k\ell}\delta)$. Finally, $\partial k/\partial\gamma > 0$ if h_k, $h_\ell < 0$ and $F_{k\ell} > 0$. Analogous results may be obtained similarly for labor. The major portion of these results are summarized in the following proposition.

PROPOSITION I. If stochastic complementarity ($F_{k\ell} > 0$) is assumed and both inputs marginally increase (reduce) risk, then factor use declines (increases) as risk increases. Similarly, capital (labor) decreases if stochastic substitution prevails ($F_{k\ell} < 0$) but only labor (capital) marginally decreases risk.

COROLLARY I. If production is of the stochastic multiplicative form in (3), then both inputs marginally increase risk and are reduced as risk increases (assuming $F_{k\ell} > 0$).

Corollary I is established by noting that when $f = h$ as in (3), then $\delta = f_\ell/f_k > 0$ when f_ℓ, $f_k > 0$.

Absolute Risk Aversion

Pratt (1964) has characterized the following class of utility functions exhibiting decreasing risk aversion

$$(21) \qquad U' = (\pi^a + \Psi)^{-c} \qquad a > 0,\ c > 0.$$

A particularly convenient utility function is derived from (21),

$$(22) \qquad U = (\pi + \Psi)^{\alpha} \qquad \Psi \geq 0 \qquad 0 < \alpha < 1.$$

The Arrow-Pratt measure of absolute risk aversion is

$$(23) \qquad R_a = -U''/U' = -\frac{(\alpha - 1)\alpha(\pi + \Psi)^{\alpha - 2}}{\alpha(\pi + \Psi)^{\alpha - 1}} = -(\alpha - 1)(\pi + \Psi)^{-1}.$$

Further,

$$(24) \qquad \partial R_a/\partial\Psi = (\alpha - 1)(\pi + \Psi)^{-2} < 0;$$

therefore, Ψ is inversely related to absolute risk aversion. Because $\partial U/\partial\Psi = \partial U/\partial\pi$ and $\partial^2 U/\partial\Psi^2 = \partial^2 U/\partial\pi^2$, the application of Lemmas 1-3 follows directly in comparative static results involving $d\Psi$.

Differentiating (11) and (12) with respect to Ψ yields

$$(25) \qquad \begin{bmatrix} \partial k/\partial\Psi & & E(U''\pi_k) \\ & = -H^{-1} & \\ \partial \ell/\partial\Psi & & \delta E(U''\pi_k) \end{bmatrix}$$

where $\pi_\ell = \delta\pi_k$ from (14). Applying (19) to (25) gives

$$(26) \qquad \partial k/\partial\Psi = -\frac{1}{|H|}\{E(U''\pi_k)[E(U'PF_{\ell\ell}) - E(U'PF_{k\ell}\delta)]\}.$$

By Lemma 2, sgn $E(U''\pi_k)$ = sgn h_k. By the concavity assumptions $|H| > 0$ and $E(U'PF_{\ell\ell}) < 0$. Hence, $F_{k\ell}$, h_k, $\delta > 0$ implies $\partial k/\partial\Psi > 0$. When $F_{k\ell} > 0$, $h_k < 0$, $\delta < 0$, then $\partial k/\partial\Psi > 0$ if $|E(U'PF_{\ell\ell})| > E(U'PF_{k\ell}\delta)$. If $F_{k\ell}$, $h_k > 0$ and $\delta < 0$, then $\partial k/\partial\Psi > 0$ when $|E(U'PF_{\ell\ell})| > E(U'PF_{k\ell}\delta)$. Finally, $\partial k/\partial\Psi < 0$ if $F_{k\ell}$, $\delta > 0$ and $h_k < 0$, or $F_{k\ell} < 0$, $\delta < 0$ and $h_k < 0$.[7/] Summarizing the major results of the above analyses leads to Proposition II.

PROPOSITION II. Assuming stochastic complementarity ($F_{k\ell} > 0$), if both inputs marginally decrease (increase) risk, then the firm with greater risk aversion will utilize larger (smaller) quantities of both inputs. Further, if only capital (labor) marginally reduces risk under stochastic substitution, then increased risk aversion implies an increase in capital (labor) use.

COROLLARY II. In the case of the multiplicative specification, in (3), an increase in risk aversion implies decreased use of both inputs.

Here as earlier, Corollary II is obtained by noting that $\delta > 0$ in the multiplicative case as in (3).

Regarding both propositions I and II, it should be emphasized that ambiguous results are obtained in the cases not considered in the propositions: that is, incentives are not complementary. In these cases, digressing to a mean-variance interpretation, the firm must weigh a factor's contribution to expected profit against the marginal increase or decrease in risk (variance).

Factor Prices

In general, unambiguous comparative static results involving factor price changes are difficult to achieve under risk aversion. When considering demand risks, Batra and Ullah found that $F_{k\ell} > 0$, F_{kk}, $F_{\ell\ell} < 0$ and decreasing absolute risk aversion were sufficient to yield downward sloping factor demand curves. Here, it is determined that in addition to the assumptions made by Batra and Ullah, restrictions regarding h_k and h_ℓ must be imposed in order to obtain similar comparative static results. These additional restrictions, however, were not sufficient to give determinate results for cross price effects (i.e., $\partial\ell/\partial r$, $\partial k/\partial w$).

Differentiating (10) and (11) with respect to r yields

$$(27)\quad \begin{bmatrix} \partial k/\partial r \\ \\ \partial \ell/\partial r \end{bmatrix} = H^{-1} \begin{bmatrix} E(U') + kE(U''\pi_k) \\ \\ E(k\delta\pi_k U'') \end{bmatrix} .$$

Using (19), we obtain from (27)

$$(28)\quad \partial k/\partial r = \frac{1}{|H|} \{E[\delta^2 U''\pi_k^2 + U'PF_{\ell\ell}]E(U') + E(U''\pi_k)Pk[E(U'F_{\ell\ell}) - \delta E(U'F_{k\ell})]\}.$$

By concavity, the first square bracketed term is negative, and $|H| > 0$. By Lemma 2, $E(U''\pi_k)$ and the second square bracketed term is unambiguously negative provided $F_{k\ell}$, δ, $h_k > 0$. Hence, these assumptions imply $\partial k/\partial r < 0$. However, if $h_k < 0$ and δ, $F_{k\ell} > 0$, then $E(U''\pi_k)Pk[E(U'F_{\ell\ell}) - \delta E(U'F_{\ell\ell})] > 0$ while the first square bracketed term is negative. If h_k, $\delta < 0$, $F_{k\ell} > 0$, then the sign of the second square bracketed term is ambiguous while $E(U''\pi_k) < 0$. Finally, δ, $F_{k\ell} < 0$, $h_k > 0$ also implies $\partial k/\partial r < 0$.[8/]

Looking now at the cross price effects, (27) yields

$$(29)\quad \partial \ell/\partial r = \frac{1}{|H|} \{E(kU''\pi_k)[\delta E(U'PF_{kk}) - E(U'PF_{k\ell})] - E(U')[E(U'PF_{k\ell}) + \delta E(U''\pi_k^2)]\}.$$

It is seen from (29) that unambiguous results are obtained only when the square bracketed terms are unambiguous. The first square bracketed term requires sgn δ = sgn $F_{k\ell}$ given concavity of F. The second bracketed term requires $-$sgn δ = sgn $F_{k\ell}$, since $E(U''\pi_k^2) < 0$ under risk aversion. Hence, given concavity and decreasing absolute risk aversion the ambiguity of $\partial \ell/\partial r$ remains as in Batra and Ullah. Proposition III contains the essentials of the above arguments.

PROPOSITION III. If both factors marginally increase (decrease) risk under stochastic complementarity (substitution), then factor demand curves are downward sloping.

COROLLARY III. For the multiplicative specification factor complementarity ensures downward sloping input demand curves.

In all other cases, incentives are not complementary and ambiguous results are obtained.[9/]

Output Price

Sandmo has shown that decreasing absolute risk aversion and a convex cost function imply $\partial q/\partial E(P) > 0$ under price uncertainty. Batra and Ullah, extending this result, determined that $\partial k/\partial E(P)$, $\partial \ell/\partial E(P) > 0$ given concavity of production, decreasing absolute risk aversion and $F_{k\ell} > 0$. Under production uncertainty, we are unable to unambiguously specify $\partial k/\partial E(P)$, $\partial \ell/\partial E(P) > 0$ given the assumptions made previously. However, a sufficient condition is developed indicating these results.

Differentiating (11) and (12) with respect to P yields

$$\begin{bmatrix} \partial k/\partial P \\ \partial \ell/\partial P \end{bmatrix} = -H^{-1} \begin{bmatrix} E(U'F_k) + E(U''\pi_k q) \\ E(U'F_\ell) + E(U''\delta\pi_k q) \end{bmatrix} .$$

or utilizing (19)

$$(30) \quad \partial k/\partial P = - \frac{1}{|H|} \{E(\pi_k U''q)[E(U'PF_{\ell\ell}) - \delta E(U'PF_{k\ell})] + E(U'F_k)$$
$$[E(\delta^2\pi_k^2 U'') + E(U'PF_{\ell\ell})] - E(U'F_\ell)[E(U'PF_{k\ell})$$
$$+ E(\delta\pi_k^2 U'')]\}.$$

Note, that given $E(\pi_k U''q) > 0$ and F_k, $F_\ell > 0$, then $\partial k/\partial P \geq 0$ if

$$(31) \quad E(U'PF_{\ell\ell}) - \delta E(U'PF_{k\ell}) < 0$$

and

$$(32) \quad E(U'PF_{k\ell}) + \delta E(\pi_k^2 U'') > 0,$$

since $E(\delta^2\pi_k^2 U'') + E(U'PF_{\ell\ell}) < 0$ given our assumptions. If $E(U'PF_{\ell\ell}) < 0$, then (31) is unambiguously signed if $F_{k\ell}$ and δ have the same sign. However, (32) is unambiguously positive if $\delta < 0$ and $F_{k\ell} > 0$, since $E(\pi_k^2U'') < 0$. Hence, from (30)-(32) one can only conclude that given f_k, f_ℓ, $E(U''\pi_\ell q)$, $E(U''\pi_k q) > 0$, then $\partial k/\partial P > 0$, if $E(U'PF_{k\ell} + \delta\pi_k^2U'') > 0$. This latter restriction is the counterpart of the sufficiency condition under certainty ($F_{k\ell} > 0$) which rules out inferior factors, or implies $\partial k/\partial P > 0$ (Bear (1965)).

The restriction that $E(U''\pi_k q) > 0$ also poses an ambiguity. Explicitly, it can be written

$$E(\pi_k U''q) = \frac{1}{h_k}\{E[U''\pi_k h_k f] + hE[U''\pi_k h_k \varepsilon]\}.$$

By Lemma 2, the sign of the first square bracketed term is negative provided $f > 0$. By Lemma 3, the second square bracketed term is negative. Hence, the assumptions in the Lemmas are not sufficient to sign $E[U''\pi_k q]$.[10/] One is accustomed under certainty to speculating that $\partial k/\partial P$, $\partial \ell/\partial P > 0$, i.e., inferior inputs are not present. Thus, the above result may appear strange. However, the major difficulty is that as price rises, under risk aversion, the additional profits are valued less and one must weigh a factor's contribution to output and risk. In the case where the factor contributes little to f but has a large marginal risk reduction, one might suppose that demand for the input would fall as price rose.[11/]

Symmetry Conditions

It can be verified by direct differentiation or the application of Silberberg's method that the Hicksian symmetry restrictions, $\partial\ell/\partial r = \partial k/\partial w$, $\partial k/\partial P = -\partial E(q)/\partial r$, $\partial\ell/\partial P = -\partial E(q)/\partial w$, generally do not hold under risk aversion but are obtained under risk neutrality. Thus, restrictions based upon risk averse behavior are substantially diminished as compared to the risk neutral case. However, for a class of commonly used utility functions, further results are obtained.

Empirical risk analyses often utilize utility functions based upon moments of the distribution of profit.[12/] Such functions are often rationalized on the basis of Taylor's series expansions of U about $E(\pi)$ (Farrar (1962)). Consider the following class of functions,

$$(33)\quad E(U) = E(\pi) + \sum_{j=2}^{J} a_j\sigma_j \equiv E(\pi) + g\,(\sigma_2 \ldots \sigma_J),$$

where $\sigma_j = E[\pi - E(\pi)]^j$ and a_j is a constant. Freund's (1952) exponential utility function (under normality) is a member of this class where $a_2 < 0$ and $\sigma_3 \ldots \sigma_J = 0$.

Given the utility function in (33), the corresponding first order conditions are:

$$(34)\quad Pf_\ell - w + \sum_{j=2}^{J} a_j jP^j h^{j-1} h_\ell \hat{\sigma}_j = 0$$

$$Pf_k - r + \sum_{j=2}^{J} a_j jP^j h^{j-1} h_k \hat{\sigma}_j = 0,$$

where $\hat{\sigma}_j$ is the jth moment of ε.

Differentiating (34) with respect to prices yields

$$(35)\quad \begin{bmatrix} \frac{\partial\ell}{\partial P} & \frac{\partial\ell}{\partial w} & \frac{\partial\ell}{\partial r} \\ \frac{\partial k}{\partial P} & \frac{\partial k}{\partial w} & \frac{\partial k}{\partial r} \end{bmatrix} = \frac{1}{|H|} \begin{bmatrix} H_{22}T_1 - H_{12}T_2 & H_{22} & -H_{12} \\ H_{11}T_2 - H_{12}T_1 & -H_{12} & H_{11} \end{bmatrix}$$

where H_{ij} denotes the i - jth element of the Hessian, H, and

$$T_1 = -f_\ell - \sum_{j=2}^{J} a_j j^2 P^{j-1} h^{j-1} h_\ell \hat{\sigma}_j \text{ and } T_2 = -f_k - \sum_{j=2}^{J} a_j j^2 P^{j-1} h^{j-1} h_k \hat{\sigma}_j.$$

Noting that by concavity $H_{ii} < 0$ $(i = 1, 2)$, it is apparent from (35) that

$$\frac{\partial\ell}{\partial w}, \frac{\partial k}{\partial r} \leq 0$$

and

$$\frac{\partial \ell}{\partial r} = \frac{\partial k}{\partial w} .$$

However, other ambiguities (e.g., $\partial E(q)/\partial P$) remain. Clearly, factor demands and output supplies are not homogeneous of degree zero in nominal income given the utility function in (30). However, for homothetic utility functions of the form $U(K\pi) = \phi(K)U(\pi)$, where K represents the factor of proportionality prices and $\phi(K) > 0$, the symmetry conditions would be violated but homogeneity would be preserved.[13/]

Conclusions

We shall not restate the results indicated in Propositions I-III of this paper, but merely conclude that the above analysis suggests that the effects of factors of production on the probability distribution of output have important consequences for firm behavior. It was found that all comparative static results are more ambiguous but richer when the limiting multiplicative stochastic specification of production is generalized. In particular, empirical studies have found evidence of risk reducing inputs. In these cases, it seems clear that risk aversion may lead to increased factor use as compared to the risk neutral case. Similarly, marginal increases in risk and risk aversion may lead to increased factor use. These intuitive results were obtained when inputs marginally decrease risk--such results are not derived under stochastic complementarity with the case of a multiplicative disturbance as utilized in most empirical analyses. Finally, the marginal impact of prices on factor demands are also sensitive to the marginal risk effects of various inputs. Thus, referring to an earlier example, the qualitative and quantitative aspects of the demand for pesticides could be substantially altered depending on the input's risk increasing or reducing nature. These results may also enrich the understanding of capital and investment demands when capital marginally decreases risk. The essential point is that for sectors or problems where risk "matters," an important aspect to be studied is the means of risk reduction through instruments internal to the firm. In this paper, we have considered a small part of this problem; that is, how do inputs affect risk and how does this relate to factor demand? Since the multiplicative specification in (3) is a special case of the stochastic formulation considered in this paper and is utilized extensively in empirical and theoretical work, the begging question is "does the generalization discussed in this paper more adequately represent technology for the problem confronting a researcher." We conclude from limited available information that the generalization enables greater understanding of observed behavior.

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Footnotes

1/ For examples of what is later defined as marginal risk increasing formulations of production, see, e.g., Batra (1974), (1975), Blair and Lusky (1975), Stiglitz (1974), Rothenberg and Smith (1971). Walters (1970), Ratti and Ullah (1976), Smith (1970) and others have considered the case where factor service flows are proportional to factor stocks. For many commonly used production functions, all inputs are used less under risk aversion than under risk neutrality. In nearly all cases, it is seen that inputs are marginally risk increasing because of the proportionality assumption.

2/ $R_a(\pi)$ is the Arrow-Pratt measure of absolute risk aversion. Note that the argument (π) of $U' = \partial U/\partial\pi$ and $U'' = \partial^2 U/\partial\pi^2$ is suppressed. This practice will be continued throughout the paper.

3/ The use of covariance in the marginal conditions follows popular usage; since $E(\varepsilon) = 0$, it follows that $\text{cov}(U', \varepsilon) = E(U'\varepsilon)$.

4/ $E(U''\pi_k^2) < 0$ under risk aversion. In order that $H_{11} < 0$, we must assume $-E(U'PF_{kk}) > E(U''\pi_k^2)$. Therefore, $E(U'PF_{kk}) = Pf_{kk}E(U') + Ph_{kk}E(U'\varepsilon)$ may actually be positive and yet satisfy the second order condition.

5/ $|H|$ under concavity of production implies no restrictions on the signs of h_k, h_ℓ, or $F_{k\ell}$. $|H|$ is given by

$$|H| = \delta^2 E[U''\pi_k^2]^2 + E[U''\pi_k^2]E[U'PF_{\ell\ell}] + E[U'PF_{kk}]E[\delta^2 U''\pi_k^2] + E[U'PF_{kk}]E[U'PF_{\ell\ell}]$$
$$- (E[U'PF_{k\ell} - \delta U''\pi_k^2])^2.$$

6/ Since U', $P \geq 0$, then F_{kk}, $F_{\ell\ell} < 0$ implies $E(U'PF_{\ell\ell})$, $E(U'PF_{kk}) \leq 0$. In general, strict concavity and a positive price are assumed throughout such that the strict inequality prevails.

7/ Under stochastic substitution ($F_{k\ell} < 0$), $\frac{\partial k}{\partial \Psi} > 0$ if $h_k > 0$ and $\delta < 0$, or h_k, $\delta > 0$ and $|E(U'PF_{\ell\ell})| > E(U'PF_{k\ell}\delta)$. When $F_{k\ell}$, $h_k < 0$, $\delta > 0$, then $\frac{\partial k}{\partial \Psi} < 0$ if $|E(U'PF_{\ell\ell})| > E(U'PF_{k\ell}\delta)$.

8/ One might be tempted to conclude that capital demands are less responsive to cost of capital changes when capital marginally decreases risk as opposed to the case where h_k, $h_\ell > 0$. However, $|H|$ also changes for each situation.

9/ We note that $E(U'F_{\ell\ell}) > \delta E(U'F_{k\ell})$ with capital marginally reducing risk and $F_{k\ell} > 0$ implies capital demands and unit capital cost are inversely related. Generally we would expect input demand curves to be negatively sloped and. hence, expect the ambiguity to be resolved.

10/ For the multiplicative case (3),

$$E(\pi_k U''q) = \frac{f}{f_k} E(\pi_k U'' f_k \varepsilon) < 0$$

by Lemma 3. Therefore a sufficiency condition such that $\partial k/\partial P > 0$ could be stated as

$$E(U'PF_{k\ell} + \delta\pi_k^2 U'') > 0$$

$$E(U'F_k + U''\pi_k q) > 0$$

$$E(U'F_\ell + U''\pi_\ell q) > 0,$$

where the first condition states that the off diagonal element of the Hessian matrix be positive as in the certainty case.

11/ One would also need to consider factor interactions as well such as $h_{k\ell}$ and $f_{k\ell}$.

12/ Though many theorists have argued convincingly against such methodology (e.g., Hadar and Russell (1969)), it appears that statistical pragmatics often dictate that moment analysis be used.

13/ An example of a commonly used utility function which is decreasing risk averse and homothetic is $U = \pi^{\alpha}$, $0 < \alpha < 1$ in which case $\phi(K) = K^{\alpha}$.

Appendix

Proof of Lemma 1:

Profit is written as

$$\pi = Pf + Ph\varepsilon - rk - w\ell = E(\pi) + Ph\varepsilon.$$

For $\varepsilon > 0$, $\pi > E(\pi)$ since $P, h > 0$. Given risk aversion $U'[E(\pi)] > U'(\pi)$. For $\varepsilon < 0$, $\pi < E(\pi) \rightarrow U'[E(\pi)] < U'(\pi)$. Hence, $E[U'(\pi)\varepsilon] < 0$ and the Lemma is established.

Proof of Lemma 2:

Let ε^* be that value of ε such that $\pi_k = 0$ (denoted by π_k^*). Defining $R_a(\pi) = -U''/U'$, it follows that decreasing absolute risk aversion $[R_a'(\pi) < 0]$ implies

$$U'' \leq (\geq) - R_a(\pi^*)U' \text{ as } \varepsilon \leq (\geq)\varepsilon^*.$$

Multiplying this result by π_k yields

$$\text{(A.1)} \quad U''\pi_k \leq (\geq) - R_a(\pi)U'\cdot\pi_k \text{ as } h_k \leq (\geq)0.$$

For example, when $\varepsilon \geq \varepsilon^*$, then $U'' \leq -R_a(\pi^*)U'$ and $\pi_k = Pf_k + Ph_k\varepsilon - r \leq 0$ if $h_k \leq 0$. Hence, $\pi_k U'' \leq -R_a(\pi^*)U'\pi_k$. $\varepsilon \leq \varepsilon^*$ implies $U'' \geq -R_a(\pi^*)U'$, and $\pi_k \geq 0$ if $h_k \leq 0$; hence,

$$U''\pi_k \leq -R_a(\pi^*)U'\pi_k$$

for all ε. Similar analysis verifies (A.1) when $h_k \geq 0$. Taking expectations of both sides of (A.1) yields $E[U''\pi_k] \geq (\leq)0$ as $h_k \geq (\leq)0$, since $E(U'\pi_k) = 0$ by first order conditions. An identical proof, mutatis mutandis, establishes the result for labor.

Proof of Lemma 3:

We note that $\pi_k = Pf_k + Ph_k\varepsilon - r$, or $Ph_k\varepsilon = \pi_k + (r - Pf_k)$. Hence,

$$\text{(A.2)} \quad E[U''\pi_k Ph_k\varepsilon] = E[U''\pi_k^2] + (r - Pf_k)E[U''\pi_k].$$

$E[U''\pi_k^2] < 0$ under risk aversion; by Lemma 2 and (12), sgn $(r - Pf_k) = -$sgn h_k and sgn $E[U''\pi_k] =$ sgn h_k. Hence, sgn$\{(r - Pf_k)E[U''\pi_k]\}$ is negative. Note that since $P > 0$, the Lemma may also be stated $E[U''\pi_k h_k\varepsilon] < 0$. The result for labor may be similarly obtained using (A.2) when ℓ replaces k.

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