UCD Department
of Agricultural Economics

REVENUE AND COST UNCERTAINTY, GENERALIZED
MEAN-VARIANCE AND THE LINEAR
COMPLEMENTARITY PROBLEM*

by

Quirino Paris

Working Paper No. 78-2

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Revenue and Cost Uncertainty, Generalized Mean-Variance and the Linear Complementarity Problem

In recent years a number of papers have reproposed in this Journal the mean-variance (E-V) approach as a means of dealing with uncertainty both at the firm (Wiens) and the market level (Hazell and Scandizzo). The same method was originally pioneered by Freund about 20 years ago and applied by McFarquar and Camm in more realistic contexts.

As it is usually known, the E-V approach required the restriction of uncertainty to elements of the primal activities (either yields or net unit revenues). The papers by Wiens and Hazell and Scandizzo follow explicitly this tradition. On the other hand, it is generally recognized that an important source of uncertainty is represented by risky supplies of limiting inputs. This area of inquiry has received wide attention since the beginning of the mathematical programming era. The innumerable contributions associated with it can conveniently be classified into the two categories of stochastic programming--proposed by Tintner, Charnes and Cooper--and of penalty cost programming which is principally associated with the names of Dantzig and Madansky.

Relevant papers about stochastic programming familiar to agricultural economists are those by Cocks, Maruyama, Rae, Hazell and How. They all proposed significant methodological improvements in dealing with farm planning under uncertainty. The apparent drawback of such approaches, however, is that they have not been sufficiently tested by means of empirical studies of some realism, possibly because the required dimensionality of the associated problems is still regarded as a heavy computational burden.

In this paper, a class of hitherto unexplored stochastic programming structures is presented and analyzed. Although the problems admitted by these structures are not among the most general stochastic programs (a nonstochastic linear technology is postulated), they seem to be of considerable interest for both methodological and computational reasons. Firstly, they treat stochastic limiting resources in a way analogous to stochastic net revenues and yields. This formulation, therefore, allows an entirely symmetric interpretation of quadratic programming problems and leads to a substantial reduction in computational effort when compared with other stochastic specifications. These savings are achieved because the proposed formulation does not require either additional constraints or variables over and above those involved in the traditional E-V approach when uncertainty is confined to only revenues and yields.

The first stochastic structure presented in this paper is based upon a methodological improvement of quadratic programming that occurred in 1963. In that year,[1/] Cottle presented the symmetric version of quadratic programming. Although he discussed it from a purely formal point of view and did not suggest any empirical use of the new specification, it is one of considerable empirical potential. Furthermore, another unexplored but flexible programming structure, called the linear complementarity problem, will be shown to represent farm planning under uncertain revenues and costs (of limiting inputs) including interaction between the two components of profits.

Risky Revenues and Costs Without Interaction

The setting is that of a farm whose entrepreneur operates in a competitive but uncertain environment. In general, uncertainty affects revenues in two ways: through output prices, p, and yields. To keep the description to its maximum of simplicity only output prices are considered aleatory.[2/] The sources of uncertain costs we are especially interested in analyzing here are the supplies of "fixed" or limiting inputs, s, that is of those inputs which acts as constraints on the production plan. As examples relevant to a farm environment, such aleatory supplies might be the amount of family labor determined by the number of days allowed by weather conditions; ground water for irrigation as determined by drought conditions; timing of custom operations as determined by the service availability; machine availability as determined by the probability of repairs and losses. Uncertain prices and supplies of nonlimiting inputs are dealt quite easily in the conventional E-V framework. With output activity levels indicated by the letter x and input (shadow) prices by the letter y, profits are simply $\pi = p'x - s'y = d'z$, where, by obvious correspondence, $d' \equiv [p', -s']$ and $z' \equiv [x', y']$; the dimensions of both d and z are taken to be $[(n+m)x1]$. Under these assumptions, only the d vector is stochastic. To adopt the E-V method it is convenient to assume also that the d vector is normally distributed as $d \sim N(E(d), \Sigma)$, where $\Sigma = \begin{bmatrix} \Sigma_p & 0 \\ 0 & \Sigma_s \end{bmatrix}$. In turn, profits will also be normally distributed as $\pi \sim N(E(\pi), z'\Sigma z) = N(E(p)'x - E(s)'y, x'\Sigma_p x + y'\Sigma_s y)$, since by assumption, there is no interaction between output prices and input supplies.

The dual pair of symmetric quadratic problems describing the uncertain problem of the firm can now be stated as follows:

(1) Primal $\max \{E(p)'x - (\phi/2)x'\Sigma_p x - (\phi/2)y'\Sigma_s y\}$

subject to $Ax - \phi\Sigma_s y \leq E(s)$

$y \geq 0, \quad x \geq 0$

(2) Dual $\min \{E(s)'y + (\phi/2)y'\Sigma_s y + (\phi/2)x'\Sigma_p x\}$

subject to $A'y + \phi\Sigma_p x \geq E(p)$

$y \geq 0, \quad x \geq 0.$

The matrix A, of dimensions (mxn), represents the nonstochastic technology of the firm. The reason for referring to problems (1) and (2) as "symmetric" ought to be clear. Each relation of either problem exhibits the same formal structure as the corresponding dual relation. If the variance matrix of input supplies, Σ_s, is the null matrix, the dual pair (1) and (2) reduces to the familiar asymmetric quadratic programming commonly used to analyze the revenue uncertainty in the conventional E-V method.

The unusual and interesting property of the above model is that, if a solution exits, the vector variable y appearing in the primal problem represents also the vector of dual variables. To show this, it is sufficient to define the Lagrangean function of problem (1) using y as the vector of Lagrange multipliers, and verify that the resulting Kuhn-Tucker conditions are indeed identical to the constraints of problems (1) and (2). Hence, from the following Lagrangean function

(3) $L = E(p)'x - (\phi/2)x'\Sigma_p x - (\phi/2)y'\Sigma_s y + y'[E(s) + \phi\Sigma_s y - Ax]$,

the Kuhn-Tucker conditions are $y \geq 0$, $x \geq 0$ and

(4) $\frac{\partial L}{\partial x} = E(p) - \phi\Sigma_p x - A'y \leq 0$

(5) $x'\frac{\partial L}{\partial x} = x'E(p) - \phi x'\Sigma_p x - x'A'y = 0$

(6) $\frac{\partial L}{\partial y} = E(s) + \phi\Sigma_s y - Ax \geq 0$

(7) $y'\frac{\partial L}{\partial y} = y'E(s) + \phi y'\Sigma_s y - y'Ax = 0$.

Clearly, the systems of constraints (4) and (6) are identical to those in problems (2) and (1), respectively.

The economic interpretation of problems (1) and (2) can be based upon the conventional specification of E-V programming and on a novel reinterpretation of the constraints in terms of chance constrained programming.[3/] According to the first scheme, the primal objective function requires the maximization of the firm's expected net revenue minus the risk premium that a risk averse entrepreneur may be willing to pay for a certain level of monetary receipts. The risk premium is composed of two elements reflecting the double source of uncertainty: $(\phi/2)x'\Sigma_p x$ is the subjective cost the entrepreneur is willing to pay as a consequence of uncertainty in output prices; $(\phi/2)y'\Sigma_s y$ is the analogous cost associated with the uncertain input supplies.

The primal constraints represent the technological possibilities of the firm under a risky environment. When restated as $Ax \leq E(s) + \phi\Sigma_s y$ they clearly indicate the requirement that the input use, Ax, must be less than or equal to expected supplies $E(s)$, modified by a term $\phi\Sigma_s y$, which constitutes a marginal risk adjustment directly related to the existence of uncertain input supplies. In general, nothing can be affirmed regarding the sign of this term, implying that risky conditions and risk aversion may dictate either a larger or a smaller procurement of inputs.

The performance function of the dual problem (2) can conveniently be viewed as the objective of an alternative entrepreneur who wishes to buy out the original owner. In this case, the new entrepreneur's goal is to minimize the total expected cost of the firm's aleatory input supplies, $E(s)'y$, as well as the amount of money that he should reimburse the original owner for the payment of the risk premium.

The dual constraints of (2) are more familiar and correspond to the traditional E-V analysis. They can be rewritten as $A'y \geq E(p) - \phi\Sigma_p x$, indicating that an equilibrium solution is reached when the marginal activity cost, $A'y$, is greater than or equal to expected price, $E(p)$, adjusted by a marginal risk premium due to uncertain output prices.

The stochastic programming interpretation of problems (1) and (2) further supports the use of these structures for dealing with uncertain economic problems. In particular, it fully justifies the adoption of expected supplies $E(s)$ and the presence of the covariance matrix Σ_s of input supplies in the primal constraints. Suppose in fact that the entrepreneur's maximization of (1) is subject to the following chance constraint

(8) $\text{Prob}\ (y'Ax - y's \leq 0) \geq \alpha$,

where the input supplies s are random variables distributed as $s \sim N(E(s), \Sigma_s)$. The probability statement indicates that the imputed cost of factor use, $y'Ax$, must be less than or equal to the inputed value of available resources with at least an α probability. In the terminology of stochastic programming, the probability α is referred to as a confidence level, while its counterpart $(1 - \alpha)$ is called risk level. It is postulated that the entrepreneur, confronted by a risky environment, chooses risk levels acceptable to him. In other words, the risk levels are related to his disliking (or preference) for risk: the smaller $(1 - \alpha)$, the smaller the propensity for risk-taking of the entrepreneur. From the theory of chance constrained programming (Vajda, p. 78) it can be shown that

$$(9) \quad \alpha = \text{Prob}\ [y'Ax - s'y \leq 0]$$

$$= \text{Prob}\ [(-s'x + E(s)'y)/(y'\Sigma_s y)^{1/2} \leq \tau_s]$$

$$= \text{Prob}\ [E(s)'y - \tau_s(y'\Sigma_s y)^{1/2} \leq s'y].$$

The choice of τ_s is made to satisfy $(1 - \alpha) = (1/\sqrt{\pi}) \int_{-\infty}^{\tau_s} \exp[-(1/2)w^2]dw$ where w is a standardized normal variate. When $\alpha > 1/2$, $\tau_s < 0$. Quoting Vajda [p. 80], "if $y'Ax$ is not larger than $E(s)'y - \tau_s(y'\Sigma_s y)^{1/2}$, then it is not larger than any of those $s'y$ which are not smaller than $E(s)'y - \tau_s(y'\Sigma_s y)^{1/2}$, and the probability of such $s'y$ is α. Hence, the constraint Prob $[y'Ax - s'y \leq 0] \geq \alpha$ is equivalent to the nonstochastic constraint

$$(10) \quad y'Ax \leq E(s)'y - \tau_s(y'\Sigma_s y)^{1/2}."$$

The relationship between (10) and the primal constraints of problem (1) is established via the Kuhn-Tucker condition associated with (1) and given in (7). Hence, $\phi \leq -\tau_s/(y'\Sigma_s y)^{1/2}$. If the E-V problem (1) is solved first, the probability α of the associated chance constrained program can be derived by computing the value of τ_s as $\tau_s = -\phi(y'\Sigma_s y)^{1/2}$ and then reading α from a table of the standardized normal variate. Notice that since $(y'\Sigma_s y)^{1/2}$ and ϕ are both positive (for a risk averse entrepreneur), the parameter τ_s is negative and the probability α of satisfying constraint (8) is greater than .5.

It should be emphasized that the stochastic specification is offered here as a further justification for the structure of constraints (1) rather than as a computational framework. In fact, solving the chance constrained problem directly requires the solution of the nonlinear constraint (10), a task not easily performed. Here, it is suggested to make explicit the relationship between the E-V approach and stochastic programming by first solving problem (1) for a given coefficient of risk aversion, ϕ, and then computing the risk level $(1 - \alpha)$ compatible with such a risk aversion. If this is done, the input use, Ax, is guaranteed to be feasible for all those input supply outcomes, s, that will be greater than or equal to $E(s) + \phi\Sigma_s y$. Therefore, the economic-technological interpretation of the primal constraints (1) can be summarized as follows: an entrepreneur with constant risk aversion, ϕ, who faces uncertain supplies of limiting inputs (normally distributed), may choose to replace the unknown constraints ($Ax \leq$ random s) with the structure $Ax \leq E(s) + \phi\Sigma_s y$, requiring the knowledge of the first two moments of the probability distributions of the input supplies. This problem possesses the stochastic programming interpretation given above.

By analogy, a chance constrained programming representation is readily available also for the dual constraints of the E-V problem. The stochastic specification corresponding to constraints (2) is thus

(11) $\text{Prob}\ (p'x - y'Ax \leq 0) \leq 1 - \beta$

where the output prices, p, are distributed as $p \sim N[E(p),\Sigma_p]$. The economic interpretation is that an entrepreneur would accept events where total revenue, $p'x$, may be less than or equal to total imputed cost, $y'Ax$, with a probability $(1 - \beta)$, or smaller. Repeating the logical process outlined above, choose τ_p such that

$$(12) \quad 1 - \beta = \text{Prob}[(p'x - E(p)'x)/(x'\Sigma_p x)^{1/2} \leq \tau_p]$$

$$= \text{Prob}[p'x \leq E(p)'x + \tau_p (x'\Sigma_p x)^{1/2}].$$

Therefore, the stochastic constraint $\text{Prob}(p'x \leq y'Ax) \leq (1 - \beta)$ is equivalent to the nonstochastic constraint

$$(13) \quad y'Ax \geq E(p)'x + \tau_p (x'\Sigma_p x)^{1/2}.$$

Again, the relationship between (12) and constraint (2) of the E-V problem is obtained via the complementary slackness condition of problem (1) given by (5). It follows that $\phi \leq - \tau_p/(x'\Sigma_p x)^{1/2}$.

Solving the Generalized E-V Problem

The solution of problem (1) may be attempted with standard QP packages based, for example, on the Frank-Wolf algorithm, but this procedure is inefficient because it requires treating the dual variables y as primal variables. The size of the problem, then, becomes $[(2m+n)x(2m+n)]$ with the introduction of auxiliary dual variables, say w, which in the end, will turn out to be equal to y.

Fortunately, a more efficient algorithm developed by Lemke is available. This algorithm was proposed for solving the linear complementarity problem (LCP). Hence, to use this solution procedure it is necessary to restate problem (1) in the form of the LCP.

The linear complementarity problem is defined as follows: find vector z such that

(14) $q - Mz \leq 0, \quad z \geq 0$

and

(15) $z'(q - Mz) = 0,$

where M is a square matrix of dimensions $[(m+n)x(m+n)]$ and q is any real vector, $q \subset R^{m+n}$. For positive semidefinite matrices M, Lemke's algorithm guarantees to find a solution (if it exists) of the LCP.

To transform problem (1) into the structure of (14) and (15) it is sufficient to notice that the Kuhn-Tucker conditions associated with it are necessary and sufficient for a global maximum since the covariance matrices Σ_p and Σ_s are positive semidefinite. Hence, the K-T conditions (4) through (7) can be rearranged into the structure of (14) and (15) by making the following correspondence:

$$z = \begin{bmatrix} x \\ y \end{bmatrix}, \quad q = \begin{bmatrix} E(p) \\ -E(s) \end{bmatrix}, \quad M = \begin{bmatrix} \phi\Sigma_p & A' \\ -A & \phi\Sigma_s \end{bmatrix}.$$

The matrix M is positive semidefinite for any matrix A.

Constant Risk Aversion Utility Functions and the Generalized E-V Approach

The traditional E-V model, as proposed originally by Freund, was associated with a class of utility functions exhibiting constant absolute risk aversion. The generalized E-V model presented here is related to the same class, via the expected utility hypothesis.

Following Freund, the entrepreneur is assigned a concave utility function conveniently specified as

(16) $U(\pi) = 1 - \exp(-\phi\pi), \quad \phi > 0$

where ϕ is a subjective and constant coefficient of risk aversion and π represents profits as defined above. The entrepreneur is assumed to choose those levels of inputs and outputs which maximize the expected utility of his profits; that is

(17) $EU(\pi) = 1 - \exp(-\phi[E(\pi) - \frac{\phi}{2}\,VAR(\pi)])$

with E indicating the expectation operator.

Maximizing this monotonically increasing function is equivalent to finding $z' = (x', y') \geq 0$ such that

$$(18)\quad \max\ \{E(\pi) - \frac{\phi}{2}\,VAR(\pi)\} = \max\ \{E(d)'z - \frac{\phi}{2}\,z'\Sigma z\}$$

$$= \max\ \{E(p)'x - E(s)'y - \frac{\phi}{2}\,x'\Sigma_p x - \frac{\phi}{2}\,y'\Sigma_s y\}$$

$$= \max\ \{q'z - (1/2)\ z'Mz\}.$$

Problem (18) is equivalent to problem (14) and (15) which, in turn, is equivalent to problem (1). This proposition is easily demonstrated by indicating that the K-T conditions of problem (18), taken in the form of the last expression, are precisely relations (14) and (15). Hence, the dual of problem (18) is

$$(19)\quad \min\ \{\frac{\phi}{2}\,x'\Sigma_p x + \frac{\phi}{2}\,y'\Sigma_s y\}$$

subject to (14). The objective function (19) is naturally interpreted as the minimization of the risk premium the risk averse entrepreneur would be willing to pay for the certainty of being guaranteed a level of satisfaction equivalent to the maximum utility of expected profits. In other words, the quantity (19) is the difference between $E(U(\pi^*))$ and $U(E(\pi^*))$ where π^* is the value of profits which maximizes (18) subject to $z \geq 0$.

A Digression on the Coefficient of Risk Aversion

The only characterization of the parameter ϕ made so far is that, for a risk averse entrepreneur, it is a positive constant and represents his subjective evaluation of the importance of uncertainty. Freund pointed to the difficulty of obtaining adequate estimates of this coefficient, simply because of its subjective nature. In his empirical work he adopted a value of ϕ equal to 1/1250, rushing to advise that "any chosen value is exceedingly difficult to defend" (p. 258). This opinion seems overly pessimistic. Recently, Wiens has proposed than an average estimate of the same coefficient may be obtained by using the dual constraints of the E-V method in association with market as well as actual farm information. He noticed that for those activities operated at positive levels the corresponding constraints of type (4) are binding and, therefore, can be solved for the ϕ coefficient as

$$(20) \quad \phi_F \simeq [E(p_j) - a_j' r]/\Sigma_{pj} x_F$$

where a_j is the jth activity (assumed to be operated at positive level), $E(p_j)$ is the jth expected output price, and Σ_{pj} is the jth row of the variance matrix Σ_p; r is the vector of market prices for the limiting inputs and x_F is the vector of actual levels of activities operated by the entrepreneur. The subjective information derived from the entrepreneur is entirely incorporated in the vector x_F of personal choices of activity levels carried out in the uncertain environment as perceived by him.

The suggestion of Wiens has some merit if the information about the individual activity levels is available; but it requires also that the estimated ϕ not be used in conjunction with the data it was estimated from, as Wiens did. To begin with, his exclusive reliance on the dual constraints for the estimation of the risk aversion coefficient is incomplete. The same coefficient can be estimated from the primal constraints (2) corresponding to positive shadow prices of the limiting inputs. In this case, the following relation is obtained:

$$(21) \quad \phi_F = [a_i' x_F - E(s_i)] / \Sigma_{si} r,$$

where a_i' is the ith row of the technological matrix A, s_i is the ith input supply, Σ_{si} is the ith row of the variance matrix Σ_s; r and x_F are the same as in (20). This second, or better, primal way to estimate ϕ, either may reinforce the consistency of the estimates of ϕ obtained from (20) or it may generate an embarrassment of choice. Obviously, this is an empirical dilemma. In any event, the two relations (20) and (21) taken together constitute a rather stringent test of the consistency of the risk averse entrepreneur. They contain, in fact, all the information necessary to make an optimal decision. If the estimates of ϕ obtained from them are consistent (that is, are almost the same), it should be concluded that the entrepreneur's actual choices of output levels are optimal and no need exists to perform further optimizations. In other words, the utilization of the estimated ϕ in the quadratic programming model (in conjunction with the same data used to compute ϕ) merely corresponds to a tautological exercise.[4/] The assumption of constancy of ϕ implies that a meaningful use of it requires a variation in either the technological or economic environment.

The reinterpretation of the E-V model in terms of chance constrained programming offers an alternative--and perhaps, more interesting--way to determine the coefficient of risk aversion ϕ. Such measures would naturally be defined as

$$\phi_F = - \tau_F/(r'\Sigma_s r)^{1/2}$$

and

$$\phi_F = - \theta_F/(x_F'\Sigma_p x_F)^{1/2},$$

where r and x_F have the same meaning as above, while τ_F and θ_F are parameters chosen to correspond to the subjective levels of probabilities α and β set by the firm's entrepreneur as a requirement for the fulfillment of constraints (8) and (11), respectively. Notice that in the above measures, the problem of multiple estimates of ϕ encountered in Wiens' method is avoided. Furthermore, the direct implications of using a linear technology (usually defined by the researcher rather than by the entrepreneur) are also eliminated. The parameters τ_F and θ_F replace the specification of the technology as an information tradeoff. Thus, it may be easier to elicit information from the entrepreneur concerning acceptable (to him) risk levels $(1 - \alpha)$ and $(1 - \beta)$, rather than either a detailed description of his input-output technology or a direct estimate of ϕ.

Risky Revenues and Costs with Interaction

The assumption of zero covariance between revenues and costs was convenient for maintaining a certain degree of simplicity. It is not, however, empirically satisfactory. If it had to be maintained because of the structure of either the symmetric QP or of the LC problem, it would almost nullify the significance of both programming frameworks. All published papers dealing with either market or farm planning in an E-V context have always glossed over the subject. This is a consequence of the structure of traditional E-V analysis and asymmetric QP problem. Uncertain supplies of limiting inputs will presumably interact with aleatory yields, rendering farm decisions even more challenging. In this section, therefore, the environmental uncertainty is redefined to include nonzero covariances between revenues and costs. Specifically, the covariances involve output prices and input quantities. Thus, the relevant variance matrix is now specified as $\Sigma = \begin{bmatrix} \Sigma_p & -\Sigma_{ps} \\ -\Sigma_{sp} & \Sigma_s \end{bmatrix}$, where, as before p and s are output price and input supply subscripts, respectively. The procedure to incorporate the nonzero covariance matrix Σ_{ps} into the generalized E-V method is slightly different from that presented in the previous sections. It is convenient to consider the expected utility model first. Then, the bilinear form $\phi x'\Sigma_{ps}y$, involving the covariance matrix appears in both primal amd dual objective functions corresponding to (18) and (19) above, as an additional component of risk premium. The crucial aspect of this more general description of uncertain economic environments is the way the new matrix Σ_{ps} enters the primal and dual constraints. The relevant primal problem (analogous to (18) above) is now to find $z \geq 0$ such that

$$(22) \quad \max\ [E(\pi) - \frac{\phi}{2}\,\text{VAR}(\pi)] = \max\ \{E(d)'z - \frac{\phi}{2}\,z'\Sigma z\}$$

$$= \max\ \{E(p)'x - E(s)'y - \frac{\phi}{2}\,x'\Sigma_p x - \frac{\phi}{2}\,y'\Sigma_s y + \phi x'\Sigma_{ps} y\},$$

$$= \max\ \{q'z - (1/2)z'Mz\}$$

where now the M matrix is defined as $M = \begin{bmatrix} \phi\Sigma_p & A'-\phi\Sigma_{ps} \\ -A-\phi\Sigma_{sp} & \phi\Sigma_s \end{bmatrix}$. Relation (22) remains a concave function because $z'\Sigma z$ (or, equivalently, $z'Mz$) is a positive semidefinite quadratic form. The K-T conditions associated with this specification are again given by (14) and (15) or, more explicitly: $x \geq 0$, $y \geq 0$

(23) $Ax + \phi\Sigma_{sp}x - \phi\Sigma_s y \leq E(s)$

(24) $A'y - \phi\Sigma_{ps}y + \phi\Sigma_p x \geq E(p)$

(25) $y'[Ax + \phi\Sigma_{sp}x - \phi\Sigma_s y - E(S)] = 0$

(26) $x'[A'y - \phi\Sigma_{ps}y + \phi\Sigma_p x - E(p)] = 0.$

The difference between this set of restrictions and those presented in (4) through (7) is that the covariance term Σ_{sp} appears explicitly either to impose further restrictions on the input use, or to relax their binding availabilities, depending upon the sign of the term $\Sigma_{sp}x$. In a perhaps more illustrative expression, constraints (23) may be restated as $Ax + \phi\Sigma_{sp}x \leq E(s) + \phi\Sigma_s y$. The left hand side indicates that the input use Ax, must now be properly adjusted to account for the interaction between output prices and input quantities. It should be transparent that, in general, the level of output, x, admissable by the system of constraints will be either larger or smaller than that corresponding to $\Sigma_{ps} = 0$, depending upon the sign of $\Sigma_{sp}x$. In a similar restatement of constraints (24), $A'y - \phi\Sigma_{ps}y \geq E(p) - \phi\Sigma_p x$, the marginal activity cost must be adjusted by the marginal risk factor $\phi\Sigma_{ps}y$, depending exclusively upon the nonnegligible interrelation between input supplies and output prices. The right hand side is, again, unchanged.

The dual to problem (22) is easily obtained by using the K-T conditions associated with it. It can easily be verified that this calculation corresponds to

$$(27)\quad \min \{(\phi/2)x'\Sigma_p x + (\phi/2)y'\Sigma_s y - \phi x'\Sigma_{ps} y\}$$

subject to (23) and (24), together with $x \geq 0$, $y \geq 0$. As before, (27) is to be interpreted as the minimization of the risk premium corresponding to this more complex environment.

The check of consistency of the risk averse entrepreneur according to Wiens' suggestion becomes even more formidable than in the previous case. As before, for binding primal constraints and positive output activity levels, it is possible to compute the risk aversion coefficient of the individual entrepreneur in two quite different but strictly related ways: from (23) and the ith (assumed) binding constraint the following is obtained

$$(28)\quad \phi_F \simeq [a_i' x_F - E(s_i)]/(\Sigma_{si} r - \Sigma_{spi} x_F).$$

Similarly, from (24) and the jth (assumed) positive activity level:

$$(29)\quad \phi_F \simeq [E(p_j) - a_j' r]/(\Sigma_{pj} x_F - \Sigma_{psj} r),$$

where r and x_F have the meaning already established. Caution ought to be exercised in the use and interpretation of these relations. Failure to obtain close values of ϕ from the two formulas should not be immediately considered as an inconsistency of the entrepreneur; it may be due to the particular technological matrix, A, chosen by the researcher.

Notice that introduction of nonzero covariances between input quantities and output prices prevents the restatement of problem (22) in the form of symmetric quadratic programming. It does not prevent, however, its transformation into a corresponding linear complementarity problem. The matrix M remains a positive semidefinite matrix since $z'Mz = \phi z'\Sigma z \geq 0$, for any z. Hence, Lemke's algorithm is guaranteed to find a solution to the LC problem, if one exists.

Empirical Issues and Conclusions

The analysis presented above was focused entirely on methodological issues. The empirical implementation of any of the models, however, was always kept in mind. Indeed, this is the main reason for dealing with uncertain environments within an E-V framework rather than with more flexible but more demanding approaches. The inevitable question, therefore, looms on the horizon: how can the variance of input supplies be estimated? One would feel tempted to reply: in the same way the variances of either output prices or yields are computed. Upon further reflection, though, it must be admitted that, typically, data collectors pay a disproportionately large attention to the revenue side of the economy with the cost aspect well underdeveloped. The question is, therefore, legitimate although only a few hints can be offered to alleviate the informational imbalance. It may be hoped that once models of cost uncertainty will be developed in large numbers, the need for accurate and suitable cost information will become self evident also to the data gathering bureaucracies. In the meantime, one must use ingenious devices such as Boisvert and Jenson's idea for estimating the variance of family labor on small Minnesota farms. They suggest that the expected field days of the farm family and their variance be computed from records of monthly weather information. Another suggestion deals with water. With drought having affected various parts of the country in recent years, an analogous method can be perfected for estimating means and variances of available water supplies. At the aggregate level—not discussed in this note—other appropriate suggestions can be formulated for the variances of, say, energy, labor supply, financial capital and so on. One thing ought to be emphasized. The initial lack of empirical procedures cannot deter any researcher from developing and presenting unusual but more realistic models.

The great surprise of this development is the realization that either quadratic programming or the linear complementarity problems are very suitable structures to deal with rather complex problems. Their flexibility to handle a variety of meaningful economic problems has been barely tapped by the treatment presented in this note. I would expect that many other and more meaningful uses of both symmetric QP and LC problems are awaiting to be dusted from the shelves of intellectual neglect.

gg
9/18/78

FOOTNOTES

1/ During the sixties, Cottle presented the same symmetric QP specification in at least five different papers.

2/ One simple way to include stochastic yields into the framework was suggested by Hazell and Scandizzo.

3/ At first sight it might seem natural to interpret the primal problem in terms of penalty cost programming, but this viewpoint is misleading for several reasons. First of all, penalty cost programming presupposes that decisions in the face of uncertainty are made in two stages: the first decision occurs prior to the eventuating of the uncertain outcome (input supplies) when a level $\bar{x}$, of activities x is chosen; the second decision, concerning an adjustment between the realized input supplies, $\bar{s}$, and the prior decision, $A\bar{x}$, is permissible only at a cost. This structure requires the equality of primal constraints and, eventually, the knowledge of the realized input supplies, $\bar{s}$. None of these conditions are present in the above formulation where all decisions are made without the exact knowledge of the eventuating outcome.

4/ Wiens concludes (p. 633) ". . . the risk aversion model accords quite well with the average behavior of sampled peasants. Primal solutions in general call for full diversification among the three crops in proportions close to those observed." This statement represents a clear illustration of the tautological use of the estimated ϕ. The discrepancy between actual and optimal activity allocations reported by Wiens is due, on one level, to the use of only the dual constraints in the estimation of ϕ (in other words, he did not utilize the available information on input supply E(s); on the other, to the averaging performed on the sample information.

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